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L^p REGULARITY OF THE DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS WITH SINGULAR DRIFT

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Abstract

Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two elliptic operators in nondivergence form, with coefficients $\mathbf{A}_\ell$ and drift terms $\mathbf{b}_\ell$, $\ell = 0, 1$ satisfying

$$\sup_{|Y-X|\leq\frac{\delta(X)}{2}} \frac{|\mathbf{A}_0(Y) - \mathbf{A}_1(Y)|^2 + \delta(X)^2 |\mathbf{b}_0(Y) - \mathbf{b}_1(Y)|^2}{\delta(X)} \, dX$$

is a Carleson measure in a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, (here $\delta(X) = \operatorname{dist}(X, \partial\Omega)$). If the harmonic measure $d\omega_{\mathcal{L}_0} \in A_\infty$, then $d\omega_{\mathcal{L}_1} \in A_\infty$. This is an analog to Theorem 2.17 in [**8**] for divergence form operators. As an application of this, a new approximation argument and known results we are able to extend the results in [**10**] for divergence form operators while obtaining totally new results for nondivergence form operators. The theorems are sharp in all cases.

1. Introduction and Background

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and an operator $\mathcal{L}$ given by

$$\mathcal{L} = \begin{cases} \operatorname{div} \mathbf{A}\nabla & \text{(divergence fom)} \qquad \text{or} \\ \mathbf{A} \cdot \nabla^2 & \text{(nondivergence form),} \end{cases}$$

the harmonic measure at $X \in \Omega$, $d\omega_{\mathcal{L}}^X$, is the unique Borel measure on $\partial\Omega$ such that for all continuous functions $g \in \mathcal{C}(\partial\Omega)$,

$$u(X) = \int_{\partial\Omega} g(Q) \, d\omega_{\mathcal{L}}^X(Q)$$

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is continuous in $\overline{\Omega}$ and it is the unique solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Where we assume that the equality $\mathcal{L}u = 0$ holds in the weak sense for divergence form operators and in the strong *a.e.* sense for nondivergence form operators. Here $\mathbf{A} = \mathbf{A}(X)$ is a symmetric $(n+1) \times (n+1)$ matrix with bounded measurable entries, satisfying a uniform ellipticity condition

$$\lambda |\xi|^2 \leq \xi \cdot \mathbf{A}(X)\, \xi \leq \Lambda |\xi|^2, \quad X, \xi \in \mathbb{R}^{n+1}, \tag{1.2}$$

for some positive constants λ, Λ. In the nondivergence case the entries of the matrix $\mathbf{A}$ are assumed to belong to $\mathrm{BMO}(\Omega)$ with small enough norm. For a given operator $\mathcal{L}$, the harmonic measures $d\omega_{\mathcal{L}}^X$, $X \in \Omega$, are regular probability measures which are mutually absolutely continuous with respect to each other. That is,

$$k(X, Y, Q) = \frac{d\omega_{\mathcal{L}}^Y}{d\omega_{\mathcal{L}}^X}(Q) \in L^1\left(d\omega_{\mathcal{L}}^X, \partial\Omega\right), \quad X, Y \in \Omega, \quad Q \in \partial\Omega.$$

By the Harnack's principle the kernel function $k(X, Y, Q)$ is positive and uniformly bounded in compact subsets of $\Omega \times \Omega \times \partial\Omega$. As a consequence, to study differentiability properties of the family $\left\{d\omega_{\mathcal{L}}^X\right\}_{X\in\Omega}$ with respect to any other Borel measure $d\nu$ on $\partial\Omega$, it is enough to fix a point $X_0 \in \Omega$ and study $\frac{d\omega}{d\nu}$, where $d\omega = d\omega_{\mathcal{L}}^{X_0}$ is referred as *the* harmonic measure of $\mathcal{L}$ on $\partial\Omega$. If there is unique solvability of the continuous Dirichlet problem and a boundary Maximum Principle is available, then the well definition of the harmonic measure follows from Riesz's representation theorem.

Definition 1.1 (Continuous Dirichlet problem - $\mathcal{CD}$)**.** Given an elliptic operator $\mathcal{L}$, we say that the continuous Dirichlet problem is uniquely solvable in Ω, and we say that $\mathcal{CD}$ holds for $\mathcal{L}$, if for every continuous function g on $\partial\Omega$, there exists a unique solution u of (1.1), such that $u \in \mathcal{C}^0\left(\overline{\Omega}\right) \bigcap W^{2,p}(\Omega)$ for some $1 \leq p \leq \infty$.

Remark 1.2. By Theorem 3.2 below [**16**], a sufficient condition for $\mathcal{CD}$ to hold for a nondivergence form operator $\mathcal{L} = \mathbf{A} \cdot \nabla^2$ is that there exists $\rho > 0$ depending on n and the ellipticity constants such that

$$\|\mathbf{A}\|_{\mathrm{BMO}(\Omega)} \leq \rho, \tag{1.3}$$

where $\|\cdot\|_{\mathrm{BMO}(\Omega)}$ denotes the BMO norm in Ω (see Definition 3.1 below). It is not known whether or not the continuous Dirichlet problem is uniquely solvable in the case of elliptic nondivergence form operators with just bounded measurable coefficients. On the other hand, even under the restrictions (1.3) for any $\rho > 0$ it is known [**18**] that the continuous Dirichlet problem has non-unique "good solutions". That is, for any $\rho > 0$ there exists $\mathbf{A}(X) \in \mathrm{BMO}(\Omega)$ with $\|\mathbf{A}\|_{\mathrm{BMO}(\Omega)} \leq \rho$ and two sequences of $\mathcal{C}^\infty$ symmetric matrices $\mathbf{A}_{0,j}$ and $\mathbf{A}_{1,j}$ with the same ellipticity constants as $\mathbf{A}$, such that $\mathbf{A}_{\ell,j}(X) \to \mathbf{A}(X)$ as $j \to \infty$ for *a.e.* X, $\ell = 0, 1$, and such that for some continuous function g on $\partial\Omega$ the solutions $u_{0,j}$ and $u_{1,j}$ to the Dirichlet problems

$$\begin{cases} \mathcal{L}_{0,j} u_{0,j} = 0 & \text{in } \Omega \\ u_{0,j} = g & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_{1,j} u_{1,j} = 0 & \text{in } \Omega \\ u_{1,j} = g & \text{on } \partial\Omega, \end{cases}$$

converge uniformly in $\bar{\Omega}$ to *different* continuous limits u_0 and u_1.

Given two regular Borel measures μ and ω in $\partial\Omega$, $d\omega \in A_\infty(d\sigma)$ if there exist constants $0 < \varepsilon, \delta < 1$ such that for any boundary ball $\Delta = \Delta_r(Q)$ and any Borel set $E \subset \Delta$,

$$\frac{\mu(E)}{\mu(\Delta)} < \delta \Longrightarrow \frac{\omega(E)}{\omega(\Delta)} < \varepsilon.$$

The relation $d\omega \in A_\infty(d\mu)$ is an equivalence relation [**15**], and any two measures related by the A_∞ property are mutually absolutely continuous with respect to each other. From classic theory of weights, if $d\omega \in A_\infty(d\mu)$ then there exists $1 < q < \infty$ such that the density $h = \frac{d\omega}{d\mu}$ satisfies a reverse Hölder inequality with exponent q:

$$\left\{ \frac{1}{\mu(\Delta)} \int_\Delta h^q \, d\mu \right\}^{\frac{1}{q}} \leq C \frac{1}{\mu(\Delta)} \int_\Delta h \, d\mu.$$

This property is denoted $d\omega \in B_q(d\mu)$, and $d\omega \in B_q(d\mu)$ is equivalent to the fact that the Dirichlet problem (1.1) for the operator $\mathcal{L}$ is solvable in $L^p(d\mu, \partial\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$ (see [**7**] for details). When $\mu = \sigma$, the Euclidean measure, we write A_∞ for $A_\infty(d\sigma)$.

Definition 1.3 (L^p - Dirichlet problem, $\mathcal{D}_p$)**.** Let $\mathcal{L}$ be an elliptic operator that satisfies $\mathcal{CD}$ and let μ be a doubling measure in $\partial\Omega$. We say that the $L^p(d\mu)$-Dirichlet problem is uniquely solvable in Ω, and we write that $\mathcal{D}_p(d\mu)$ holds for $\mathcal{L}$, if for every continuous function g on $\partial\Omega$, the unique solution u of (1.1) satisfies

$$\|Nu\|_{L^p(d\mu)} \leq C \|g\|_{L^p(d\mu)},$$

for some constant independent of g. Here Nu denotes the nontangential maximal function of u on $\partial\Omega$. When $\mu = \sigma$ is the Lebesgue measure on $\partial\Omega$ we simply say that $\mathcal{D}_p$ holds for $\mathcal{L}$.

Definition 1.4 (Carleson measure)**.** Let Ω be an open set in $\mathbb{R}^{n+1}$ and let μ be a nonnegative Borel measure on $\partial\Omega$. For $X \in \partial\Omega$ and $r > 0$ denote by $\triangle_r(X) = \{Z \in \partial\Omega : |Z - X| < r\}$ and $T_r(X) = \{Z \in \Omega : |Z - X| < r\}$. Given a nonnegative Borel measure ν in Ω, we say that ν is a *Carleson measure in* Ω *with respect to* μ, if there exist a constant C_0 such that for all $X \in \partial\Omega$ and $r > 0$,

$$\nu(T_r(X)) \leq C_0 \mu(\triangle_r(X)).$$

The infimum of all the constants C_0 such that the above inequality holds for all $X \in \partial\Omega$ and $r > 0$ is called *the Carleson norm of* ν *with respect to* μ *in* Ω. For conciseness, we will write $\nu \in \mathfrak{C}(d\mu, \Omega)$ when v is a Carleson measure in Ω, and we denote by $\|\nu\|_{\mathfrak{C}(d\mu,\Omega)}$ its Carleson norm. When $\mu = \sigma$ is the Lebesgue measure on $\partial\Omega$ we just say that ν is a Carleson measure in Ω.

Definition 1.5. Throughout this work, $Q_\gamma(X)$ denotes a cube centered at X with faces parallel to the coordinate axes and sidelength γ; i.e.

$$Q_\gamma(X) = \left\{ Y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : |y_i - x_i| < \frac{\gamma}{2},\ i = 1, \ldots, n+1 \right\}.$$

When X belongs to a domain Ω, we write $\delta(X) = \operatorname{dist}(X, \partial\Omega)$. In particular, when $\Omega = \mathbb{R}^{n+1}_+$ and $X \in \Omega$, it follows that $\delta(X) = x_{n+1}$.

In the remarkable work [**7**], the authors established a perturbation result relating the harmonic measures of two operators in divergence form. The analogue result was later obtained by the author in [**17**] for nondivergence form operators.

Theorem 1.6 ([**7**]–[**17**])**.** *Let* $\mathcal{L}_{d,0} = \operatorname{div} \mathbf{A}_0 \nabla$ *and* $\mathcal{L}_{d,1} = \operatorname{div} \mathbf{A}_1 \nabla$ *be two elliptic operators with bounded measurable coefficients in* Ω*, and let* $\omega_{d,0}$ *and* $\omega_{d,1}$ *denote their respective harmonic measures. Let* σ *be a doubling measure on* $\partial\Omega$ *and suppose that*

$$\delta(X)^{-1} \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}} |\mathbf{A}_0(Y) - \mathbf{A}_1(Y)|^2 \, dX \in \mathfrak{C}(d\sigma, \Omega). \tag{1.4}$$

If $d\omega_{d,0} \in A_\infty$ *then* $d\omega_{d,1} \in A_\infty$*. Also, if* $\mathcal{CD}$ *holds for the operators* $\mathcal{L}_{n,0} = \mathbf{A}_0 \cdot \nabla^2$ *and* $\mathcal{L}_{n,1} = \mathbf{A}_1 \cdot \nabla^2$ *(see Definition 1.1), then their respective harmonic measures* $\omega_{n,0}$*,* $\omega_{n,1}$*, satisfy* $d\omega_{n,0} \in A_\infty \Rightarrow d\omega_{n,1} \in A_\infty$*.*

The theorems above were stated in terms of the supremum of the differences $|\mathbf{A}_0(Y) - \mathbf{A}_1(Y)|$ for Y in Euclidean balls $|Y - X| < \frac{\delta(X)}{2}$. The above formulation is equivalent. In [**8**], Theorem 1.6 was extended to elliptic divergence form operators with a singular drift:

Theorem 1.7 (Theorem 1.9, Chapter III of [**8**])**.** *If* $\mathcal{L}_{D,0} = \operatorname{div}\mathbf{A}_0\nabla + \mathbf{b}_0 \cdot \nabla$ *and* $\mathcal{L}_{D,1} = \operatorname{div}\mathbf{A}_1\nabla + \mathbf{b}_1 \cdot \nabla$ *where* $\mathbf{A}_0$ *and* $\mathbf{A}_1$ *satisfy* (1.4)*, and* $\mathbf{b}_i = \left(b^i_j\right)_{j=1}^{n+1}$, $i = 0, 1$ *satisfy*

$$\delta(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}_1(Y) - \mathbf{b}_0(Y)|^2 \, dX \in \mathfrak{C}(d\sigma, \Omega), \tag{1.5}$$

then $d\omega_{D,0} \in A_\infty \Rightarrow d\omega_{D,1} \in A_\infty$.

