ALGEBRAIC DEGREES FOR ITERATES OF MEROMORPHIC SELF-MAPS OF $\mathbb{P}^k$

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Abstract

We first introduce the class of quasi-algebraically stable meromorphic maps of $\mathbb{P}^k$. This class is strictly larger than that of algebraically stable meromorphic self-maps of $\mathbb{P}^k$. Then we prove that all maps in the new class enjoy a recurrent property. In particular, the algebraic degrees for iterates of these maps can be computed and their first dynamical degrees are always algebraic integers.

1. Introduction

Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a meromorphic self-map. It can be written $f := [F] := [F_0 : \cdots : F_k]$ in homogeneous coordinates where the F_j's are homogeneous polynomials in the $k+1$ variables $z_0, \ldots, z_k$ of the same degree d with no nontrivial common factor. The polynomial F will be called *a lifting* of f in $\mathbb{C}^{k+1}$. The number $\mathrm{d}(f) := d$ will be called *the algebraic degree* of f. Moreover f is said to be *dominating* if it is generically of maximal rank k, in other words, its jacobian determinant does not vanish identically (in any local chart). The *indeterminacy locus* $\mathcal{I}(f)$ of f is the set of all points of $\mathbb{P}^k$ where f is not holomorphic, or equivalently the common zero set of $k + 1$ component polynomials $F_0, \ldots, F_k$. Observe that $\mathcal{I}(f)$ is a subvariety of codimension at least 2. From now on, we always consider dominating meromorphic self-maps f of $\mathbb{P}^k$ with $k \geq 2$. For such a map f, its *first dynamical degree* $\lambda_1(f)$ is given by

$$(1.1) \qquad \lambda_1(f) := \lim_{n \to \infty} \mathrm{d}(f^n)^{\frac{1}{n}}.$$

Computing $\lambda_1(f)$ and other dynamical degrees associated to f (see, for example, [5] for a definition of the later degrees) is a fundamental problem in Complex Dynamics. Indeed, this is a necessary step to determine the topological entropy of f (see [5], [10]).

Recall the following definition (see [8], [9], [11]).

Definition 1. A meromorphic self-map $f\colon \mathbb{P}^k \to \mathbb{P}^k$ is said to be algebraically stable (or AS for short) if there is no hypersurface of $\mathbb{P}^k$ which would be sent, by some iterate f^N, to $\mathcal{I}(f)$.

In other words, f is AS if and only if $\mathrm{d}(f^n) = \mathrm{d}(f)^n$, $n \in \mathbb{N}$.

For all AS maps f with $\mathrm{d}(f) > 1$, Sibony proves in [11] that the following limit in the sense of current

$$\lim_{n\to\infty} \frac{(f^n)^*\omega}{\mathrm{d}(f^n)}$$

exists, where ω denotes the Fubiny-Study Kähler form on $\mathbb{P}^k$ so normalized that $\int_{\mathbb{P}^k} \omega^k = 1$ (see Section 2 below). This limit is called *the Green current* associated to f. The Green current contains many important dynamical informations of the corresponding map. We address the reader to the recent works of Sibony [11], Dinh-Sibony [3] [4], Fornæss-Sibony [8], [9], and Favre [6] for further explanations.

However, the situation becomes much harder in the case of non AS maps. In general, for a non AS map f, $(f^m)^* \circ (f^n)^* \neq (f^{m+n})^*$ when these operators act on currents. One of the first works in this direction is the article of Bonifant-Fornæss [2] where some special non AS maps are thoroughly studied. In her thesis [1] Bonifant constructs an appropriate Green current for these maps and then writes down the functional equation. Favre and Jonsson [7] have studied the case of polynomial maps in $\mathbb{C}^2$. These works show that there is a deep connection between the construction of a good invariant current for a non AS meromorphic self-map f and the sequence $(\mathrm{d}(f^n))_{n=1}^\infty$ of algebraic degrees for iterates of f. Moreover, this sequence is, in general, very complicated.

