ON $F'$- CLOSURE OF $\tilde{F}$- HOMOGENEOUS GROUPS

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ABSTRACT: Given a homomorph $F$, a finite group $G$ is a $D_F$ group if $G$ has an $F$-projector $F$ such that every solvable $F$-subgroup is contained in some conjugate of $F$. $G$ is $\tilde{F}$-homogeneous if $N_G(X)/C_G(X) \subseteq F$ for every solvable $F$-subgroup of $G$. The following theorem is proved. Assume that $F$ is an s-closed extensible homomorph and $G$ is a $D_F$ group which is $\tilde{F}$-homogeneous, then $G \in F' F$.

This theorem generalizes results about $D_\pi$ $\pi$-homogeneous groups and $\pi'$-closure.

Introduction

All groups considered in this paper are finite. In [1], it is shown that if $\pi$ is a set of prime numbers, then every $\pi$-homogeneous $D_\pi$-group is $\pi'$-closed. The following equivalence is then trivial: $G/0_n(G)$ is a solvable $\pi$-group if and only if $G$ is a $\pi$-homogeneous $D_\pi$-group with solvable Hall $\pi$-subgroups.

1980 Mathematics Subject Classification 20D99

Key words and phrases, s-closed extensible homomorph, $D_F$ group, $\tilde{F}$-homogeneity.

* The second author was partially supported by a National Science Foundation Grant.
In this paper, as in [4], we consider an extensible and s-closed homomorph $F$ and generalize these results.

We recall that a non-empty class of groups $F$ is a homomorph if whenever $G \in F$, then all homomorphic images of $G$ are contained in $F$. A homomorph is s-closed if whenever $G \in F$, then all subgroups of $G$ are contained in $F$. A homomorph $F$ is extensible if whenever both $G/N$ and $N$ are contained in $F$, then $G \in F$. Let $F$ denote any homomorph which is closed under normal subgroups; then $F'$, the derived class of $F$, is defined by $F' = \{G | S/N \in F \text{ implies that } S = N \}$ for each subgroup $S$ of $G$. (See [4]). For such a homomorph $F$, the radical $G_F$ is defined for a group $G$ by $G_F = \langle N | G/N \in F' \rangle$ is defined for a group $G$. A group $G$ is defined to be $F'$-closed if $G/G_F \in F$ or equivalently if $G \in F'F$.

Let $F$ be a homomorph, a group $G$ is defined to be a $D_F$ group if $G$ has an $F$-projector $P$ such that every solvable $F$-subgroup of $G$ is contained in some conjugate of $F$.

$G$ is defined to be a $D_F$ group if $G$ has exactly one conjugacy class of $F$-projectors and every $F$-subgroup of $G$ is contained in an $F$-projector.

$G$ is defined to be $F$-homogeneous if $N_G(X)/C_G(X) \in F$ for every solvable $F$-subgroup $X$ of $G$, is $F$-homogeneous if $N_G(X)/C_G(X) \in F$ for every $F$-subgroup $X$ of $G$.

We note that if $G$ is a $\tau$-homogeneous $D_F$ group, then it is direct to see that $G$ is an $F$-homogeneous $D_F$ group where $F$ is s-closed extensible homomorph of $\tau$-groups. We show
(Lemma 3) that $G_{F^1} = O_{F^1}(G)$ for this $F$. Thus the following theorem is a generalization of the result mentioned in the first paragraph.

**Theorem A**

Assume that $F$ is an $s$-closed extensible homomorph and $G$ is a $D_F$ group which is $F$-homogeneous, then $G \in F' F$.

The following corollary generalizes the second remark in the first paragraph and characterizes the product class $F'(F \cap H)$ when $H$ is a solvable class.

**Corollary B**

Let $F$ be an $s$-closed and extensible homomorph and $H$ a class of solvable groups. Then the following are equivalent:

1. $G \in F'(F \cap H)$.
2. $G$ is a $D_F$-group with $F$-projectors belonging to $H$ and $G$ is $F$-homogeneous.
3. $G$ is a $D_F$-group with an $F$-projector $F$ belonging to $H$ and $G$ is $F$-homogeneous.