The results in Theorems 1.6 and 1.7 concern perturbation of elliptic operators. They provide solvability for the L^q Dirichlet problem (for some $q > 1$) for an operator $\mathcal{L}_1$ given that there exists an operator $\mathcal{L}_0$ for which the L^p Dirichlet problem is solvable for some $p > 1$ *and* the disagreement of their coefficients satisfy the Carleson measure conditions (1.4) and (1.5). In [**10**], the authors answer a different question: What are sufficient conditions on the coefficients $\mathbf{A}$ and $\mathbf{b}$ so that a given operator $\mathcal{L}_D = \operatorname{div}\mathbf{A}\nabla + \mathbf{b}$ has unique solutions for the L^p-Dirichlet problem for some $p > 1$? See also [**9**].

Theorem 1.8 ([**10**])**.** *Let* $\mathcal{L}_D = \operatorname{div}\mathbf{A}\nabla + \mathbf{b} \cdot \nabla$*, where* $\mathbf{A}$ *satisfies*

$$\delta(X) \sup_{|Y-X| \leq \frac{\delta(X)}{2}} |\nabla\mathbf{A}(Y)|^2 \, dX \in \mathfrak{C}(d\sigma, \Omega), \tag{1.6}$$

and $\mathbf{b}$ *satisfies*

$$\delta(X) \sup_{Y,Z \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}(Y) - \mathbf{b}(Z)|^2 \, dX \in \mathfrak{C}(d\sigma, \Omega).$$

Then $d\omega_{\mathcal{L}_D} \in A_\infty$.

2. Statement of the results

One of the main results in this work is an analog of Theorem 1.7 for nondivergence form operators.

Theorem 2.1. *Let* $\mathcal{L}_{N,0}=\mathbf{A}_0\cdot\nabla^2+\mathbf{b}_0\cdot\nabla$ *and* $\mathcal{L}_{N,1}=\mathbf{A}_1\nabla^2+\mathbf{b}_1\cdot\nabla$ *where* $\mathbf{A}_\ell=\left(A^\ell_{ij}\right)_{i,j=1}^{n+1}$ *and* $\mathbf{b}_\ell=\left(b^\ell_j\right)_{j=1}^{n+1}$ $\ell=0,1$ *are measurable coefficients and* $\mathbf{A}_\ell$ *satisfy the ellipticity condition* (1.2) *for* $\ell=0,1$ *in a bounded Lipschitz domain* Ω*. Suppose that* $\mathcal{CD}$ *holds for* $\mathcal{L}_{N,\ell}$*,* $\ell=0,1$ *and that*

(2.1)
$$\left\{\sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)}\frac{\left|\mathbf{A}_1(Y)-\mathbf{A}_0(Y)\right|^2+\delta^2(X)\left|\mathbf{b}_1(Y)-\mathbf{b}_0(Y)\right|^2}{\delta(X)}\right\}dX\in\mathfrak{C}\left(d\sigma,\Omega\right),$$

where σ *is Lebesgue's measure on* $\partial\Omega$*. Then* $d\omega_{N,0}\in A_\infty\Rightarrow d\omega_{N,1}\in A_\infty$*. That is, if the* L^p*-Dirichlet problem is uniquely solvable for* $\mathcal{L}_{N,0}$ *in* Ω*, for some* $1<p<\infty$*, then there exists* $1<q_1<\infty$ *such that the* L^q*-Dirichlet problem is uniquely solvable for* $\mathcal{L}_{N,1}$ *in* Ω*,* $1\le q\le q_1$*.*

As an application of this result, Theorem 1.6 (nondivergence case) and a simple averaging of the coefficients argument we obtain an analog to Theorem 1.8. Moreover, this averaging argument and Theorem 1.7 yield an extension of Theorem 1.8 to the case when condition (1.6) is replaced by the weaker assumption (1.4).

Remark 2.2. In the divergence form case considered in [**8**], it was only necessary to assume that condition $\mathcal{CD}$ held for just the operator $\mathcal{L}_{D,0}$, and not both of $\mathcal{L}_{D,0}$ and $\mathcal{L}_{D,1}$. As mentioned in Remark 1.2, in the nondivergence case there is no well defined notion of solution for operators with measurable coefficients satisfying the hypotheses of Theorem 2.1. Hence, without the $\mathcal{CD}$ assumption the harmonic measure for $\mathcal{L}_{N,1}$ would not be defined in general.

To make the statements of the results below more concise, we introduce the following definition.

Definition 2.3 (*Oscillation*)**.** For $r>0$, the r-oscillation of a measurable function $f(X)$ (scalar or vector-valued) at a point X, denoted $\operatorname{osc}_r f(X)$, is given by

$$\operatorname{osc}_r f(X)=\sup_{Y,W\in Q_r(X)}\left|f(W)-f(Z)\right|.$$

Theorem 2.4. *Let* $\mathcal{L}_D = \operatorname{div} \mathbf{A}\nabla + \mathbf{b}\cdot\nabla$ *and* $\mathcal{L}_N = \mathbf{A}\cdot\nabla^2 + \mathbf{b}\cdot\nabla$ *be uniformly elliptic operators in divergence form and nondivergence form, respectively, with bounded measurable coefficient matrix* $\mathbf{A}$ *and drift vector* $\mathbf{b}$ *in a bounded Lipschitz domain* Ω. *In the nondivergence case we assume that* $\mathcal{CD}$ *holds for* $\mathcal{L}_N$. *Suppose that the coefficients* $\mathbf{A}$, $\mathbf{b}$ *satisfy*

$$\frac{\left(\operatorname{osc}_{\frac{\delta(X)}{2\sqrt{n}}} \mathbf{A}\,(X)\right)^2 + \delta^2\,(X)\left(\operatorname{osc}_{\frac{\delta(X)}{2\sqrt{n}}} \mathbf{b}\,(X)\right)^2}{\delta\,(X)}\, dX \in \mathfrak{C}\,(d\sigma,\Omega)\,. \tag{2.2}$$

Then $d\omega_{\mathcal{L}_D} \in A_\infty$ *and* $d\omega_{\mathcal{L}_N} \in A_\infty$. *That is, there exist indexes* $1 < p_D, p_N < \infty$ *such that the* L^p*-Dirichlet problem is uniquely solvable for* $\mathcal{L}_D$ *in* Ω, $1 \le p \le p_D$, *and the* L^q*-Dirichlet problem is uniquely solvable for* $\mathcal{L}_N$ *in* Ω, $1 \le q \le p_N$.

Remark 2.5. The following extensions and generalizations can be obtained (and might be subject for a subsequent work):

(1) The techniques used to obtain Theorems 2.1 and 2.4 also yield analog results with the Euclidean measure $d\sigma$ replaced by any *doubling* measure $d\mu$ on $\partial\Omega$.

(2) Appropriate parabolic versions of Theorems 2.1 and 2.4 are possible in the nondivergence case (the divergence case was considered in [**8**]).

(3) Stronger conclusions can be obtained in Theorems 2.1 and 2.4 if the Carleson measure conditions are replaced by *vanishing* Carleson measures (the Carleson norm vanishes as the radius of the regions goes to zero). Under such assumptions, it can be shown that the harmonic measures $d\omega_{\cdot,0}$ and $d\omega_{\cdot,1}$ from Theorem 2.1 preserve the A_p condition for any $1 \le p \le \infty$. That is, $d\omega_{\cdot,0} \in A_p \Rightarrow d\omega_{\cdot,1} \in A_p$. In the case of Theorem 2.4, it can be shown that under vanishing Carleson measure conditions $d\omega_{\mathcal{L}_D} \in A_1$ and $d\omega_{\mathcal{L}_N} \in A_1$.

2.1. The theorems are sharp. In [**7**] it was shown that Theorem 1.6 is sharp for divergence form equations in two fundamental ways (see Theorems 4.11 and 4.2 in [**7**]). The examples provided in that work were constructed using Beurling-Ahlfors quasiconformal mappings on the half plane $\mathbb{R}^2_+$ [**2**]. Quasi-conformal mappings preserve the divergence form-structure of an elliptic operator $\mathcal{L}_D$, but when composed with a nondivergence form operator $\mathcal{L}_N$ the transformed operator has first order drift terms.

The more recent work [**10**] for divergence form operators (Theorem 1.8) can be applied to obtain regularity of elliptic equation in *nondivergence* form in the special case that the coefficient matrix satisfies (1.6).

The approximation technique to be introduced in the next section, together with Theorem 2.1, show that Theorem 1.8 can be extended to operators in divergence and nondivergence form with coefficients satisfying the weaker condition (2.2). At the same time, this opens the door to extend the scope of the examples provided in Theorem 1.6 to this wider class of operators. The following theorem is a nondivergence analog to Theorem 4.11 in [**7**].

Theorem 2.6. *Given any nonnegative function $\alpha(X)$ in $\mathbb{R}^2_+ = \{(x,t) : t > 0\}$ such that $\alpha(X)$ satisfies the doubling condition: $\alpha(X) \leq C\alpha(X_0)$ for all $X = (x,t)$, $X_0 = (x_0,t_0) : |X - X_0| < \frac{t_0}{2}$ and such that*

$$\sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} \frac{\alpha(Y)^2}{\delta(X)}\, dX \notin \mathfrak{C}\left(d\sigma, [0,1]^2\right),$$

where $d\sigma$ is the Euclidean measure in $\partial\left([0,1]^2\right)$ and $\delta((x,t)) = t$. There exists a coefficients matrix $\mathbf{A}$ such that

(1) *the function $a(X) = \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{A}(Y) - \mathbf{I}|$, satisfies that for all $I \subset \mathbb{R}$,*

$$\frac{1}{|I|}\iint_{T(I)} a^2(x,y)\, dx\frac{dy}{y} \leq C\left[\frac{1}{|I|}\iint_{T(2I)} \alpha^2(x,y)\, dx\frac{dy}{y} + 1\right];$$

(2) *the function $\tilde{a}(X) = \operatorname{osc}_{Q_{\frac{\delta(X)}{2\sqrt{n}}}} \mathbf{A}(X)$, satisfies that for all $I \subset \mathbb{R}$,*

$$\frac{1}{|I|}\iint_{T(I)} \tilde{a}^2(x,y)\, dx\frac{dy}{y} \leq C\left[\frac{1}{|I|}\iint_{T(2I)} \alpha^2(x,y)\, dx\frac{dy}{y} + 1\right];$$

and

(3) *if $\mathcal{L}_N = \mathbf{A} \cdot \nabla^2$ on $\mathbb{R}^2_+$, the elliptic measure $d\omega_{\mathcal{L}_N}$ is* not *in $A_\infty(dx, [0,1])$.*

The above theorem shows that the Carleson measure condition (1.4) in Theorem 1.6 is sharp also in the nondivergence case. In particular, it shows that the main result in [**17**] is sharp. The proof is a simple application of Theorem 4.11 in [**7**] and the approximation argument given in the section below (Lemma 3.9). Indeed, by Theorem 4.11 in [**7**] there exists a coefficients matrix $\mathbf{A}$ such that (1) and (2) hold for $\mathbf{A}$ and (3) holds for $d\omega_{\mathcal{L}_D}$, with $\mathcal{L}_D = \operatorname{div} \mathbf{A}\nabla$. For simplicity, let say that two elliptic operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are *simultaneously* in A_∞, and write $\mathcal{L}_0 \approx \mathcal{L}_2$ if their respective harmonic measures $d\omega_{\mathcal{L}_0}$ and $d\omega_{\mathcal{L}_1}$ satisfy:

$d\omega_{\mathcal{L}_0} \in A_\infty \Leftrightarrow d\omega_{\mathcal{L}_1} \in A_\infty$. Using Lemma 3.9 we can construct a coefficient matrix $\mathbf{A}^*$ such that if $\mathcal{L}^* = \operatorname{div} \mathbf{A}^* \nabla = \mathbf{A}^* \cdot \nabla^2 + \mathbf{b} \cdot \nabla$ and $\mathcal{L}_N^* = \mathbf{A}^* \cdot \nabla^2$ then Theorems 2.1 and 2.4 can be applied to show that $\mathcal{L}_D \approx \mathcal{L}^* \approx \mathcal{L}_N^* \approx \mathcal{L}_N$. See the proof of Theorem 2.1 in Section 4 for a detailed application of this technique.

2.2. Organization of the paper. In the following section we define several function spaces where the coefficients or the solutions to our equations will belong. The basic objects associated to the geometry of Lipschitz domains are also introduced. In this section we also list useful known properties of solutions and the harmonic measure, and we establish some auxiliary results that will allow us to treat the problems locally. In Section 4 we prove Theorem 2.1 by reducing it to a special case (Theorem 2.1). Theorem 2.4 then follows from the result just established and previous theory. Finally, in Section 4.1 we prove Theorem 2.1 by implementing the techniques from [**17**] (originally adapted from [**7**]) to this special case.

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3. Preliminary results

Given a weight w in the Muckenphout class $A_p(\Omega)$, we denote by $L^q(\Omega, w)$, $1 \leq p \leq q < \infty$, the space of measurable functions f such that

$$\|f\|_{L^q(\Omega,w)} = \left(\int_\Omega |f(x)|^q \, w(x) \, dx\right) < \infty.$$

And for a nonnegative integer k, we define the Sobolev space $W^{k,q}(\Omega, w)$ as the space of functions f in $L^q(\Omega, w)$ such that f has weak derivatives up to order k in $L^q(\Omega, w)$. Under the assumption $w \in A_p$, the space $W^{k,q}(\Omega, w)$ is a Banach space and it is also given as the closure of $\mathcal{C}_0^\infty(\Omega)$ (smooth functions of compact support in Ω) under the norm

$$\|f\|_{W^{k,q}(\Omega,w)} = \sum_{\ell=0}^{k} \left\|\nabla^\ell f\right\|_{L^q(\Omega,w)},$$

see [**6**], [**11**]. We recall now some definitions.