The purpose of this paper is to study the sequence $(\mathrm{d}(f^n))_{n=1}^\infty$ for all f in a new class of meromorphic self-maps of $\mathbb{P}^k$: the class of *quasi-algebraically stable self-maps* (or QAS for short). This class contains strictly that of all AS self-maps. The QAS self-maps share a *recurrent property* with the AS ones. Let us explain this more explicitly. For an AS self-map f we may define a sequence of polynomial maps $(F_n)_{n=0}^\infty\colon \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ such that F_n is a lifting of f^n, $n \geq 0$, in the

following recurrent way:

$$F_n := F_1 \circ F_{n-1}, \quad n \geq 1,$$

where F_1, F_0 are arbitrarily fixed liftings of f, $f^0 := \mathrm{Id}$ respectively. Following the same pattern, the *recurrent law* for a QAS self-map f which is not AS may be stated as follows:

$$F_n := \frac{F_1 \circ F_{n-1}}{H_0 \circ F_{n-n_0-1}}$$

for all $n > n_0$. Here $n_0 \geq 1$ is an integer and H_0 is a homogeneous polynomial. The recurrent phenomenon happens when the orbits of the hypersurfaces which are sent to $\mathcal{I}(f)$ by some iterate f^N (see Definition 1 above) are, in some sense, not so complicated. That is the main point of our observation.

This paper is organized as follows.

We begin Section 2 by collecting some background and introducing some notation. This preparatory is necessary for us to state the results afterwards.

Section 3 starts with the definition of quasi-algebraically stable meromorphic self-maps. Then we provide some examples illustrating this definition.

The last section is devoted to the formulation and the proof of the main theorem. Some examples are also analyzed in the light of this theorem. Finally, we conclude the paper with some remarks.

2. Background and notation

Let f be a dominating meromorphic self-map of $\mathbb{P}^k$, $\Gamma(f)$ the graph of f in $\mathbb{P}^k \times \mathbb{P}^k$ and π_1, π_2 the natural projections of $\Gamma(f) \subset \mathbb{P}^k \times \mathbb{P}^k$ onto its factors. Then $\mathcal{I}(f)$ is exactly the set of points $z \in \mathbb{P}^k$ where π_1 does not admit a local inverse. Let A be a (not necessarily irreducible)

analytic subset of $\mathbb{P}^k$. We define the following analytic sets

$$f(A) := \overline{f|_{\mathbb{P}^k \setminus \mathcal{I}(f)}(A)} = \overline{\pi_2(\pi_1^{-1}(A \setminus \mathcal{I}(f)))} \quad \text{and}$$

$$f^{-1}(A) := \pi_1(\pi_2^{-1}(A)).$$

If g is another dominating meromorphic self-map of $\mathbb{P}^k$, the graph $\Gamma(g \circ f)$ of the composite map $g \circ f$ is the closure of the set

$$\left\{ (z, g(f(z)) \in \mathbb{P}^k \times \mathbb{P}^k : z \notin \mathcal{I}(f) \text{ and } f(z) \notin \mathcal{I}(g) \right\}.$$

Consequently, we can define the sequence $(f^n)_{n=1}^{\infty}$ of n-iterates of f by the induction formula

$$\Gamma(f^n) := \Gamma(f^{n-1} \circ f), \quad n \geq 2.$$

In the sequel, $\mathrm{codim}(A)$ denotes the codimension of A. Moreover, we recall that a *hypersurface* is an analytic set of pure codimension 1 in $\mathbb{P}^k$. Let $\mathrm{Crit}(f)$ denote the critical set of f (i.e. the hypersurface defined outside $\mathcal{I}(f)$ by the zero set of the jacobian of f in any local coordinates). The following result is very useful.

