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ON A THEOREM OF M. FUJII

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INTRODUCTION:

In 1967 M. Fujii [2] computed the KO^{-i} - rings of the complex projective spaces. We give a modified proof here using some results by S.G. Hoggar [3]. Our method seems direct and easier to handle and it has been applied to compute the KO^{-i} - groups of the complex flag manifolds of lengths 2 and 3 [1].

The result we reproved is Theorem 2 of Fujii [2].

Theorem [2, p. 142]

The KO^{-i} - groups of $\mathbb{P}^{n-1}(\mathbb{C})$ are as follow

i	$n \equiv 2 \pmod 4$	$n \equiv 0 \pmod 4$	n odd
0	$(2t+1)\mathbb{Z} + \mathbb{Z}_2$	$(2t)\mathbb{Z}$	$(t+1)\mathbb{Z}$
	$n = 4t+2$	$n = 4t$	$n = 2t+1$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$(t+1)\mathbb{Z} + \mathbb{Z}_2$	$(t+1)\mathbb{Z} + \mathbb{Z}_2$	$(t)\mathbb{Z} + \mathbb{Z}_2$
		$t = [\frac{n-1}{2}]$	
3	0	$\mathbb{Z}_2$	0
4	$(t+1)\mathbb{Z}$	$(t+1)\mathbb{Z} + \mathbb{Z}_2$	$(t+1)\mathbb{Z}$
		$t = [\frac{n-1}{2}]$	

5	0	0	0
6	$(t+1)\mathbb{Z}$	$(t+1)\mathbb{Z}$	$(t)\mathbb{Z}$
		$t = [\frac{n-1}{2}]$	
7	$\mathbb{Z}_2$	0	0 .

<u>Cohomology of $\mathbb{P}^{n-1}(\mathbb{C})$</u>

Let λ_R be the realified bundle of the canonical bundle, λ, over $\mathbb{P}^{n-1}(\mathbb{C})$. Then the second Stiefel-Whitney class $w_2(\lambda_R)$ is the mod 2 reduction of $C_1(\lambda)$. Put $x = w_2(\lambda_R)$, then an additive basis for $H^*(\mathbb{P}^{n-1}(\mathbb{C});\ \mathbb{Z}_2)$ is given by x^i subject to the condition $x^n = o$.

The Poincaré polynomial for $\mathbb{P}^{n-1}(\mathbb{C})$ is given by

(1) $$P(\mathbb{P}^{n-1}(\mathbb{C}), t) = 1 + t^2 + t^4 + \ldots\ldots + t^{2(n-1)}$$

<u>$KO^{-i}(\mathbb{P}^{n-1}(\mathbb{C}))$.</u>

From (1), it is clear that the $2k^{th}$ Betti number, $\beta_{2k} = 1$ for $0 \leqslant k \leqslant n-1$, thus the ranks of $KO^*(\mathbb{P}^{n-1}(\mathbb{C}))$ are determined as follows using lemma (2.4) of [3]:

$$\text{rank } KO^o = \text{rank } KO^{-4} = \sum_{k=o}^{[\frac{n-1}{2}]} \beta_{4k} = [\tfrac{n-1}{2}] + 1$$

for all values of n.

Also, $$\text{rank } KO^{-2} = \text{rank } KO^{-6} = \sum_{k=o}^{[\frac{n-1}{2}]} \beta_{4k+2} = [\tfrac{n-1}{2}] + 1$$

for n even and

$$\text{rank } KO^{-2} = \text{rank } KO^{-6} = \sum_{k=o}^{[\frac{n-2}{2}]} \beta_{4k+2} = [\tfrac{n-2}{2}] + 1 = \frac{n-1}{2}$$

for n odd and this completes the free part.

For the torsion part, consider the Atiyah-Hirzebruch spectral sequence which converges to $KO^{p+q}(\mathbb{P}^{n-1}(\mathbb{C}))$, see [2]. Consider the sequence of differentials

$$(2) \qquad E_2^{p-2,q+1} \xrightarrow{d_2} E_2^{p,q} \xrightarrow{d_2} E_2^{p+2,q-1}.$$

For $q \equiv 0,4 \pmod 8$, $E_2^{p,q}$ gives the free part of KO^{p+q} which is determined. For the torsion part, we need only consider $q \equiv -1, -2 \pmod 8$. For $q \equiv -1 \pmod 8$ (2) becomes

$$E_2^{p-2,8t} \xrightarrow{d_2} E_2^{p,8t-1} \xrightarrow{d_2} E_2^{p+2,8t-2}.$$

The map $E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$ is zero for $p \equiv 0,4 \pmod 8$ (2a) and is an isomorphism for $p \equiv 2,6 \pmod 8$ if $E_2^{p,q} \neq 0$.

Thus $E_3^{0,-1} \cong E_2^{0,-1} = \mathbb{Z}_2$ for all n. For n even, (2b) $2(n-2) \equiv 0,4 \pmod 8$ and for n odd, $2(n-2) \equiv 2,6 \pmod 8$, thus the differential

$$E_2^{2(n-2),8t} \xrightarrow{d_2^{2(n-2),8t)}} E_2^{2(n-1),-1}$$

is an isomorphisms for n odd and zero for n even using (2a) and (2b). Hence

$$(3) \qquad E_3^{2(n-1),-1} = \begin{cases} 0 ; & n \text{ odd} \\ \mathbb{Z}_2 ; & n \text{ even} \end{cases}$$

and $E_3^{p,-1} = 0$ otherwise.