Definition 3.1 (BMO)**.** Given a locally integrable function f in $\Omega \subset \mathbb{R}^n$, the BMO modulus of continuity of f, $\eta_{\Omega,f}(r)$, is given by

$$\eta_{\Omega,f}(r) = \sup_{x\in\Omega} \sup_{0<s\leq r} \frac{1}{|B_s(x)\bigcap\Omega|} \int_{B_s(x)\bigcap\Omega} |f(y) - f_s(x)|\, dy, \tag{3.1}$$

where

$$f_s(x) = \frac{1}{|B_s(x)\bigcap\Omega|} \int_{B_s(x)\bigcap\Omega} f(z)\, dz.$$

The space $\mathrm{BMO}(\Omega)$ of functions of *bounded mean oscillation* in Ω is given by

$$\mathrm{BMO}(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|\eta_{\Omega,f}\|_{L^\infty(\mathbb{R}_+)} < \infty \right\}.$$

For $\varrho \geq 0$, we let $\mathrm{BMO}_\varrho(\Omega)$ be given by

$$\mathrm{BMO}_\varrho(\Omega) = \left\{ f \in \mathrm{BMO}(\Omega) : \liminf_{r\to 0^+} \eta_{\Omega,f}(r) \leq \varrho \right\}.$$

It is easy to check that $\mathrm{BMO}_\varrho(\Omega)$ is a closed convex subset of $\mathrm{BMO}(\Omega)$ under the BMO norm $\|f\|_{\mathrm{BMO}(\Omega)} = \|\eta_{\Omega,f}\|_{L^\infty(\mathbb{R}_+)}$. When $\varrho = 0$, $\mathrm{BMO}_0(\Omega)$ is the space $\mathrm{VMO}(\Omega)$ of functions of *vanishing mean oscillation.* We say that a vector or matrix function belongs to a space of scalar functions $\mathcal{X}$ if each component belongs to that space $\mathcal{X}$. For example, we sat that the coefficient matrix $\mathbf{A} \in \mathrm{BMO}$ if $A_{ij} \in \mathrm{BMO}$ for $1 \leq i,j \leq n+1$. The following theorem establishes the solvability of the continuous Dirichlet problem for a large class of elliptic operators in nondivergence form. In particular, for such operators the harmonic measure is well defined.

Theorem 3.2 ([**16**])**.** *Let $\mathcal{L}_n = \mathbf{A} \cdot \nabla^2$ be an elliptic operator in nondivergence form in $\Omega \subset \mathbb{R}^{n+1}$ with ellipticity constants (λ, Λ). There exists a constant $\varrho = \varrho(n,\lambda,\Lambda)$, such that if $\mathbf{A} \in \mathrm{BMO}_\varrho(\Omega)$ then for every $g \in \mathcal{C}(\partial\Omega)$ there exists a unique*

$$u \in \mathcal{C}(\bar{\Omega}) \bigcap_{1\leq p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega)$$

such that $\mathcal{L}_n u = 0$ in Ω and $u \equiv g$ on $\partial\Omega$. Moreover, for any subdomain $\Omega' \Subset \Omega$ and $1 \leq p < \infty$, there exists $C = C(n,\lambda,\Lambda,p,\Omega,\mathrm{dist}(\Omega',\partial\Omega))$ such that the above solution satisfies

$$\|u\|_{W^{2,p}(\Omega')} \leq C \|u\|_{L^p(\tilde{\Omega}')}, \quad \text{where } \tilde{\Omega}' = \left\{ X \in \Omega : \delta(X) > \frac{1}{2}\,\mathrm{dist}(\Omega',\partial\Omega) \right\}.$$

Theorem 3.3. *Let $w \in A_p$, $p \in [n,\infty)$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. For any $0 < \lambda \leq \Lambda < \infty$ there exist a positive $\varrho = \varrho(n,p,\lambda,\Lambda,|[w]|_{A_p})$, such that if $\mathcal{L}_N = \mathbf{A}(X)\cdot\nabla^2$, $\mathbf{A} \in \mathrm{BMO}_\rho(\Omega)$ then for any $f \in L^p(\Omega,w)$, there exists a unique $u \in \mathcal{C}(\overline{\Omega}) \bigcap W^{2,p}_{\mathrm{loc}}(\Omega,w)$ such that $\mathcal{L}_N u = f$ in Ω and $u = 0$ on $\partial\Omega$. Moreover, if $\partial\Omega$ is of class $\mathcal{C}^2$, then $u \in W^{1,p}_0(\Omega,w) \bigcap W^{2,p}(\Omega,w)$ and there exists a positive $c = c(n,p,\lambda,\Lambda,|[w]|_{A_p},\eta_{\Omega,\mathbf{A}},\partial\Omega)$, with $\eta_{\Omega,\mathbf{A}}$ given by* (3.1)*, such that*

$$\|u\|_{W^{2,p}(\Omega,w)} \leq c\|f\|_{L^p(\Omega,w)}.$$

The following comparison principle is the main tool that allows us to treat nondivergence form equations with singular drift. Since we assume that our operator satisfies $\mathcal{CD}$, the result from [**1**] originally stated in a $\mathcal{C}^2$ domain and for continuous coefficients extends to the more general case stated here. Before stating the theorem, we introduce more notation.

If $\Omega \subset \mathbb{R}^{n+1}$ is a Lipschitz domain and $Q \in \partial\Omega$, $r > 0$, we define the *boundary ball of radius r at Q* as

$$\triangle_r(Q) = \{P \in \partial\Omega : |P - Q| < r\}.$$

The *Carleson region* associated to $\triangle_r(Q)$ is

$$T_r(Q) = \{X \in \Omega : |X - Q| < r\}.$$

The *nontangential cone of aperture α and height r at Q*, α, $r > 0$, is defined by

$$\Gamma_{\alpha,r}(Q) = \{X \in \Omega : |X - Q| < (1+\alpha)\,\delta(X) < (1+\alpha)\,r\}.$$

Theorem 3.4 (Comparison Theorem for Solutions [**1**])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain and $\mathcal{L} = \mathbf{A}\cdot\nabla^2 + \mathbf{b}\cdot\nabla$, be a uniformly elliptic operator such that $\mathbf{b}$ satisfies*

$$\delta(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}(Y)|^2 \, dX \in \mathfrak{C}(d\sigma,\Omega) \tag{3.2}$$

and $\mathcal{L}$ satisfies $\mathcal{CD}$. There exists a constant C, $r_0 > 0$ depending only on $\mathcal{L}$ and Ω such that if u and v are two nonnegative solutions to $\mathcal{L}w = 0$ in $T_{4r}(Q)$, for some $Q \in \partial\Omega$, and such that $u \equiv v \equiv 0$ continuously on $\triangle_{2r}(Q)$, then

$$C^{-1}\frac{u(X)}{u(X_r(Q))} \leq \frac{v(X)}{v(X_r(Q))} \leq C\frac{u(X)}{u(X_r(Q))},$$

for every $X \in T_r(Q)$, $0 < r < r_0$. Here $X_r(Q) \in \Omega$ with $\mathrm{dist}(X_r(Q)) \approx |X - Q| \approx r$.

Remark 3.5. In [**1**] the hypothesis on $\mathbf{b}$ is $\delta(X)|\mathbf{b}(X)| \leq \eta(\delta(X))$ where η is an non-decreasing function such that $\eta(0) = \lim_{s\to 0^+}\eta(s) = 0$. Nevertheless, the main tools used to obtain Theorem 3.4 were

(1) $\mathcal{L}$ satisfies property $\mathcal{DP}$
(2) a maximum principle
(3) Harnack inequality.

We assume (1) holds, while (2) and (3) follow as in [**1**] once we notice that (3.2) implies that $\mathbf{b}$ is locally bounded in Ω. Therefore, Theorem 3.4 also holds under our assumptions.

We list now some consequences of the above result that will be useful to us.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain and $\mathcal{L} = \mathbf{A}\cdot\nabla^2 + \mathbf{b}\cdot\nabla$, be a uniformly elliptic operator such that $\mathbf{b}$ satisfies (3.2) and $\mathcal{L}$ satisfies $\mathcal{CD}$. Let $\triangle_r(Q)$, $T_r(Q)$, $X_r(Q)$ and r_0 be as in Theorem 3.4. Let $\triangle = \triangle_r(Q)$ for some $0 < r \leq r_0$ and $Q \in \partial\Omega$.*

(1) $\omega^{X_r(Q)}(\triangle) \approx 1$.
(2) *If $E \subset \triangle$ and $X \in \Omega\backslash T_{2r}(Q)$, then*
$$\omega^{X_r(Q)}(E) \approx \frac{\omega^X(E)}{\omega^X(\triangle)}.$$
(3) *(Doubling property) $\omega^X(\triangle_r(Q)) \approx \omega^X(\triangle_{2r}(Q))$ whenever $X \in \Omega\backslash T_{2r}(Q)$.*

An important consequence of the properties listed in Lemma 3.6 is the following analog to the "main lemma" in [**4**]. Before stating the result we need to introduce the concept of *saw-tooth* region.

Definition 3.7 (Saw-tooth region)**.** Given a Lipschitz domain Ω and $F\subset\partial\Omega$ a closed set, a "saw-tooth" region Ω_F above F in Ω of height $r>0$ is a Lipschitz subdomain of Ω with the following properties:

(1) for some $0<\alpha<\beta$, $0<c_1<c_2$ and all $\alpha<\alpha'<\alpha''<\beta$,
$$\bigcup_{P\in F}\overline{\Gamma}_{\alpha',c_1 r}(P) \subset \Omega_F \subset \bigcup_{P\in F}\overline{\Gamma}_{\alpha'',c_2 r}(P);$$
(2) $\partial\Omega\bigcap\partial\Omega_F = F$;
(3) there exists $X_0 \in \Omega_F$ (the center of Ω_F) such that $\operatorname{dist}(X_0,\partial\Omega_F) \approx r$;
(4) for any $X \in \Omega$, $Q \in \partial\Omega$ such that $X \in \Gamma_{\alpha_0,r_0}(Q)\bigcap\Omega_F \neq \emptyset$, $\exists\, P \in F : Q_{\frac{\delta(X)}{2\sqrt{n}}}(X) \subset \Gamma_{\alpha_0,r_0}(P)$;

(5) Ω_F is a Lipschitz domain with Lipschitz constant depending only on that of Ω.

With these provisos, now we state the following:

Lemma 3.8. *Let Ω be a Lipschitz domain and $\mathcal{L} = \mathbf{A} \cdot \nabla^2 + \mathbf{b} \cdot \nabla$, be a uniformly elliptic operator that satisfies $\mathcal{CD}$. Let $F \subset \partial\Omega$ be a closed set, and let Ω_F be a saw-tooth region above F in Ω. Let $\omega = \omega_{\mathcal{L},\Omega}$ and let $\nu = \omega^{X_0}_{\mathcal{L},\Omega_F}$, where X_0 is the center of Ω_F. There exists $\theta > 0$ such that*

$$\frac{\omega\left(E \bigcap F\right)}{\omega\left(\triangle\right)} \leq \nu\left(E \bigcap F\right)^{\theta}, \quad \textit{for } E \subset \triangle = \triangle_r\left(Q\right).$$

Here θ depends on the Lipschitz character of Ω, but not on E or $\triangle$.

The following result will allow us to relax the hypothesis (1.6) in Theorem 1.8. We adopt the following notation

$$\begin{aligned} W &= (w, r), \quad \text{where} \quad w = (w_1, \ldots, w_n), \quad r = w_{n+1} \\ Y &= (y, s), \quad \text{where} \quad y = (y_1, \ldots, y_n), \quad s = y_{n+1} \\ X &= (x, t), \quad \text{where} \quad x = (x_1, \ldots, x_n), \quad t = x_{n+1}. \end{aligned}$$

Given a measurable function $f\left(X\right)$ and $\gamma > 0$, we recall that the oscillation of f in the cube $Q_\gamma\left(X\right)$ (see Definition 1.5) is given by

$$\operatorname{osc}_\gamma f\left(X\right) = \sup_{Y,W \in Q_\gamma(X)} \left|f\left(W\right) - f\left(Z\right)\right|.$$

Lemma 3.9. *Let Ω be a Lipschitz domain in $\mathbb{R}^{n+1}$, μ be a doubling measure on $\partial\Omega$, $\delta\left(X\right) = \operatorname{dist}\left(X, \partial\Omega\right)$, and suppose that g is a measurable function (scalar or vector valued) such that for some constant $0 < \alpha < 1$ it satisfies*

$$d\nu = \delta\left(X\right) \operatorname{osc}_{\alpha\delta(X)} g\left(X\right)^2 \, dX \in \mathfrak{C}\left(d\mu, \Omega\right).$$

The for every Lipschitz subdomain $\tilde{\Omega} \subset \Omega$, there exists a doubling measure $\tilde{\mu}$ on $\partial\tilde{\Omega}$, with doubling constant depending only on the doubling constant of μ, such that $\tilde{\mu} = \mu$ in $\partial\Omega \bigcap \partial\tilde{\Omega}$, and

$$d\tilde{\nu} = \tilde{\delta}\left(X\right) \operatorname{osc}_{\alpha\delta(X)} g\left(X\right)^2 \, dX \in \mathfrak{C}\left(d\tilde{\mu}, \tilde{\Omega}\right),$$

where $\tilde{\delta}\left(X\right) = \operatorname{dist}\left(X, \partial\tilde{\Omega}\right)$. Moreover, the Carleson norm $\|d\tilde{\nu}\|_{\mathfrak{C}\left(d\tilde{\mu}, \tilde{\Omega}\right)}$ depends only on the Carleson norm of $d\nu$, the doubling constant of μ, and the Lipschitz character of $\tilde{\Omega}$. If μ is the Lebesgue measure on $\partial\Omega$ then $\tilde{\mu}$ can be taken as the Lebesgue measure on $\partial\tilde{\Omega}$.