Lemma 2.1. *Let f be as above. Then, for every irreducible analytic set $A \subset \mathbb{P}^k$ such that $A \not\subset \mathcal{I}(f)$, $f(A)$ is also an irreducible analytic set.*

Proof: Suppose in order to get a contradiction that $f(A) = B_1 \cup B_2$, where B_1, B_2 are analytic sets in $\mathbb{P}^k$, distinct from $f(A)$. It follows that $(A \cap f^{-1}(B_1)) \cup (A \cap f^{-1}(B_2)) \subset A$ and the two sets $A \cap f^{-1}(B_1)$, $A \cap f^{-1}(B_2)$ are distinct analytic components of A. We therefore get the desired contradiction. This finishes the proof. □

We denote by $\mathcal{C}_1^+(\mathbb{P}^k)$ the set of positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. A current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$ can be written locally as $T = \mathrm{dd}^c u$ for some plurisubharmonic function u (which is called a *local potential* of T). The mass of T is defined by $\|T\| := \int_{\mathbb{P}^k} T \wedge \omega^{k-1}$. Fix a point $a \in \mathbb{P}^k$ and local coordinates sending a to the origin in $\mathbb{C}^k$. Choose a local plurisubharmonic potential u for T defined around 0 in these coordinates. We can define the *Lelong number* of u at 0 as follows

$$\nu(u, 0) := \max \left\{ c \geq 0 : u(z) \leq c \log|z| + \mathcal{O}(1) \right\}$$

which is a finite nonnegative real number. We then set $\nu(T, a) := \nu(u, 0)$, which does not depend on any choice we made.

For a current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$, we use local potentials to define the induced pull-back $f^*T \in \mathcal{C}_1^+(\mathbb{P}^k)$. More precisely, for any $z \in \mathbb{P}^k \setminus \mathcal{I}(f)$, T has a local potential u in a neighborhood of $f(z)$, and we define $\widetilde{f^*T} := \mathrm{dd}^c(u \circ f)$ in a neighborhood of z. This yields a well-defined, positive

closed $(1,1)$-current on the set $\mathbb{P}^k \setminus \mathcal{I}(f)$. Since $\mathrm{codim}(\mathcal{I}(f)) > 1$, we can extend $\widetilde{f^*T}$ to a current $f^*T \in \mathcal{C}_1^+(\mathbb{P}^k)$ by assigning zero mass on the set $\mathcal{I}(f)$ to the coefficients measures of $\widetilde{f^*T}$.

Any hypersurface $\mathcal{H}$ of $\mathbb{P}^k$ defines a current of integration $[\mathcal{H}] \in \mathcal{C}_1^+(\mathbb{P}^k)$, and $\|[\mathcal{H}]\| = \deg(H)$, where $H : \mathbb{C}^{k+1} \to \mathbb{C}$ is any homogeneous polynomial defining $\mathcal{H}$. Finally, for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$, it holds that

$$(2.1) \qquad \|f^*T\| = \mathrm{d}(f) \cdot \|T\|.$$

For further information on this matter, the reader is invited to consult the works [**8**] and [**11**].

3. Quasi-algebraically stable meromorphic self-maps

In [**9**] Fornæss and Sibony establish the following definition.

Definition 3.1. A hypersurface $\mathcal{H} \subset \mathbb{P}^k$ is said to be a *degree lowering hypersurface* of f if, for some (smallest) $n \geq 1$, $f^n(\mathcal{H}) \subset \mathcal{I}(f)$. The integer n is then called the *height* of $\mathcal{H}$.

The following proposition gives us the structure of a non AS self-map.

Proposition 3.2. *Let f be a meromorphic self-map of $\mathbb{P}^k$. Then there are exactly an integer $M \geq 0$, M degree lowering hypersurfaces $\mathcal{H}_j$ with height n_j, $j = 1, \ldots, M$, satisfying the following properties:*

 (i) *all the numbers n_j, $j = 1, \ldots, M$, are pairwise different;*

 (ii) *$\mathrm{codim}\,(f^m(\mathcal{H}_j)) > 1$ for $m = 1, \ldots, n_j$, and $j = 1, \ldots, M$;*

 (iii) *for any degree lowering irreducible hypersurface $\mathcal{H}$ of f, there are integers $n \geq 0$ and $1 \leq j \leq M$ such that $f^n(\mathcal{H})$ is a hypersurface and $f^n(\mathcal{H}) \subset \mathcal{H}_j$.*

In particular, f is AS if and only if $M = 0$.