Corollary B strengthens Theorem III 2.6 of [4] by replacing the nilpotency hypothesis in the $F$-projectors of that theorem by solvability.
Section One

Lemma 1

Let $F$ be an $s$-closed homomorph. Every subgroup $H$ of an $\widetilde{F}$-homogeneous group $G$ is an $\widetilde{F}$-homogeneous group.

Proof:

Let $X$ be a solvable $F$-subgroup of $H$, then $X$ is an $F$-subgroup of $G$. Whence $N_G(X)/C_G(X) \in F$. Now $N_H(X)/C_H(X) = (N_G(X) \cap H)/(C_G(X) \cap H) \cong (N_G(X) \cap H)C_G(X)/C_G(X)$ implies that $N_H(X)/C_H(X)$ is isomorphic to a subgroup of $N_G(X)/C_G(X)$. Since $F$ is $s$-closed, $N_H(X)/C_H(X) \in F$.

Lemma 2

Let $F$ an extensible homomorph of finite groups and let $G$ be an $F$-homogeneous group. Then:

(i) $G/K$ is an $\widetilde{F}$-homogeneous group for each normal solvable $F$-subgroup $K$ of $G$.

(ii) If $F$ is also an $s$-closed homomorph, with $G/K \in F'$ and $K \in F$ where $K$ is solvable, then $G \in F'$.

Proof:

(i) Let $X/K$ be a solvable $F$-subgroup of $G$, then $X$ is solvable so $N_G(X)/C_G(X) \in F$. Now $\rho : N_G(X) \rightarrow N_{G/K}(X/K)/C_{G/K}(X/K)$ defined by $\rho(g) = gK$ is an epimorphism whose kernel
contains $C_G(X)$.

Therefore, $N_{G/K}(X/K) / C_{G/K}(X/K)$ is an epimorphic image of the $F$-group $N_G(X)/C_G(X)$. Since $F$ is a homomorph, $N_{G/K}(X/K) / C_{G/K}(X/K)$ lies in $F$. Hence $G/K$ is a $\tilde{F}$-homogeneous group.

(ii) Let $M/K$ be the $F'$-radical $(G/K)_F$ of $G/K$. Since $K \in F$ and $M/K \in F'$, $([K], |M:K|) = 1$ as $F$ is s-closed. Now the Schur-Zassenhaus theorem yields a subgroup $L$ of $M$ such that $M = KL$ and $M/K \cong L$. Let $K_p$ denote a Sylow $p$ subgroup of $K$. By the Frattini argument $M = N_M(K_p)K$. Thus $|L|/|N_M(K_p)|$.

Further, $N_M(K_p) \cap K$ is a normal Hall subgroup of $N_M(K_p)$. Hence by the Schur-Zassenhaus theorem, $N_M(K_p)$ has a Hall subgroup $L_1$ of order $|L|$ and $L_1$ and $L$ are conjugate in $M$. Thus, by Sylow theory, we may choose notation so that $L \subseteq N_M(K_p)$.

Therefore, $LC_M(K_p)/C_M(K_p)$ is contained in $N_M(K_p)/C_M(K_p)$. Since $G$ is $\tilde{F}$-homogeneous, Lemma 1 implies that $N_M(K_p)/C_M(K_p) \in F$. Now $LC_M(K_p)/C_M(K_p)$ must be an $F$-group because $F$ is s-closed. However, $L \in F'$ implies that $LC_M(K_p)/C_M(K_p) \in F'$.

Thus, $|LC_M(K_p)/C_M(K_p)| = 1$ and $[L, K_p] = 1$. Repeating the argument for all primes $p$ dividing $|K|$, we conclude that $M = L \times K$ and $L = M_F$.

Now $L$ is a characteristic subgroup of $M$ so $L \leq G$.

Finally, $G/L / M/L \cong G/M$, $M/L \cong K \in F$, and $G/M \cong G/K / M/K \in F$. Since $F$ is extensible, $G/L \in F'$.