Proof: In the case that $d\sigma$ is the Euclidean measure $d\sigma$ on $\partial\Omega$, the result follows as in the proof of Lemma 3.1 in [**10**]. To obtain the Lemma 3.9 for an arbitrary doubling measure σ, given a Lipschitz subdomain $\tilde{\Omega} \subset \Omega$, we will construct an appropriate extension of σ from $\partial\Omega \bigcap \partial\tilde{\Omega}$ to $\partial\tilde{\Omega}$. We may assume that $\Omega = \mathbb{R}^{n+1}_{+} = \{(x,t) : x \in \mathbb{R}^n,\, t > 0\}$, the general case follows by standard change of variables techniques. For $X = (x,t) \in \mathbb{R}^{n+1}_{+}$, let $\triangle_X = \{(y,0) : |x-y| \le t\}$ and define $M(X) = \frac{\mu(\triangle_X)}{\sigma(\triangle_X)}$, where σ is the Euclidean measure in $\mathbb{R}^n = \partial\Omega$. Since μ is doubling, $M(X)$ is continuous in $\mathbb{R}^{n+1}_{+}$ and $M \to \frac{d\mu}{d\sigma}$ as $t \to 0$ in the weak* topology of measures. Now, for any Borel set $E \subset \partial\tilde{\Omega}$, we define

$$\tilde{\mu}(E) = \mu\left(E \bigcap \partial\Omega\right) + \int_{E \setminus \partial\Omega} M(X)\, d\tilde{\sigma}(X),$$

where $d\tilde{\sigma}(X)$ denotes the Euclidean measure on $\partial\tilde{\Omega}$. Then obviously $\tilde{\mu}$ is an extension of μ from $\partial\Omega \bigcap \partial\tilde{\Omega}$ to $\partial\tilde{\Omega}$. It remains to check that $\tilde{\mu}$ is a doubling measure and that $d\tilde{\nu} \in \mathfrak{C}\left(d\tilde{\mu}, \tilde{\Omega}\right)$. Let $\triangle \subset \partial\tilde{\Omega}$ be a surface ball centered at $X_0 = (x_0, t_0) \in \partial\tilde{\Omega}$ and let $T(\triangle) \subset \tilde{\Omega}$ be the associated Carleson region. Following the ideas in [**10**], we consider two cases.

In case 1, we assume that $d_0 = \operatorname{diam} T(\triangle)$ is smaller than $\frac{t_0}{10}$. If $X = (x,t) \in \mathbb{R}^{n+1}_{+}$ belongs to $T(\triangle)$, then $|t_0 - t| \le \frac{t_0}{10}$ and so $\delta(X_0) = t_0 \le \frac{10}{9} t = \frac{10}{9}\delta(X)$. From the definition of $M(X)$, and the doubling property of μ, it follows that for all $X \in B_{\frac{t_0}{10}}(X_0)$, $M(X) \approx M(X_0)$ with constants depending only on the doubling constant of μ. Since $\tilde{\Omega}$ is a Lipschitz domain, it follows that

$$\tilde{\mu}(\triangle) = \int_{\triangle} M(X)\, d\tilde{\sigma}(X) \approx M(X_0) \int_{\triangle} d\tilde{\sigma}(X) \approx d_0^n M(X_0) \approx \frac{d_0^n}{t_0^n} \mu(\triangle_{X_0}).$$

Similarly, $\tilde{\mu}(2\triangle) \approx \frac{2^n d_0^n}{t_0^n}\mu(2\triangle_{X_0})$, which shows that $\tilde{\mu}(\triangle) \approx \tilde{\mu}(2\triangle)$. On the other hand, because $d\nu(X) \in \mathfrak{C}(d\mu, \Omega)$, we have

$$\sup_{Y,Z \in B_{\alpha\tilde{\delta}(X)}(X)} |g(Y) - g(Z)| \le \frac{C}{t}\left(\frac{\mu(\triangle_X)}{t^n}\right)^{\frac{1}{2}},$$

where C depends only on n, α, the doubling constant of σ and $\|d\nu\|_{\mathfrak{C}(d\sigma,\Omega)}$. Since $\tilde{\Omega}\subset\Omega$, we have $\tilde{\delta}(X)\le\delta(X)=t$ for all $X\in\tilde{\Omega}$. Then

$$\begin{aligned}\int_{T(\triangle)} d\tilde{\nu}(X) &= \int_{T(\triangle)} \tilde{\delta}(X) \sup_{Y,Z\in B_{\alpha\tilde{\delta}(X)}(X)} |g(Y)-g(Z)|^2\, dX\\ &\le C\int_{T(\triangle)} \tilde{\delta}(X)\frac{C}{t^2}\frac{\mu(\triangle_X)}{t^n}\, dX\\ &\le C\frac{\mu(\triangle_{X_0})}{t_0^{n+1}}|T(\triangle)|\\ &\le C\frac{d_0^{n+1}}{t_0^{n+1}}\mu(\triangle_{X_0})\le C\tilde{\mu}(\triangle).\end{aligned}$$

In case 2, when $d_0=\operatorname{diam}T(\triangle)>\frac{t_0}{10}$, let $Q_0=Q_{cd_0}(x_0,0)$ be the cube in $\mathbb{R}^{n+1}$ centered at $(x_0,0)$, with faces parallel to the coordinate axes and side-length cd_0. For c big enough depending only on the Lipschitz character of $\tilde{\Omega}$, we have $T(\triangle)\subset\tilde{T}=Q_0\bigcap\mathbb{R}^{n+1}_+$. Then, since $\tilde{\delta}(X)\le\delta(X)$ for all $X\in\tilde{\Omega}$ and $d\nu(X)\in\mathfrak{C}(d\sigma,\Omega)$,

$$\begin{aligned}\int_{T(\triangle)} d\tilde{\nu}(X) &= \int_{T(\triangle)} \tilde{\delta}(X)\operatorname{osc}_{\alpha\tilde{\delta}(X)} g(X)^2\, dX\\ &\le \int_{\tilde{T}} \delta(X)\operatorname{osc}_{\alpha\delta(X)} g(X)^2\, dX\\ &= \int_{\tilde{T}} d\nu(X)\\ &\le C\mu\left(Q_0\bigcap\mathbb{R}^n\times\{0\}\right)\\ &\le C\mu\left(Q_{d_0}(x_0,0)\bigcap\mathbb{R}^n\times\{0\}\right),\end{aligned}\tag{3.3}$$

where we used the doubling property of μ. Let $\{\mathcal{Q}_i\}_{i=1}^{\infty}$ be a Whitney decomposition of $\mathbb{R}^{n+1}_+$ into cubes, i.e. for each i, $\operatorname{diam}(\mathcal{Q}_i)\approx\operatorname{dist}\left(\mathcal{Q}_i,\partial\mathbb{R}^{n+1}_+\right)$, for different indexes ι and j, $\mathcal{Q}_i\bigcap\mathcal{Q}_j$ has no interior, and $\mathbb{R}^{n+1}_+=\bigcup_{i=1}^{\infty}\mathcal{Q}_i$. Let $\{\mathcal{P}_j\}_{j=1}^{\infty}\subset\{\mathcal{Q}_i\}_{i=1}^{\infty}$ be the collection of cubes such that $\mathcal{P}_j\bigcap\triangle\ne\emptyset$, and let X_j be an arbitrary point in $\mathcal{P}_j\bigcap\triangle$. Then from the definition of $\tilde{\mu}$ and since $\operatorname{diam}(\mathcal{Q}_i) \approx \operatorname{dist}\left(\mathcal{Q}_i, \partial\mathbb{R}^{n+1}_+\right)$,

$$\tilde{\mu}\left(\mathcal{P}_j \bigcap \triangle\right) = \int_{\mathcal{P}_j \bigcap \triangle} M(X)\, d\tilde{\sigma}(X) \approx \frac{\mu\left(\triangle_{X_j}\right)}{\sigma\left(\triangle_{X_j}\right)} \int_{\mathcal{P}_j \bigcap \triangle} d\tilde{\sigma}(X)$$

$$\approx C \frac{\mu\left(\triangle_{X_j}\right)}{\operatorname{diam}(\mathcal{P}_i)^n} \int_{\mathcal{P}_j \bigcap \triangle} d\tilde{\sigma}(X) \approx C\mu\left(\triangle_{X_j}\right).$$

Then

$$\begin{aligned}\tilde{\mu}(\triangle) &= \mu\left(\triangle \bigcap \Omega\right) + \sum_{j=1}^{\infty} \tilde{\mu}\left(\mathcal{P}_j \bigcap \triangle\right) \approx \mu\left(\triangle \bigcap \Omega\right) + \sum_{j=1}^{\infty} \mu\left(\triangle_{X_j}\right) \\ &\geq C\mu\left(Q_{d_0}(x_0, 0) \bigcap \mathbb{R}^n \times \{0\}\right),\end{aligned} \tag{3.4}$$

the last inequality follows from a simple geometrical argument. From this and (3.3) we have $\int_{T(\triangle)} d\tilde{\nu}(X) \leq C\tilde{\mu}(\triangle)$ as wanted. From (3.4) we also have $\tilde{\mu}(\triangle) \approx \tilde{\mu}(2\triangle)$ in this case. □

The following lemma estates the local character of the regularity of the harmonic measure.

Lemma 3.10. *Let $\mathcal{L}$ be an elliptic operator in divergence form or nondivergence form with drift $\mathbf{b}$ in a Lipschitz domain Ω; i.e. $\mathcal{L} = \mathbf{A} \cdot \nabla + \mathbf{b} \cdot \nabla$ or $\mathcal{L} = \mathbf{A} \cdot \nabla^2 + \mathbf{b} \cdot \nabla$ where $\mathbf{A}$ satisfies the ellipticity condition* (1.2). *Suppose that $\mathbf{b}$ is locally bounded in Ω and it satisfies*

$$\delta(X) \operatorname{osc}_{\frac{\delta(X)}{2\sqrt{n}}} \mathbf{b}(X)^2 \in \mathfrak{C}(\Omega). \tag{3.5}$$

Then $d\omega_{\mathcal{L}} \in A_\infty$ if and only if *there exists a finite collection of Lipschitz domains $\{\Omega\}_{i=1}^N$ and compact sets $K_i \Subset \partial\Omega_i \bigcap \partial\Omega$ such that $\bigcup_{i=1}^n \Omega_i \subset \Omega$, $\partial\Omega \subset \bigcup_{i=1}^n K_i$, and*

$$d\omega_{\mathcal{L}_i|K_i} \in A_\infty(d\sigma), \quad i = 1, \dots, N,$$

where $\mathcal{L}_i$ denotes the restriction of $\mathcal{L}$ to the subdomain Ω_i.

Proof: If $\mathbf{b} = 0$, the result is an immediate consequence of the "main lemma" in [**4**] for the divergence case and the analog to the main lemma in the nondivergence case, contained in [**5**] (see also Lemma 3.8). The case $\mathbf{b} \neq 0$ then follows from Theorem 1.7 for the divergence case and Theorem 4.1 from next section for the nondivergence case. Indeed, by the mentioned theorems, if $\mathcal{L}$ is the operator with drift $\mathbf{b}$ and $\mathcal{L}_0$ is the operator with the same second order coefficients but *without* a drift term, then $d\omega_{\mathcal{L}} \in A_\infty(d\sigma) \Leftrightarrow d\omega_{\mathcal{L}_0} \in A_\infty(d\sigma)$. On the other hand, by Lemma 3.9 with $g(X) = \mathbf{b}(X)$, the restriction of $\mathbf{b}$ to any Lipschitz subdomain $\Omega' \subset \Omega$ also satisfies (3.5) in Ω'. Hence, by Theorems 1.7 and 4.1, for any doubling measure $d\sigma'$ on $\partial\Omega'$, we have $d\omega_{\mathcal{L}|\Omega'} \in A_\infty(d\sigma') \Leftrightarrow d\omega_{\mathcal{L}_0|\Omega'} \in A_\infty(d\sigma')$. Now, for Ω_i, K_i as in the statement of Lemma 3.10, $d\sigma|_{K_i}$, the restriction of $d\sigma$ to the compact set K_i, can be extended to a doubling measure $d\sigma_i$ on $\partial\Omega_i$ with the same doubling constant. Then $d\omega_{\mathcal{L}|\Omega_i} \in A_\infty\left(d\sigma^i\right) \Leftrightarrow d\omega_{\mathcal{L}_0|\Omega_i} \in A_\infty\left(d\sigma^i\right)$, which implies that

$$d\omega_{\mathcal{L}|\Omega_i}|_{K_i} \in A_\infty(d\sigma|_{K_i}) \Leftrightarrow d\omega_{\mathcal{L}_0|\Omega_i}|_{K_i} \in A_\infty(d\sigma|_{K_i}),$$

where $d\omega_{\mathcal{L}|\Omega_i}|_{K_i}$ denotes the restriction to K_i of the harmonic measure of $\mathcal{L}$ in Ω_i, with a similar definition for $d\omega_{\mathcal{L}_0|\Omega_i}|_{K_i}$. This shows that Lemma 3.10 in the case $\mathbf{b} \neq 0$ follows from the case $\mathbf{b} = 0$. $\square$

4. Proofs of the Theorems

We recall that $Q_r(X)$ denotes a cube centered at X with sidelength r, and $\delta(x)$ denotes the distance of X to the boundary (see Definition 1.5). The proof of Theorem 2.1 relies on the following special case.