For $q \equiv -2 \pmod 8$, (2) becomes

$$E_2^{p-2,-1} \longrightarrow E_2^{p,-2}$$ and using lemma

(2.4) in [3] we have

$$(4) \qquad E_3^{p,-2} \cong \begin{cases} \mathbb{Z}_2 \ ; & \text{for } p \equiv 2,6 \pmod 8, \quad p \neq 0 \\ 0 \ ; & \text{otherwise.} \end{cases}$$

and $E_3^{0,-2} = \mathbb{Z}_2$.

Now, for $n \equiv 1 \pmod 4$, $2(n-1) \equiv 0 \pmod 8$,

for $n \equiv 3 \pmod 4$, $2(n-1) \equiv 4 \pmod 8$,

for $n \equiv 2 \pmod 4$, $2(n-1) \equiv 2 \pmod 8$, and

for $n \equiv 0 \pmod 4$, $2(n-1) \equiv 6 \pmod 8$.

Thus, by comparing (3) with (4) and using lemma (2.1) in [3], we have for n odd,

$$KO^{-3} = \mathbb{Z}_2 - \text{part } KO^{-4} = 0$$
$$KO^{-5} = \mathbb{Z}_2 - \text{part } KO^{-6} = 0 \text{ and}$$
$$KO^{-7} = \mathbb{Z}_2 - \text{part } KO^{0} = 0$$

For $n \equiv 0 \pmod 4$, we have

$$KO^{-5} = \mathbb{Z}_2 - \text{part } KO^{-6} = 0 \text{ and}$$
$$KO^{-7} = \mathbb{Z}_2 - \text{part } KO^{0} = 0$$

Also for $n \equiv 2 \pmod 4$, we have

$$KO^{-3} = \mathbb{Z}_2 \text{ - part } KO^{-4} = 0 \text{ and}$$
$$KO^{-5} = \mathbb{Z}_2 \text{ - part } KO^{-6} = 0.$$

Now, we show that the E_3-terms survive to E_∞, for $q \equiv -1 \pmod 8$.
Let $E_3^{0,-1} = {}^{\alpha}\mathbb{Z}_2$, $E_3^{2(n-1),-1} = {}^{\beta}\mathbb{Z}_2$ (n even)
Consider the differential

$$E_r^{0,-1} \xrightarrow{d_r} E_r^{r,-r}, \; E_r^{r,-r} = 0 \text{ except } r \equiv 0,2,4 \pmod 8)$$

and $d_r = 0$ for $r = 0, 4 \pmod 8$ because it maps a finite group to a free group. Thus, we are left with the case $r \equiv 2 \pmod 8$. In this case, we claim that $d_r = 0$ for $r \equiv 2 \pmod 8$.

Proof of claim: It suffices to show that $d_{10} = 0$. From the zero differential

$$E_2^{0,-1} \xrightarrow{d_2} E_2^{2,-2}, \text{ we see that } E_3^{0,-1} = \mathbb{Z}_2 \text{ is}$$

generated by x^o and since d_{10} is a derivation we have $d_{10}(x^o) = 0$ from the formula

$$d_r(x^s) = sx^{s-1}\, d_r(x), \text{ finishing the claim.}$$

Thus $E_\infty^{0,-1} = {}^{\alpha}\mathbb{Z}_2$.
Also for n even, we consider the differential

$$E_r^{2(n-1)-r,r-2} \xrightarrow{d_r} E_r^{2(n-1),\,-1} \quad E_r^{2(n-1)-r,r-2} = 0$$

except $r \equiv 0,2,6 \pmod 8$ using the property of $KO^*(*)$ and $d_r = 0$ for $r \equiv 0,6 \pmod 8$, see [3]. When $r \equiv 2 \pmod 8$ $E_r^{2(n-1)-r,r-2}$ is a free group which survives to E_∞. Thus $d_r = 0$ for all $r \geqslant 3$.

We consider the filtrations

$$KO^{-1} = F^{0,-1} \supset F^{1,-2} \supset \dots \supset F^{n-1,-n} \supset F^{n,-n-1} = 0$$

$$\text{and} \quad KO^{2n-3} = F^{0,2n-3} \supset F^{1,2n-4} \supset \dots \supset F^{n-1,n-2} \supset 0$$

where

$$\alpha \mathbb{Z}_2 = E_\infty^{0,-1} = F^{0,-1}/F^{1,-2}, \quad \beta \mathbb{Z}_2 = E_\infty^{2(n-1),-1} = F^{2(n-1),-1}/F^{2n-1,-2}$$

$$F^{p,q} = \mathrm{Ker}(KO^{p+q}(X) \longrightarrow KO^{p+q}(X^{p-1}))$$

$$X = \mathbb{P}^{n-1}(\mathbb{C}), \quad \text{and} \quad E_\infty^{p,q} = 0 \quad \text{for either } p \text{ or } q \text{ odd.}$$

Thus $KO^{-1} = \mathbb{Z}_2$ for all n.

Also $KO^{2n-3} = KO^{-3}$ for $n \equiv 0 \pmod 4$

and $KO^{2n-3} = KO^{-7}$ for $n \equiv 2 \pmod 4$

finishing the proof of the theorem.

References

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