We state Lemma 3 and Proposition 4 in greater generality
than needed for independent interest. We note that every s-closed extensible homomorph $F$ is an s-closed and saturated formation by [5, I (1.2), (2.1), I 2.5]) and the proof of [5, I (1.14)].

**Lemma 3**

Assume $F$ is an s-closed and saturated formation and $G$ is a $D_F$-group with $F$-projector $F$ such that every solvable $F$-group in $G$ lies in some $F^g$, $g \in G$. Let $\pi$ denote the set of prime divisors of $|F|$, then $F$ is a Hall $\pi$-subgroup of $G$ and $G_{F^g} \subseteq 0_{\pi'}(G)$.

**Proof:**

Let $F_p$ denote a non-trivial Sylow p-subgroup of $F$. If $F_p$ is not a Sylow p-subgroup of $G$, there is a p-group $K$ such that $F_p \triangleleft K$ and $[K : F_p] = p$.

Since $F$ is s-closed and saturated, $K$ must belong to $F$ following [5, I: (3.1)]. But $K$ is solvable so $K \subseteq F^g$ which is a contradiction. Hence $F$ is a Hall $\pi$-subgroup of $G$.

Let $R$ be any $\pi'$-subgroup of $G$. If $R \not\subseteq F'$, there is $N \triangleleft T$ with $T/N \in F$ and $R \supseteq T \supseteq N$. Let $v$ be a prime dividing $[T:N]$, then $F$ contains $Z_v$, a cyclic group of order $v$. However, $R$ also contains $\langle x \rangle$ a cyclic group of order $v$ and $\langle x \rangle \subseteq F$.

Since $\langle x \rangle$ is solvable, $\langle x \rangle \subseteq F^g$ which contradicts $(|F|, |R|) = 1$. Thus $R \subseteq F'$ and in particular $0_{\pi'}(G) \subseteq G_{F^g}$. If $vl(|G_{F^g}|, |F|)$, then both $F$ and $F'$ contain a cyclic
group of order \( v \) since \( F \) and \( F' \) are s-closed. This contradicts \( F \cap F' = \{1\} \). Hence, \( G_{F'} = 0_{\pi}(G) \).

**Proposition 4**

Let \( F \) be an s-closed saturated formation. Assume \( G \) is a \( D_F \)-group with \( F \)-projector \( F \) such that every solvable \( F \)-subgroup of \( G \) lies in some \( F^g \). If whenever two elements in \( F \) are conjugate in \( G \) then they are conjugate in \( F \), then \( G \in F' F \).

**Proof:**

Let \( \pi \) denote the set of prime divisors of \( |F| \). By Lemma 3, \( F \) is a Hall \( \pi \)-subgroup of \( G \). Let \( E \) be an elementary subgroup of \( G \) such that \( |E| = |F| \), then \( E = Z \times P \) where \( P \) is a p-group and \( Z \) is cyclic. Following [5, I. (3.1)], \( P \) and every Sylow subgroup of \( Z \) lie in \( F \). Now \( F \) a formation yields \( E \in F \). Hence, \( E \subseteq F^g \) for some \( g \in G \). The Brauer-Suzuki Theorem [3, Th. (8.22)] implies that \( G = 0_{\pi}(G)F \).

By Lemma 3, \( 0_{\pi}(G) = G_{F'} \), whence \( G \in F' F \).

**Proof of Theorem A:**

The proof is divided into three parts. Let \( G \) be a minimal counterexample to the theorem, \( F \) be a \( F \)-projector such that every solvable \( F \)-subgroup lies in a conjugate of \( F \), and let \( \pi \) denote the set of prime divisors of \( |F| \).