Theorem 4.1. *Let $\tilde{\mathcal{L}}_{N,0} = \mathbf{A} \cdot \nabla^2$ and $\tilde{\mathcal{L}}_{N,1} = \mathbf{A}\nabla^2 + \mathbf{b} \cdot \nabla$ where $\mathbf{A} = (A_{ij})_{i,j=1}^{n+1}$ and $\mathbf{b} = (b_j)_{j=1}^{n+1}$ are bounded, measurable coefficients and $\mathbf{A}$ satisfy the ellipticity condition* (1.2) *in a Lipschitz domain Ω. Suppose that $\mathcal{CD}$ holds for $\tilde{\mathcal{L}}_{N,\ell}$, $\ell = 0, 1$ in Ω and that*

$$\delta(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}(Y)|^2 \, dX \in \mathfrak{C}(d\sigma, \Omega). \tag{4.1}$$

Then $d\omega_{\tilde{\mathcal{L}}_{N,0}} \in A_\infty \Rightarrow d\omega_{\tilde{\mathcal{L}}_{N,1}} \in A_\infty$.

We defer the proof of this result (which contains the main substance of Theorem 2.1) to next section. Now we obtain Theorem 2.1 from Theorem 4.1.

Proof of Theorem 2.1: Let $\mathcal{L}_{N,0} = \mathbf{A}_0 \cdot \nabla^2 + \mathbf{b}_0 \cdot \nabla$ and $\mathcal{L}_{N,1} = \mathbf{A}_1 \nabla^2 + \mathbf{b}_1 \cdot \nabla$ where $\mathbf{A}_\ell$, $\mathbf{b}_\ell$ and $\mathcal{L}_{N,\ell}$ satisfy the hypotheses of Theorem 2.1 for $\ell = 0, 1$. Then if $d\omega_{\mathcal{L}_{N,0}} \in A_\infty$, by Theorem 4.1 it follows that $d\omega_{\mathcal{L}^*_{N,0}} \in A_\infty$ where $\mathcal{L}^*_{N,0} = \mathbf{A}_0 \cdot \nabla^2$. By Theorem 1.6 [**17**] and Theorem 4.1 again, we have that if $\mathcal{L}^*_{N,1} = \mathbf{A}_1 \cdot \nabla^2$, then

$$d\omega_{\mathcal{L}^*_{N,0}} \in A_\infty \Rightarrow d\omega_{\mathcal{L}^*_{N,1}} A_\infty \Rightarrow d\omega_{\mathcal{L}_{N,1}} \in A_\infty,$$

as wanted. $\square$

Theorem 2.4 will follow by reduction to the special case $\Omega = \mathbb{R}^{n+1}_+$, the localization given by Lemma 3.10 and an approximation of the coefficients matrix $\mathbf{A}$ by appropriate smooth matrices.

Proof of Theorem 2.4: Given $P_0 \in \partial\Omega$ let $X = (x,t)$ be a coordinate system such that $P_0 = (x_0, t_0) \in \partial\Omega$ and there exists a Lipschitz function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ defining a local coordinate system of Ω in a neighborhood of P_0. That is, for some $r_0 = r_0(\Omega) > 0$ we have

$$\begin{aligned} &\partial\Omega \bigcap \{|x-x_0| < r_0\} \times \mathbb{R} = \{(x, \psi(x)) : |x-x_0| < r_0\} \\ &\Omega' = \{(x,t) : |x-x_0| < r_0,\, \psi(x) < t < \psi(x) + r_0\} \subset \Omega. \end{aligned} \tag{4.2}$$

Let $\eta_s(y) = s^{-n}\eta\left(\frac{y}{s}\right)$, where η is an even $\mathcal{C}^\infty$ approximate identity in $\mathbb{R}^n$ supported in $\left\{|y| \le \frac{1}{2}\right\}$. Set $\rho(y,s) = (y, c_0 s + F(y,s))$ with $F(y,s) = \eta_s * \psi(y) = \int_{\mathbb{R}^n} \eta_s(y-z)\,\psi(z)\,dz$. We have

$$\nabla_Y \rho = \begin{pmatrix} I & \nabla_y F \\ 0 & c_0 + \frac{\partial F}{\partial s} \end{pmatrix}.$$

Since $\left\|\frac{\partial F}{\partial s}\right\|_\infty \le C_n \|\nabla_y \psi\|_\infty$, where $C_n = n\int \eta(y)\,|y|\,d\sigma$, (note that, for appropriate η, C_n is a universal constant), taking $c_0 = 1 + C_n\|\nabla_y\psi\|_\infty$, ρ is a 1-1 map of $\mathbb{R}^{n+1}_+$ onto $\{(x,t) : t > \psi(x)\}$, moreover, ρ is bi-Lipschitz and $1 \le |\det \nabla_Y \rho| \le 1 + 2C_n\|\nabla_y\psi\|_\infty$. This transformation gives rise to the Dahlberg-Kenig-Stein adapted distance function. For $\alpha > 0$, let $\Phi_\alpha = \{y : |y-x_0| < \alpha\} \times \left(0, \frac{\alpha}{c_0}\right)$, and $\Omega_\alpha = \rho(\Phi_\alpha)$. For $\alpha = \alpha_0$ small enough depending only on $\|\nabla_y\psi\|_\infty$ and n, we have $\Omega_{\alpha_0} \subset \Omega'$, where Ω' is given by (4.2). Moreover, Ω_{α_0} is a Lipschitz domain with Lipschitz constant depending only on the constant of Ω.

We note that because of Theorem 1.7 from [**8**] in the divergence case and because of Theorem 2.1 in the nondivergence case we might assume $\mathbf{b} \equiv 0$. We will first consider the divergence case, suppose that $\mathcal{L}_D = \operatorname{div}\mathbf{A}\nabla$ is a uniformly elliptic operator in divergence form with bounded measurable coefficient matrix $\mathbf{A}$ satisfying (2.2), i.e.

$$\delta^{-1}(X)\operatorname{osc}_{Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} \mathbf{A}(X)^2\, dX \in \mathfrak{C}(d\sigma, \Omega). \tag{4.3}$$

Since P_0 is an arbitrary point on $\partial\Omega$ and $\partial\Omega$ is compact, by Lemma 3.10, to prove Theorem 2.4 it is enough to prove that if ω is the harmonic measure for $\mathcal{L}_D$ in Ω_{α_0}, then

$$\omega|_K \in A_\infty(d\sigma|_K), \tag{4.4}$$

where $K \subset \partial\Omega_{\alpha_0}$ is the compact set given by

$$K = \rho\left\{Y = (y,s) : |y-x_0| \le \frac{\alpha_0}{6},\, s = 0\right\} \subset \partial\Omega_{\alpha_0}. \tag{4.5}$$

For simplicity, we will write $\Omega = \Omega_{\alpha_0}$ and $\Phi = \Phi_{\alpha_0}$. Since ψ is Lipschitz, it follows that the transformation $\rho\colon \Phi \to \Omega$ can be extended to a homeomorphism from $\overline{\Phi}$ to $\overline{\Omega}$ and such that the restriction of ρ to $\partial\Phi$ is a bi-Lipschitz homeomorphism from $\partial\Omega$ to $\partial\Omega$. Indeed, since $\{\eta_s\}_{s>0}$ is a smooth approximation of the identity, it easily follows that ρ restricted to $\partial\Phi$ is given by

(4.6)

$$\rho(Y) = \rho(y,s) = \begin{cases} (y, \psi(y)), & s = 0 \\ (y, c_0 s + F(y,s)), & 0 < s < \dfrac{\alpha_0}{c_0},\ |y - x_0| = \alpha_0 \\ \left(y, c_0 \dfrac{\alpha_0}{c_0} + F\left(y, \dfrac{\alpha_0}{c_0}\right)\right) & s = \dfrac{\alpha_0}{c_0} \end{cases}$$

whenever $Y \in \partial\Phi$. If $\tilde{\delta}(Y) = \operatorname{dist}(Y, \partial\Phi)$, then for some constant $\tilde{C}$ depending only on $\|\nabla_y \psi\|_\infty$ and n, the following estimate holds for the distance functions δ and $\tilde{\delta}$:

$$\tilde{C}^{-1}\tilde{\delta}(Y) \le \delta(\rho(Y)) \le \tilde{C}\delta(Y), \quad Y \in \Phi. \tag{4.7}$$

To see this, let $Y_0 \in \Phi$ and let $\tilde{\delta}_0 = \delta(Y_0)$. Now, $\delta(\rho(Y_0)) = \operatorname{dist}(\rho(Y_0), \partial\Omega) = |\rho(Y_0) - X'|$ for some $X' \in \partial\Omega$. Let $Y' = \rho^{-1}(X')$, and $X_0 = \rho(Y_0)$, thus, $Y' \in \partial\Phi$ and since ρ^{-1} is Lipschitz in $\overline{\Phi}$, we have

$$\tilde{\delta}(Y_0) \le |Y_0 - Y'| = \left|\rho^{-1}(X_0) - \rho^{-1}(X')\right| \le C|X_0 - X'| = C\delta(\rho(Y_0)),$$

with $C = C\left(n, \|\nabla_y \psi\|_\infty\right)$. The other inequality in (4.7) follows in a similar manner.

The Lebesgue measure σ on $\partial\Omega$ induces a doubling measure $\tilde{\mu}$ on $\partial\Phi$ by the relation

$$\tilde{\mu}(E) = \sigma(\rho(E)), \quad \text{for any Borel set } E \subset \partial\Phi.$$

Let $\tilde{\sigma}$ be the Lebesgue measure on $\partial\Phi$, then from the definition of $\tilde{\mu}$ and the fact that ρ is bi-Lipschitz it easily follows that $\frac{d\tilde{\mu}}{d\tilde{\sigma}} \approx 1$. Hence we can replace $\tilde{\mu}$ by $\tilde{\sigma}$ in our calculations. Now, if $u(x,t)$ is a solution of $\mathcal{L}_D u = \operatorname{div}_X \mathbf{A}\nabla_X u = 0$ in Ω, then $v(y,s) = u(\rho(y,s))$, defined in Φ, is a solution of $\tilde{\mathcal{L}}_D v = \operatorname{div}_Y \tilde{\mathbf{A}}\nabla_Y v = 0$, where

$$\tilde{\mathbf{A}}(Y) = \left((\nabla_Y \rho)^{-1}(Y)\right)^t \mathbf{A}(\rho(Y))\,(\nabla_Y \rho)^{-1}(Y)\det(\nabla_Y \rho)(Y). \tag{4.8}$$

We claim that $\tilde{\mathbf{A}}$ satisfies

$$\tilde{\delta}(Y)^{-1} \operatorname{osc}_{Q_{\frac{\delta(Y)}{2\sqrt{n}}}(Y)} \tilde{\mathbf{A}}(Y)^2 \, dY \in \mathfrak{C}(d\tilde{\sigma}, \Phi). \tag{4.9}$$

Let $Y_0 \in \Phi$ let Y_1, Y_2 such that $|Y_1 - Y_0| \leq \frac{1}{2}\delta_0$ and $|Y_2 - Y_0| \leq \frac{1}{2}\delta_0$, where $\delta_0 = \tilde{\delta}(Y_0)$, then by (4.8)

(4.10)
$$\begin{aligned}
&\frac{\left|\tilde{\mathbf{A}}(Y_1) - \tilde{\mathbf{A}}(Y_2)\right|^2}{\tilde{\delta}_0} \\
&\quad = \tilde{\delta}_0^{-1}\left|\left((\nabla_Y\rho)^{-1}(Y_1)\right)^t \mathbf{A}(\rho(Y_1))(\nabla_Y\rho)^{-1}(Y_1)\det(\nabla_Y\rho)(Y_1)\right. \\
&\qquad \left. - \left((\nabla_Y\rho)^{-1}(Y_2)\right)^t \mathbf{A}(\rho(Y_2))(\nabla_Y\rho)^{-1}(Y_2)\det(\nabla_Y\rho)(Y_2)\right|^2 \\
&\quad \leq C\frac{\left|\left((\nabla_Y\rho)^{-1}(Y_1)\right)^t - \left((\nabla_Y\rho)^{-1}(Y_2)\right)^t\right|^2}{\tilde{\delta}_0} \\
&\qquad \times\left|\mathbf{A}(\rho(Y_1))(\nabla_Y\rho)^{-1}(Y_1)\det(\nabla_Y\rho)(Y_1)\right|^2 \\
&\qquad + C\frac{\left|(\nabla_Y\rho)^{-1}(Y_1) - (\nabla_Y\rho)^{-1}(Y_2)\right|^2}{\tilde{\delta}_0} \\
&\qquad \times\left|\left((\nabla_Y\rho)^{-1}(Y_2)\right)^t \mathbf{A}(\rho(Y_1))\right|^2 |\det(\nabla_Y\rho)(Y_1)|^2 \\
&\qquad + C\frac{|\det(\nabla_Y\rho)(Y_1) - \det(\nabla_Y\rho)(Y_2)|^2}{\tilde{\delta}_0} \\
&\qquad \times\left|\left((\nabla_Y\rho)^{-1}(Y_2)\right)^t \mathbf{A}(\rho(Y_1))(\nabla_Y\rho)^{-1}(Y_2)\right|^2 \\
&\qquad + C\frac{|\mathbf{A}(\rho(Y_1)) - \mathbf{A}(\rho(Y_2))|^2}{\tilde{\delta}_0} \\
&\qquad \times\left|\left((\nabla_Y\rho)^{-1}(Y_2)\right)^t\right|^2 \left|(\nabla_Y\rho)^{-1}(Y_2)\det(\nabla_Y\rho)(Y_2)\right|^2.
\end{aligned}$$

We will use the following fact:

Lemma 4.2. *For Ω, Φ, σ, $\tilde{\mu}$, δ, $\tilde{\delta}$ and ρ as above, the functions*

$$r_1(Y) = \frac{1}{\tilde{\delta}(Y)} \left| \operatorname{osc}_{Q_{\frac{\tilde{\delta}(Y)}{2\sqrt{n}}}(Y)} (\nabla_Y \rho)^{-1}(Y) \right|^2$$

and

$$r_2(Y) = \frac{1}{\tilde{\delta}(Y)} \left| \operatorname{osc}_{Q_{\frac{\tilde{\delta}(Y)}{2\sqrt{n}}}(Y)} (\det(\nabla_Y \rho)(Y)) \right|^2$$

defined in Φ, satisfy $[r_1(Y) + r_2(Y)]\, dY \in \mathfrak{C}(d\tilde{\sigma}, \Phi)$, where $\tilde{\sigma}$ is the Lebesgue measure on $\partial\Phi$.

The lemma above follows from the fact that

$$\tilde{\delta}(Y) \left|\nabla^2 \rho\right|^2 \, dY \in \mathfrak{C}(d\tilde{\sigma}, \Phi). \tag{4.11}$$

This property is discussed in [**10**], and it can be obtained as an application of the characterization of A_∞ in terms of Carleson measures given in [**7**].

Then (4.9) follows by applying (4.3), (4.7) and Lemma 4.2 to (4.10).

We recall that $\Phi = \Phi_{\alpha_0}$ where Φ_α is given by $\Phi_\alpha = \{y : |y - x_0| < \alpha\} \times \left(0, \frac{\alpha}{c_0}\right)$. Denote by $\Phi_\alpha^\pm = \{y : |y - x_0| < \alpha\} \times \left(-\frac{\alpha}{c_0}, \frac{\alpha}{c_0}\right)$ and let $\nu(Y) \in \mathcal{C}_0^\infty\left(\Phi_{\alpha_0}^\pm\right)$ such that $0 \le \nu \le 1$, $\nu \equiv 1$ in $\Phi_{\frac{\alpha_0}{3}}^\pm$ and $\nu \equiv 0$ in $\Phi_{\alpha_0}^\pm \backslash \Phi_{\frac{2\alpha_0}{3}}^\pm$. For $Y \in \mathbb{R}_+^{n+1}$, let

$$\tilde{\mathbf{A}}^*(Y) = \nu(Y)\, \tilde{\mathbf{A}}(Y) + (1 - \nu(Y))\, I,$$

where I is the $(n+1) \times (n+1)$ identity matrix. It follows that $\tilde{\mathbf{A}}^*(Y)$ is an elliptic matrix function, with the same ellipticity constants as $\tilde{\mathbf{A}}$. The measure $\tilde{\sigma}$ extends trivially from $\partial\Phi \bigcap \partial\mathbb{R}_+^{n+1}$ to $\partial\mathbb{R}_+^{n+1}$, we dub this extension (which is just the Euclidean measure) $d\tilde{\sigma}^*$. With this definitions, because of (4.9), for $Y = (y, t) \in \mathbb{R}_+^{n+1}$, $\tilde{\mathbf{A}}^*$ satisfies

$$\frac{\operatorname{osc}_{Q_{\frac{t}{2\sqrt{n}}}(Y)} \tilde{\mathbf{A}}^*(Y)^2}{t} \in \mathfrak{C}\left(d\tilde{\sigma}^*, \mathbb{R}_+^{n+1}\right), \tag{4.12}$$

(i.e.: is a Carleson measure in $\mathbb{R}_+^{n+1}$ with respect to $\tilde{\sigma}^*$). Where $Q_\gamma(Y)$ is the cube centered at Y with faces parallel to the coordinate axes and sidelength γt.

Now we will construct a smooth approximation of $\tilde{\mathbf{A}}^*$ via an $n+1$ dimensional approximate identity. Let

$$P_t f(x,t) = \iint t^{-n-1} \varphi\left(\frac{x-y}{t}, \frac{t-s}{t}\right) f(y,s)\, dy\, ds,$$

where $\varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}^{n+1}\right)$ is supported in the ball or radius $\alpha < 1$ at the origin (α to be chosen later) and $\iint \varphi = 1$. Since $P_t 1 \equiv 1$ we have $\nabla_{x,t} P_t 1 \equiv 0$ and therefore $\tilde{\mathbf{A}}^{**} = P_t \tilde{\mathbf{A}}^*$ satisfies

$$\begin{aligned}
\nabla \tilde{\mathbf{A}}^{**} &= \iint t^{-n-1} \nabla_{x,t} \varphi\left(\frac{x-y}{t}, \frac{t-s}{t}\right) \tilde{\mathbf{A}}^*(y,s)\, dy\, ds \\
&= -\iint t^{-n-1} \nabla_{y,s} \varphi\left(\frac{x-y}{t}, \frac{t-s}{t}\right) \tilde{\mathbf{A}}^*(y,s)\, dy\, ds \\
&= -\iint t^{-n-1} \nabla_{y,s} \varphi\left(\frac{x-y}{t}, \frac{t-s}{t}\right) \left(\tilde{\mathbf{A}}^*(y,s) - C(x,t)\right) dy\, ds
\end{aligned}$$

for any function $C(X)$ at our disposal. Taking $C(X) = \tilde{\mathbf{A}}^*(X)$ it follows that for α small enough

$$\tilde{\mathcal{E}}^*(Y) = \sup_{Z \in Q_0(Y)} \left|\nabla \tilde{\mathbf{A}}^{**}(Z)\right| \leq C \frac{\operatorname{osc}_{Q_{\frac{t}{2\sqrt{n}}}(Y)} \tilde{\mathbf{A}}^*}{t}$$

and

$$\tilde{\mathcal{E}}^{**}(Y) = \sup_{Z \in Q_0(Y)} \left|\tilde{\mathbf{A}}^{**}(Z) - \tilde{\mathbf{A}}^*(Z)\right| \leq C \operatorname{osc}_{Q_{\frac{t}{2\sqrt{n}}}(Y)} \tilde{\mathbf{A}}^*$$

where $Q_0(Y) = Q_{\frac{\delta(Y)}{6\sqrt{n}}}(Y)$ and C is a universal constant (this constant depends on the Lipschitz norm of φ, which we can assume only depends on the dimension n). From (4.12) it follows that

$$(4.13) \quad t\mathcal{E}^*(Y)^2 \in \mathfrak{C}\left(d\tilde{\sigma}^*, \mathbb{R}^{n+1}_+\right) \quad \text{and} \quad \frac{\tilde{\mathcal{E}}^{**}(X)^2}{t}\, dX \in \mathfrak{C}\left(d\tilde{\sigma}^*, \mathbb{R}^{n+1}_+\right).$$

Moreover, from the definitions it is easy to check that $\tilde{\mathbf{A}}^{**}$ is elliptic with the same ellipticity constants as $\tilde{\mathbf{A}}$.

Let now $\tilde{\mathcal{L}}_D^* = \operatorname{div}_Y \tilde{\mathbf{A}}^* \nabla_Y$ and $\tilde{\mathcal{L}}_D^{**} = \operatorname{div}_Y \tilde{\mathbf{A}}^{**} \nabla_Y$. By the Carleson measure property of $\tilde{\mathcal{E}}^{**}$, and Lemma 3.9, $\tilde{\mathcal{L}}_D^*$ and $\tilde{\mathcal{L}}_D^{**}$ satisfy the hypotheses of Theorem 1.7 in Φ (with respect to the measure $\tilde{\sigma}$), therefore, if $\tilde{\omega}^*$ and $\tilde{\omega}^{**}$ denote the harmonic measures of $\tilde{\mathcal{L}}_D^*$ and $\tilde{\mathcal{L}}_D^{**}$ in Φ, respectively, we have that $\tilde{\omega}^* \in A_\infty \Leftrightarrow \tilde{\omega}^{**} \in A_\infty$. On the other hand, because of the Carleson measure property of $\tilde{\mathcal{E}}^*$, and Lemma 3.9, $\tilde{\mathcal{L}}_D^{**}$ satisfies the hypotheses of Theorem 1.8 in Φ; and therefore $\tilde{\omega}^{**} \in A_\infty$. From what we just proved it follows that $\tilde{\omega}^* \in A_\infty$.

Let $\mathcal{L}_D^*$ denote the pull-back of $\tilde{\mathcal{L}}_D^*$ from Φ to Ω through the mapping ρ. Since the mapping $\rho\colon \partial\Phi \to \partial\Omega$ is bi-Lipschitz, and if ω^* denotes the harmonic measure of $\mathcal{L}_D^*$ in Ω, then $\omega^* \in A_\infty$. This follows directly from the definitions of the A_∞ class, the harmonic measure ω^* and $\mathcal{L}_D^*$. On the other hand, the operator $\mathcal{L}_D^*$ coincides with $\mathcal{L}_D = \operatorname{div}_X \mathbf{A}\nabla_X$ in $\rho\left(\Phi_{\frac{\alpha_0}{3}}\right)$, hence an application of Theorem 1.7 and the"main lemma" in [**4**] (see Lemma 3.10), implies that $\omega|_K \in A_\infty\left(d\omega^*|_K\right)$, where ω is the harmonic measure of $\mathcal{L}_D$ restricted to the compact set K given by (4.5). This, in turn, implies that $\omega|_K \in A_\infty\left(d\sigma|_K\right)$ and proves (4.4), hence Theorem 2.4, for the divergence case.

We now consider the nondivergence case. Let $\mathcal{L}_N = \mathbf{A}\cdot\nabla^2$ where $\mathbf{A}$ satisfies (4.3). Let $\Omega = \Omega_{\alpha_0}$, $\Phi = \Phi_{\alpha_0}$, and ρ be as before. Also, for $Y \in \Phi$, let $\tilde{\mathbf{A}}^{**}(Y)$ be as above, and define $\mathcal{L}^{**} = \mathbf{A}^{**}\cdot\nabla^2$, where

$$\mathbf{A}^{**}(X) = \left(\left(\nabla_Y\rho\right)\left(\rho^{-1}(X)\right)\right)^t \tilde{\mathbf{A}}^{**}\left(\rho^{-1}(X)\right)\left(\nabla_Y\rho\right)\left(\rho^{-1}(X)\right) \times \left(\det\left(\nabla_Y\rho\right)\left(\rho^{-1}(X)\right)\right)^{-1}.$$

Let $\Omega_{\frac{1}{3}} = \rho\left(\Phi_{\frac{\alpha_0}{3}}\right)$, we claim that $\mathbf{A}^{**}(X)$ satisfies the following

$$\sup_{Z\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} \frac{\left|\mathbf{A}(Z) - \mathbf{A}^{**}(Z)\right|^2}{\delta(X)}\,dX \in \mathfrak{C}\left(d\hat{\sigma}, \Omega_{\frac{1}{3}}\right), \quad \text{and} \tag{4.14}$$

$$\sup_{Z\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} \delta(X)\left|\nabla\mathbf{A}^{**}(Z)\right|^2 dX \in \mathfrak{C}\left(d\hat{\sigma}, \Omega_{\frac{1}{3}}\right) \tag{4.15}$$

where $\hat{\sigma}$ is the Lebesgue measure on $\partial\Omega_{\frac{1}{3}}$. Taking these properties for granted, by (4.15), (4.3) and Theorem 1.8 applied to the operator $\mathcal{L}^{**}$, we have that if ω^{**} is the harmonic measure of $\mathcal{L}^{**}$ on $\partial\Omega_{\frac{1}{3}}$, then $\omega^{**} \in A_\infty$. On the other hand, by (4.14) and Theorem 2.1 applied to the operators $\mathcal{L}$ and $\mathcal{L}^{**}$, from $\omega^{**} \in A_\infty$ we conclude that $\omega \in A_\infty$, where ω is the harmonic measure of $\mathcal{L}$ on $\partial\Omega_{\frac{1}{3}}$. This finishes the proof of Theorem 2.4 in the nondivergence case.