**(A)** There are no normal non-trivial solvable \( F \)-subgroups of \( G \).
Assume $K$ is a nontrivial normal solvable $F$-subgroup of $G$, then $K \subseteq F$ and $F/K$ is a $F$-projector of $G/K$. Suppose $X/K$ is any solvable $F$-subgroup of $G/K$, then $X$ is solvable and $X \subseteq F$. Thus, $X \subseteq F^q$ and $X/K \subseteq (F/K)^q$. Hence, $G/K$ is a $\widetilde{D}_F$-group. By Lemma 2, $G/K$ is $\widetilde{r}$-homogeneous. The minimality of $G$ yields $G/K \subseteq F'F$. Now $G \subseteq F'F$ follows from Lemma 2.

(B) Let $S$ be a non-trivial $p$-subgroup of $F$. Then

(i) $N_G(S)$ is $F'$-closed.

(ii) $N_G(S) = N_F(S) \cap (C_G(S))$, and

(iii) $S \subseteq F^w$ implies that $F^w = F^y$ where $y \in C_{N_G(S)}(C_G(S))$.

We first show that (B)(i) and (ii) hold for any $1 \neq S$ such that

(*) $S \subseteq F^w$ implies $N_{F^w}(S) = (N_F(S))^r$

for $r \in N_G(S)$.

Assume (*) holds, we will show $N_G(S)$ is a $\widetilde{D}_F$-group and that $N_F(S)$ is an $F$-projector of $N_G(S)$. If $K$ is any Sylow $v$ subgroup of $N_G(S)$ for $v$ a prime in $\pi$, then $KS$ is a $p$ group if $v = p$ or a $(p,v)$-group. In particular, $KS$ is solvable so $KS \subseteq F^w$. By (*) $KS \subseteq (N_F(S))^r$ for some $r \in N_G(S)$. Thus, $N_F(S)$ is a Hall $\pi$-subgroup of $N_G(S)$.

Let $U$ be a subgroup of $N_G(S)$ which contains $N_F(S)$ with $W \Delta U$ and $U/W \in F$. If $t$ is any prime dividing $[U:W]$, then $F$ $s$-closed implies that $F$ contains a cyclic group of
order $t$. However, $U$ also contains a cyclic subgroup $\langle x \rangle$ of order $t$. Since $\langle x \rangle$ is a solvable $F$-group, $\langle x \rangle \subseteq F^g$.

Therefore, $U/W$ is a $\pi$-group. Since $N_F(S)$ is a Hall $\pi$-subgroup of $N_G(S)$, $U = WN_F(S)$. Hence, $N_F(S)$ is an $F$-projector. It $T$ is a solvable $F$-subgroup of $N_G(S)$, then $TS$ is solvable and $TS \subseteq F^w$ for some $w \in G$.

Now (*) implies that $TS \subseteq (N_F(S))^r$ for some $r \in N_G(S)$. Hence $N_G(S)$ is a $\tilde{D}_F$-group. By Lemma 1, $N_G(S)$ is $\tilde{F}$-homogeneous. Using (A), $|N_G(S)| < |G|$ so $N_G(S)$ is $F'$-closed. Lemma 3 implies that $N_G(S)_F = 0_\pi (N_G(S))$. However, $0_\pi (N_G(S)) \subseteq C_G(S)$ since $N_G(S)/C_G(S) \subseteq F$ and is thus a $\pi$-group. Hence $N_G(S)_F = 0_\pi (C_G(S))$ and $N_G(S) = 0_\pi (C_G(S))N_F(S)$ follows directly.

We now prove (B) by induction on $[F_p : S]$ where $F_p$ is a Sylow $p$ subgroup of $F$. Assume first that $[F_p : S] = 1$, then $S$ is a Sylow $p$-subgroup of $G$. Hence, $S \subseteq F \cap F^w$ yields $S = S^{fw}$ where $f \in F$. Therefore, $fw \in N_G(S)$ and (*) is satisfied so (i) and (ii) are proved.

Now $N_G(S) = N_F(S)0_\pi (C_G(S))$ yields $fw = f_1y$ where $f_1 \in N_F(S)$ and $y \in 0_\pi (C_G(S))$. Hence $F^w = F^y$ and (iii) follows.