It only rests to establish properties (4.14) and (4.15). Let $Z \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)$ and let $W = \rho^{-1}(Z)$, then from the definitions of $\tilde{\mathbf{A}}$ and $\mathbf{A}^{**}$ we have

$$\left|\mathbf{A}(Z) - \mathbf{A}^{**}(Z)\right| = \left|\left(\left(\nabla_Y\rho\right)(W)\right)^t\left[\tilde{\mathbf{A}}(W) - \tilde{\mathbf{A}}^{**}(W)\right] \times \left(\nabla_Y\rho\right)(W)\left(\det\left(\nabla_Y\rho\right)(W)\right)^{-1}\right|.$$

From (4.7) and the fact that ρ is bi-Lipschitz, and since $\rho^{-1}\left(\Omega_{\frac{1}{3}}\right) = \Phi_{\frac{\alpha_0}{3}}$,

$$\frac{\left|\mathbf{A}(Z) - \mathbf{A}^{**}(Z)\right|^2}{\delta(Z)} \approx \frac{\left|\tilde{\mathbf{A}}(W) - \tilde{\mathbf{A}}^{**}(W)\right|^2}{\tilde{\delta}(W)} = \frac{\left|\tilde{\mathbf{A}}^{*}(W) - \tilde{\mathbf{A}}^{**}(W)\right|^2}{\tilde{\delta}(W)}.$$

Applying the proof of Lemma 3.9 to $\tilde{\delta}(W)^{-1} \operatorname{osc}_{\alpha\tilde{\delta}(W)}\left(\tilde{\mathbf{A}}^{*} - \tilde{\mathbf{A}}^{**}\right)(W)$, from the second property in (4.13) it follows that for some $0 < c < 1$

$$\frac{\sup_{W \in Q_{c\tilde{\delta}(Y)}(Y)} \left|\tilde{\mathbf{A}}^{*}(W) - \tilde{\mathbf{A}}^{**}(W)\right|^2}{\tilde{\delta}(Y)} \in \mathfrak{C}\left(d\mu, \Phi_{\frac{\alpha_0}{3}}\right),$$

where μ is the Lebesgue measure on $\partial\Phi_{\frac{\alpha_0}{3}}$; (4.14) then follows from the fact that ρ is bi-Lipschitz. Now, by the product rule of differentiation

$$\begin{aligned}
\nabla \mathbf{A}^{**} &= \nabla_X \left\{ (\nabla_Y \rho)^t \, \tilde{\mathbf{A}}^{**} (\nabla_Y \rho) \det (\nabla_Y \rho)^{-1} \right\} \\
&= \left\{ \nabla_X (\nabla_Y \rho)^t \right\} \tilde{\mathbf{A}}^{**} (\nabla_Y \rho) \det (\nabla_Y \rho)^{-1} \\
&\quad + (\nabla_Y \rho)^t \left\{ \nabla_X \tilde{\mathbf{A}}^{**} (\nabla_Y \rho) \right\} \det (\nabla_Y \rho)^{-1} \\
&\quad + (\nabla_Y \rho)^t \, \tilde{\mathbf{A}}^{**} \left\{ \nabla_X (\nabla_Y \rho) \right\} \det (\nabla_Y \rho)^{-1} \\
&\quad + (\nabla_Y \rho)^t \, \tilde{\mathbf{A}}^{**} (\nabla_Y \rho) \left\{ \nabla_X \det (\nabla_Y \rho)^{-1} \right\}.
\end{aligned}$$

Applying the chain rule in each term, we see that $\nabla \mathbf{A}^{**}$ satisfies (4.15) because of the first property in (4.13), (4.11), and the boundedness of $|\nabla_Y \rho|$. $\square$

5. Proof of Theorem 4.1

In the spirit of [**7**] (see also [**17**]) we will obtain Theorem 4.1 as a consequence of the following perturbation result.

Theorem 5.1. *Let $\tilde{\mathcal{L}}_{N,0} = \mathbf{A} \cdot \nabla^2$ and $\tilde{\mathcal{L}}_{N,1} = \mathbf{A}\nabla^2 + \mathbf{b} \cdot \nabla$ where $\mathbf{A} = (A_{ij})_{i,j=1}^{n+1}$ and $\mathbf{b} = (b_j)_{j=1}^{n+1}$ are bounded, measurable coefficients and $\mathbf{A}$ satisfy the ellipticity condition (1.2). Suppose that $\mathcal{CD}$ holds for $\tilde{\mathcal{L}}_{N,\ell}$, $\ell = 0, 1$. Let $\mathcal{G}_0(X,Y)$ denote the Green's function for $\tilde{\mathcal{L}}_{N,0}$ in Ω and set $\mathcal{G}_0(Y) = \mathcal{G}_0(0,Y)$. There exists $\varepsilon_0 > 0$ which depends only on n, λ,*

Λ and Ω such that if

$$\mathcal{G}_0(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}|^2 \, dX, \quad X \in \Omega, \tag{5.1}$$

is a Carleson measure in Ω with respect to $d\omega_{\tilde{\mathcal{L}}_{N,0}}$ on $\partial\Omega$ with Carleson norm bounded by ε_0, i.e.,

$$\mathcal{G}_0(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}|^2 \, dX \in \mathfrak{C}\left(d\omega_{\tilde{\mathcal{L}}_{N,0}}, \Omega\right),$$

$$\left\| \mathcal{G}_0(X) \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}|^2 \, dX \right\|_{\mathfrak{C}\left(d\omega_{\tilde{\mathcal{L}}_{N,0}}, \Omega\right)} \leq \varepsilon_0,$$

then $d\omega_{\tilde{\mathcal{L}}_{N,1}} \in B_2\left(d\omega_{\tilde{\mathcal{L}}_{N,0}}\right)$. Where $B_2\left(d\omega_{\tilde{\mathcal{L}}_{N,0}}\right)$ denotes the reverse Hölder class of $d\omega_{\tilde{\mathcal{L}}_{N,0}}$ with exponent 2.

We defer the proof of Theorem 5.1 to the next subsection, and prove now Theorem 4.1, we follow the argument in [**7**]. Let $\triangle_r(Q)$ be the boundary ball $\triangle_r(Q) = \{P \in \partial\Omega : |Q - P| < r\}$, and denote by $T_r(Q)$ the Carleson region in Ω associated to $\triangle_r(Q)$, $T_r(Q) = \{X \in \Omega : |X - Q| < r\}$. By Lemma 3.10 we may assume that $\mathbf{b}(X) \equiv 0$ if $\delta(X) > r_0$ for some fixed (small) $r_0 > 0$. To prove Theorem 4.1 it is enough to show that if $\tilde{\omega}_1 = \omega_{\tilde{\mathcal{L}}_{N,1}}$ with $\tilde{\mathcal{L}}_{N,1}$ as in the statement of the theorem, then for all $Q \in \partial\Omega$,

$$\tilde{\omega}_1|_{\triangle_{r_0}(Q)} \in A_\infty. \tag{5.2}$$

For $Q \in \partial\Omega$, $r > 0$, $\alpha > 0$, let $\Gamma_{\alpha,r}(Q)$ be a nontangential cone of fixed aperture α and height r, i.e.

$$\Gamma_{\alpha,r}(Q) = \{X \in \Omega : |X - Q| < (1+\alpha)\,\delta(X) < (1+\alpha)\, r\}.$$

For a fixed $\alpha_0 > 0$ to be determined later, let $\mathcal{E}_r(Q)$ be given by

$$\mathcal{E}_{\mathbf{b},r}(Q) = \left\{ \int_{\Gamma_{\alpha_0,r}(Q)} \delta(X)^{1-n} \sup_{Y \in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}|^2 \, dX \right\}^{\frac{1}{2}}, \quad Q \in \partial\Omega.$$

Fix α_0, r_0, such that $\Gamma_{\alpha_0,r_0}(Q) \subset T_{2r_0}(Q)$ for all $Q \in \partial\Omega$. Then, letting σ be the Lebesgue measure on $\partial\Omega$, by Fubini's theorem, the hypothesis

$\delta(X)\sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)}|\mathbf{b}(Y)|^2\,dX\in\mathfrak{C}(d\sigma,\Omega)$ and the doubling property of $d\sigma$, we have

$$\frac{1}{\sigma(\triangle_{r_0}(Q))}\int_{\triangle_{r_0}(Q)}\mathcal{E}_{\mathbf{b},r_0}(P)^2\,d\sigma(P)$$
$$\leq C\frac{1}{\sigma(\triangle_{r_0}(Q))}\int_{T_{2r_0}(Q)}\delta(X)\sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)}|\mathbf{b}(Y)|^2\,dX\leq C.$$

That is, the average in $\triangle_{r_0}(Q)$ of $\mathcal{E}^2_{\mathbf{b},r_0}$ is bounded. Hence, there exists a closed set $F\subset\triangle_{r_0}(Q)$ such that $\sigma(F)>\frac{1}{2}\sigma(\triangle_{r0}(Q))$ and $\mathcal{E}_{\mathbf{b},r_0}(P)\leq C$ for all $P\in F$. Let Ω_F be a "saw-tooth" region above F in Ω as given in Definition 3.7, and let $\mathbf{b}^*(X)$ be given by

$$\mathbf{b}^*(X)=\begin{cases}\mathbf{b}(X) & X\in\Omega_F\\ \mathbf{0} & X\in\Omega\backslash\Omega_F.\end{cases}$$

The drift $\mathbf{b}^*$ so defined satisfies $\mathcal{E}_{\mathbf{b}^*,r_0}(P)\leq C_0$ for *all* $P\in\triangle_{r_0}(Q)$. We claim that if C_0 is small enough then the operators $\mathcal{L}_0=\tilde{\mathcal{L}}_N=\mathbf{A}\cdot\nabla^2$ and $\mathcal{L}_1=\mathbf{A}\cdot\nabla^2+\mathbf{b}^*\cdot\nabla$ satisfy the hypotheses of Theorem 5.1 in $\Phi=T_{3r_0}(Q)$. Indeed, since $\mathbf{b}^*\equiv 0$ in $T_{3r_0(Q)}\backslash T_{2r_0}(Q)$, we only need to check the Carleson measure condition (5.1) near $\triangle_{2r_0}(Q)$. More precisely, we will show that for all $s<r_0/2$ and $P\in\triangle_{2r_0}(Q)$,

$$\int_{T_s(P)}\mathcal{G}_0(X)\sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)}|\mathbf{b}|^2\,dX\leq\varepsilon_0\omega_0(\triangle_s(P)),\tag{5.3}$$

where $\mathcal{G}_0(X,Y)$ is the Green's function for $\mathcal{L}_0$ in Φ, $\mathcal{G}_0(Y)=\mathcal{G}_0(X_0,Y)$ (where X_0 is the *center* of Φ) and $\omega_0=\omega^{X_0}_{\mathcal{L}_0,\Phi}$ in $\partial\Phi$. As in [**17**] (see Lemmas 2.8 and 2.14 there), we have

$$\frac{1}{\omega_0(\triangle_X)}\frac{\mathcal{G}_0(X)}{\delta(X)^2}\approx\frac{\mathcal{G}(X)}{\int_{Q_{\frac{\delta(X)}{2\sqrt{n}}}}\mathcal{G}(Y)\,dY},\tag{5.4}$$

where $\triangle_X=\triangle_{\delta(X)}(Q)$ for some $Q\in\partial\Phi$ such that $|X-Q|=\delta(X)$; and $\mathcal{G}(X)=\mathcal{G}\left(\overline{X},X\right)$, with $\mathcal{G}(Z,X)$ the Green's function for $\mathcal{L}_0$ in a fixed domain $\mathcal{T}\Supset\Omega$ and $\overline{X}\in\mathcal{T}\backslash\overline{\Omega}$ is a fixed point away from Ω. From (5.4), writing $\mathcal{G}\left(Q_{\frac{\delta(X)}{2\sqrt{n}}}\right)=\int_{Q_{\frac{\delta(X)}{2\sqrt{n}}}}\mathcal{G}(Y)\,dY$ and proceeding as in the proof of (5.1) in [**17**] we have

$$\begin{aligned}
&\int_{T_s(P)} \mathcal{G}_0(X) \sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}^*|^2 \, dX \\
&\le \int_{T_s(P)} \omega_0(\triangle_X) \frac{\delta(X)^2 \mathcal{G}(X)}{\mathcal{G}\left(Q_{\frac{\delta(X)}{2\sqrt{n}}}\right)} \sup_{Y\in Q_{\frac{\delta(X)}{2\sqrt{n}}}(X)} |\mathbf{b}|^2 \, dX \\
&\le C \int_{\triangle_s(P)} \mathcal{E}^2_{\mathbf{b}^*,r}(P) \, d\omega_0(P) \\
&\le C C_0^2 \omega_0(\triangle_s(P)) \le \varepsilon_0 \omega_0(\triangle_s(P))
\end{aligned}$$

if C_0 is small enough. Thus, $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy the hypotheses of Theorem 5.1 in Φ and hence $\omega_1 = \omega^{X_0}_{\mathcal{L}_1,\Phi} \in B_2(d\omega_0, \partial\Phi)$. By property (2) in Lemma 3.6 we have that for $E \subset \triangle_{2r_0}(Q)$,

$$\omega_0(E) \approx \frac{\omega_{\mathcal{L}_0,\Omega}(E)}{\omega_{\mathcal{L}_0,\Omega}(\triangle)}.$$

This together with the hypothesis from Theorem 4.1 that $\omega_{\mathcal{L}_0,\Omega} \in A_\infty$, implies $\omega_0|_{\triangle_{3r_0}(Q)} \in A_\infty$. From $\omega_1 \in B_2(d\omega_0, \partial\Phi)$ we conclude that $\omega_1|_{\triangle_{2r_0}(Q)} \in A_\infty$. Hence, for some constants $0 < \alpha_0, c_0$, we have

$$(\omega_1(F))^{\alpha_0} \approx \left(\frac{\omega_1(F)}{\omega_1(\triangle_{r_0}(Q))}\right)^{\alpha_0} \ge c_0 \frac{\sigma(F)}{\sigma(\triangle_{r_0}(Q))} \ge \frac{c_0}{2}, \tag{5.5}$$

where we applied property (1) of Lemma 3.6 and we used that $\sigma(F) > \frac{1}{2}\sigma(\triangle_{r0}(Q))$.

Now, let ν be the harmonic measure of $\mathcal{L}_1$ in Ω_F. By Lemma 3.8 we have that for some $0 < \theta < 1$,

$$\frac{\omega_1(F)}{\omega_1(\triangle_{r_0}(Q))} < \nu(F)^\theta. \tag{5.6}$$

On the other hand, since $\mathbf{b}^*$ coincides with $\mathbf{b}$ in Ω_F, any solution u of $\tilde{\mathcal{L}}_1 u = 0$ in Φ is a solution of $\tilde{\mathcal{L}}_1 u = 0$ in $\Omega_F \subset \Phi$. The boundary maximum principle implies that for all $F \subset \triangle_{r_0}(Q)$, $\nu(F) \le \omega_1^*(F)$, where $\omega_1^* = \omega_{\tilde{\mathcal{L}}_1,\Phi}$. From (5.6) and (5.5) then we obtain

$$\omega_1^*(F) > c_1 > 0.$$

By property (2) in Lemma 3.6 and the maximum principle, we have

$$\frac{\tilde{\omega}_1(F)}{\tilde{\omega}_1(\triangle_{r_0}(Q))} \ge \omega_1^*(F) > c_1 > 0.$$

Therefore, whenever $\frac{\sigma(F)}{\sigma(\triangle_{r_0}(Q))} > \frac{1}{2}$ it follow that $\frac{\tilde{\omega}_1(F)}{\tilde{\omega}_1(\triangle_{r_0}(Q))} > c_1$. This shows that (5.2) holds.

5.1. Proof of Theorem 5.1. The proof of this results closely follows the steps in [**17**]. We will sketch the main steps and refer the reader to [**17**] for the technical details omitted here. First, by standard arguments the problem is reduced to treating the case in which Ω is the unit ball $B = B_1(0)$ (this is justified as far as the methods are *preserved* under bi-Lipschitz transformation). For simplicity, we will write $\omega_0 = \omega_{\tilde{\mathcal{L}}_{N,0}}$ and $\omega_1 = \omega_{\tilde{\mathcal{L}}_{N,1}}$. To see that $d\omega_1 \in B_2(d\omega_0)$ it is equivalent to prove that if u_1 is a solution of the Dirichlet problem

$$\begin{cases} \tilde{\mathcal{L}}_{N,1} u_1 = 0 & \text{in } B \\ u_1 = g & \text{on } \partial B, \end{cases}$$

where g is continuous in ∂B, then

$$\|N u_1\|_{L^2(\partial B, d\omega_0)} \leq C \|g\|_{L^2(\partial B, d\omega_0)}, \tag{5.7}$$

where Nu is the nontangential maximal function (with some fixed aperture $\alpha > 0$) of u. We let u_0 be the solution to

$$\begin{cases} \tilde{\mathcal{L}}_{N,0} u_0 = 0 & \text{in } B \\ u_0 = g & \text{on } \partial B. \end{cases}$$

Then $u_1 - u_0 = 0$ on ∂B and we have the representation

$$u_1(X) = u_0(X) - \int_B \mathcal{G}_0(X,Y)\, \tilde{\mathcal{L}}_{N,0} u_1 \, dY = u_0(X) - F(X). \tag{5.8}$$

Then, (5.7) follows as in [**17**] from the following two lemmas.

Lemma 5.2. *Let $\mathcal{G}(X,Y)$ denote the Green's function for $\tilde{\mathcal{L}}_{N,0}$ in $B_{10}(0)$ and let $\mathcal{G}(Y) = \mathcal{G}(\bar{X}, Y)$ where $\bar{X}$ is some fixed point in $\mathbb{R}^{n+1}$ such that $|\bar{X}| = 5$. For $Y \in \mathcal{B}$ let $B_0(Y)$ and $B(Y)$ denote the Euclidean balls centered at Y of radii $\frac{\delta(Y)}{6}$ and $\frac{\delta(Y)}{2}$ respectively (in this case $\delta(Y) = \operatorname{dist}(Y, \partial B) = 1 - |Y|$). Under the hypotheses of Theorem 5.1, we have that F as in* (5.8) *satisfies*

$$N^0 F(Q) = \sup_{X \in \Gamma(Q)} \left\{ \int_{B_0(X)} F^2(Y) \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))} \, dY \right\}^{\frac{1}{2}} \leq C \varepsilon_0 M_{\omega_0}(S u_1)(Q),$$

where

$$M_{\omega_0} f(Q) = \sup_{r>0} \frac{1}{\omega_0(\triangle_r(Q))} \int_{\triangle_r(Q)} |f(P)| \, d\omega_0$$

is the Hardy-Littlewood maximal function of f with respect to the measure ω_0, and $S u_1$ is the area function of u_1.

Lemma 5.3. *Under the hypotheses of Lemma 5.2,*

$$\int_{\partial B} S^2 u_1 \, d\omega_0 \le C \int_{\partial B} \left(N u_1\right)^2 \, d\omega_0.$$

Indeed, by Lemma 2.21 in [**17**] it follows that

$$\begin{aligned}\int_{\partial B} \left(N u_1\right)^2 \, d\omega_0 &\le C \int_{\partial B} \left(N^0 u_1\right)^2 \, d\omega_0 \\ &\le C \int_{\partial B} \left(N^0 u_0\right)^2 \, d\omega_0 + C \int_{\partial B} \left(N^0 F\right)^2 \, d\omega_0\end{aligned}$$

where $N^0 u_1$ is as in Lemma 5.2. Given that Lemmas 5.2 and 5.3 hold, we have

$$\begin{aligned}\int_{\partial B} \left(N u_1\right)^2 \, d\omega_0 &\le C \int_{\partial B} \left(N u_0\right)^2 \, d\omega_0 + C\varepsilon_0 \int_{\partial B} M_{\omega_0} \left(S u_1\right)^2 (Q) \, d\omega_0 \\ &\le C \int_{\partial B} g^2 \, d\omega_0 + C\varepsilon_0 \int_{\partial B} \left(S u_1\right)^2 (Q) \, d\omega_0 \\ &\le C \int_{\partial B} g^2 \, d\omega_0 + C\varepsilon_0 \int_{\partial B} \left(N u_1\right)^2 (Q) \, d\omega_0\end{aligned}$$

and the last term on the right can be absorbed into the left if ε_0 is small enough. This proves (5.7) and hence Theorem 5.1.

We will only write in some detail the proof of Lemma 5.2. Given the big overlap with the methods in [**17**] this will suffice to indicate the proof of Lemma 5.3, which is very similar to the proof of Lemma 3.3 in [**17**].

Proof of Lemma 5.2: Fix $Q_0 \in \partial B$ and $X_0 \in \Gamma(Q_0)$. Let $B_0 = B_{\frac{\delta_0}{6}}(X_0)$ and $KB_0 = B_{\frac{K\delta_0}{6}}(X_0)$ where $\delta_0 = \delta(X_0)$ and $K > 0$. Let $\tilde{\mathcal{G}}(X,Y)$ be the Green's function for $\tilde{\mathcal{L}}_{N,0}$ on $3B_0$, set

$$\begin{aligned}F_1(X) &= \int_{2B_0} \tilde{\mathcal{G}}(X,Y)\, \tilde{\mathcal{L}}_{N,0} u_1(Y) \, dY, \\ F_2(X) &= \int_{2B_0} \left[\mathcal{G}_0(X,Y) - \tilde{\mathcal{G}}(X,Y)\right] \tilde{\mathcal{L}}_{N,0} u_1(Y) \, dY, \\ F_3(X) &= \int_{B \setminus 2B_0} \mathcal{G}_0(X,Y)\, \tilde{\mathcal{L}}_{N,0} u_1(Y) \, dY.\end{aligned}$$

So that F in (5.8) is given by $F(X) = F_1(X) + F_2(X) + F_3(X)$, and proving Lemma 5.2 is reduced to proving that

$$\int_{B_0} F_i^2(Y) \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))} \, dY \leq C\varepsilon_0^2 M_{\omega_0}^2 (Su_1)(Q_0), \quad \text{for } i = 1, 2, 3.$$

We will only prove this in some detail for $i = 1$. Even though $i = 1$ is allegedly the simplest case of the three, its proof captures the significant differences with the proof of Lemma 3.2 in [**17**] (the analog to Lemma 5.2 here), so the other two cases follow in a similar manner as in [**17**].

Let $\beta(X) = \sup_{|Z-X| \leq \frac{\delta(X)}{2}} |\mathbf{b}(Z)|$, then, given $Y \in B_0$, we have

$$\begin{aligned} |\mathbf{b}(Y)| &\leq \frac{C}{|3B_0(Y)|} \int_{3B_0(Y)} \beta(X) \, dX \\ &\leq \frac{C}{|3B_0(Y)|} \left\{ \int_{3B_0(Y)} \beta(X)^2 \frac{\mathcal{G}(X)}{\mathcal{G}(B(X))} \, dX \right\}^{\frac{1}{2}} \\ &\qquad \times \left\{ \int_{3B_0(Y)} \frac{\mathcal{G}(B(X))}{\mathcal{G}(X)} \, dX \right\}^{\frac{1}{2}}. \end{aligned} \tag{5.9}$$

Using (5.4) on the right side of (5.9), applying the Carleson measure property of $\mathcal{G}_0(X)\beta(X)^2$ and the doubling property of ω_0, we obtain

$$\begin{aligned} |\mathbf{b}(Y)| &\leq \frac{C}{|3B_0(Y)|} \left\{ \frac{1}{\omega_0(\triangle_Y)} \int_{3B_0(X)} \beta(X)^2 \frac{\mathcal{G}_0(X)}{\delta(X)^2} \, dX \right\}^{\frac{1}{2}} \\ &\qquad \times \left\{ \int_{3B_0(Y)} \frac{\mathcal{G}(B(X))}{\mathcal{G}(X)} \, dX \right\}^{\frac{1}{2}} \\ &\leq \frac{C}{|3B_0(Y)|} \left\{ \frac{1}{\omega_0(\triangle_Y)} \int_{3B_0(Y)} \beta(X)^2 \frac{\mathcal{G}_0(X)}{\delta(X)^2} \, dX \right\}^{\frac{1}{2}} \\ &\qquad \times \left\{ \int_{3B_0(Y)} \frac{\mathcal{G}(B(X))}{\mathcal{G}(X)} \, dX \right\}^{\frac{1}{2}} \\ &\leq \frac{C\varepsilon_0}{|3B_0(Y)| \, \delta_0} \left\{ \int_{3B_0(Y)} \frac{\mathcal{G}(B(X))}{\mathcal{G}(X)} \, dX \right\}^{\frac{1}{2}} \leq \frac{C\varepsilon_0}{\delta_0}, \end{aligned} \tag{5.10}$$

where the last inequality follows as (3.6) in [**17**]. Note that F_1 satisfies $\tilde{\mathcal{L}}_{N,0}F_1 = \chi(2B_0)\,\tilde{\mathcal{L}}_{N,0}u_1$ in $3B_0$, where $\chi(2B_0)$ denotes the characteristic function of $2B_0$. Hence, from the (weighted) a priori estimates for solutions (Theorem 2.5 in [**17**]),

$$\left\{\int_{3B_0} |\nabla F_1(Y)|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \leq C\delta_0 \left\{\int_{3B_0} \left|\tilde{\mathcal{L}}_{N,0}F_1(Y)\right|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}}. \tag{5.11}$$

On the other hand, by the weighted Poincaré inequality (Theorem 1.2 in [**6**]),

$$\left\{\int_{3B_0} F_1^2(Y)\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \leq C\delta_0 \left\{\int_{3B_0} |\nabla F_1(Y)|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}}.$$

Combining this with (5.11), using

$$\tilde{\mathcal{L}}_{N,0}F_1 = \chi(2B_0)\left\{\tilde{\mathcal{L}}_{N,0} - \tilde{\mathcal{L}}_{N,1}\right\}u_1 = \chi(2B_0)\,\mathbf{b}\cdot\nabla u_1,$$

and (5.10), we get

$$\begin{aligned}
&\left\{\int_{B_0} F_1^2(Y)\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \\
&\quad\leq C\delta_0^2 \left\{\int_{3B_0} \left|\tilde{\mathcal{L}}_{N,0}F_1(Y)\right|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \\
&\quad\leq C \left\{\int_{2B_0} \delta(Y)^4\,|\mathbf{b}|^2\,|\nabla u_1|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \\
&\quad\leq C\varepsilon_0 \left\{\int_{2B_0} \delta(Y)^2\,|\nabla u_1|^2 \frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \\
&\quad\leq C\varepsilon_0 S u_1(Q_0).
\end{aligned} \tag{5.12}$$

The rest of the proof proceeds as in [**17**], to obtain

$$\left\{\int_{B_0} F_2^2(Y)\,\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \leq C\varepsilon_0 Su_1(Q_0) \qquad \text{and} \tag{5.13}$$

$$\left\{\int_{B_0} F_3^2(Y)\,\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY\right\}^{\frac{1}{2}} \leq C\varepsilon_0 M_{\omega_0}(Su_1)(Q_0) \tag{5.14}$$

respectively. Since $F(Y) = F_1(Y) + F_2(Y) + F_3(Y)$, we have

$$\int_{B_0(X_0)} F^2(Y)\,\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY \leq C\sum_{i=1}^{3}\int_{B_0(X_0)} F_i^2(Y)\,\frac{\mathcal{G}(Y)}{\mathcal{G}(B(Y))}\,dY.$$

Lemma 5.2 then follows from (5.12)–(5.14) by taking supremum over all $X_0 \in \Gamma(Q_0)$. □

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