We assume (B) is proved for all $p$-subgroups $T$ of $F_p$ such that $[F_p : T] < [F_p : S]$ and $|T| > 1$. Let $T$ be a Sylow $p$ subgroup of $N_F(S)$, then $|T| > |S|$ and $S \subseteq T \subseteq S_1 \subseteq F^g$ where $S_1$ is a Sylow $p$-subgroup of $N_G(S)$.

By induction $F^g = F^y$ where $y \in 0_\pi (C_G(T))$. Hence, $N_p g(S) = (N_p(S))^y$ so $T$ is a Sylow subgroup of $N_G(S)$. If $S \subseteq F \cap F^w$, let $U$ be a Sylow $p$-subgroup of $N_F(S)$.
Then $S \subseteq U = T_r^1 \subseteq T^r$ where $r \in N_G(S)$ and $|T_r| > |S|$. Now $T_r^1 \subseteq T^{\wr -1} \cap F$ yields $T^{\wr -1} = F^r_1$ where $y_1 \in \mathfrak{S}^{-1}(C_G(T_1))$. Therefore, $N_F^w(S) = (N_F(S))^y_1r$ where $y_1r \in N_G(S)$ and $(\ast)$ is satisfied.

Hence, (B) (i) and (ii) are proved. Further, $F^w = F^{y_1r}$ where $y_1r \in N_G(S) = N_F(S)\mathfrak{S}^{-1}(C_G(S))$ yields $y_1r = fy$ where $f \in N_F(S)$ and $y \in \mathfrak{S}^{-1}(C_G(S))$. Now (B) (iii) follows.

(C) Final Contradiction.

By Lemma 3, $F$ is a Hall $\pi$-subgroup of $G$. Thus, $N_G(F) = FM$ where $M$ is a Hall $\pi'$-subgroup of $N_G(F)$. Let $p \in \pi$, then the Frattini argument and the Schur-Zassenhaus theorem imply that there is a Sylow $p$ subgroup $F_p$ of $F$ such that $N_G(F) = N_G(F_p)F$ and $M \subseteq N_G(F_p)$. By (B) (ii), $M \subseteq C_G(F_p)$. Repeating this argument for all primes $p$ in $\pi$, we see that $M \subseteq C_G(F)$ and $N_G(F) = F \times M$.

Suppose $z_1 = z_2^w$ where $z_1$ and $z_2 \in F^\#$, then $z_1 \in F^w \cap F$. Because of (B), an argument analogous to that used in the proof of [1, Lemma 5] implies that $F^w = F^y$ for some $y \in \mathfrak{S}^{-1}(C_G(z_1))$. Thus, $wy^{-1} \in N_G(F)$ and by the previous paragraph $w = fm$ where $f \in F$ and $m \in M$. Hence $z_2 = z_1^{wy^{-1}m^{-1}} = z_1^{f^{-1}}$, so $z_1$ and $z_2$ are conjugate in $F$. The theorem now follows from Proposition 4.

Proof of Corollary B:

(i) $\Rightarrow$ (ii). [4, II, (2.7)] yields that $G$ is a $D_F$-group, and [4, III (2.2)] implies that $G$ is $F$-homogeneous. Let $F$ be an $F$-projector of $G$, then $G/G_F \cong F$ and $G/G_F \in F \cap H$. 38
Therefore, $F \in H$.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i) By Theorem A, $G \in F'$.

Now $G/G_F \cong F \in F$ implies that $G \in F'(F \cap H)$.

As noted in the introduction if a group $G$ is $\pi$-homogeneous and $D_\pi$, then $G$ is $\tilde{F}$-homogeneous and $\tilde{D}_F$ where $F$ is the s-closed extensible homomorph of $\pi$-groups.

The following generalization of the theorem in [1] may be obtained easily from Theorem A and Lemma 3.

**Corollary C:**

Assume $G$ is a finite group which is $\pi$-homogeneous and has a Hall $\pi$-subgroup which contains a conjugate of every solvable $\pi$-subgroup of $G$, then $G$ is $\pi'$-closed.

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Rebut el 6 de juny del 1985

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