Proof: First, we give the construction of M and $\mathcal{H}_j$, n_j, $j = 1, \ldots, M$. To this end observe that every hypersurface $\mathcal{H}$ satisfying $\mathrm{codim}\,(f(\mathcal{H})) > 1$ should be contained in $\mathrm{Crit}(f)$. Therefore, one takes the family $\mathcal{F}$ of all degree lowering irreducible components $\mathcal{H}$ of $\mathrm{Crit}(f)$ such that $\mathrm{codim}\,(f^m(\mathcal{H})) > 1$ for $1 \leq m \leq n$, where n is the height of $\mathcal{H}$. Let $1 \leq n_1 < \cdots < n_M$ be all the heights of elements in $\mathcal{F}$ (it is easy to see that M is finite). Let $\mathcal{H}_j$ be the (finite) union of all elements in $\mathcal{F}$ with the same height n_j. Then properties (i) and (ii) are satisfied.

To prove (iii) let $\mathcal{H}$ be a degree lowering irreducible hypersurface with height h. In virtue of Lemma 2.1, let n be the greatest integer such that

$0 \leq n < h$ and $f^n(\mathcal{H})$ is a hypersurface. The choice of n implies that $\text{codim}\,(f^m(\mathcal{H})) > 1$, $m = n+1, \ldots, h$. Consequently, in virtue of the above construction, we deduce that $f^n(\mathcal{H}) \subset \mathcal{H}_j$ for some $1 \leq j \leq M$. This proves (iii).

Since the uniqueness of M and $\mathcal{H}_j$, n_j, $j = 1, \ldots, M$, is almost obvious, it is therefore left to the reader. This completes the proof. $\qquad\square$

Definition 3.3. Under the hypothesis and the notation of Proposition 3.2, for every $j = 1, \ldots, M$, $\mathcal{H}_j$ is called the primitive degree lowering hypersurface of f with the height n_j.

We are now able to define the class of quasi-algebraically stable self-maps.

Definition 3.4. A meromorphic self-map f of $\mathbb{P}^k$ is said to be quasi-algebraically stable (or QAS for short) if either it is AS or it satisfies the following properties:
  (i) there is only one primitive degree lowering hypersurface (let $\mathcal{H}_0$ be this hypersurface and let n_0 be its height);
 (ii) for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$ and every $m = 1, \ldots, n_0$, $f^m(\mathcal{H}) \not\subset \mathcal{H}_0$;
(iii) for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$, one of the following two conditions holds
    $(\text{iii})_1$ $f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all $m \geq n_0 + 1$,
    $(\text{iii})_2$ there is an $m_0 \geq n_0$ such that $f^{m_0+1}(\mathcal{H})$ is a hypersurface and $f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all m verifying $n_0 + 1 \leq m \leq m_0$.

It is worthy to remark that Proposition 3.2 allows us to check if a map is QAS.

We conclude this section by studying some examples.

Example 3.5. Consider the following meromorphic self-map of $\mathbb{P}^2$:

$$(3.1) \quad f\,([z:w:t]) := \left[2tz - \left(z^2 + w^2\right) : 2tw - \left(z^2 + w^2\right) : 2t^2 - \left(z^2 + w^2\right)\right].$$

It can be checked that $\mathcal{I}(f) = \{[1:1:1], [1:i:0], [1:-i:0]\}$, and $\text{Crit}(f) = \{t(2t^2 + w^2 + z^2 - 2zt - 2wt) = 0\}$. Moreover we have

$$f\,(\{t = 0\}) = [1:1:1] \in \mathcal{I}(f),$$

$$f\left(\left\{2t^2 + w^2 + z^2 - 2zt - 2wt = 0\right\}\right) = \{t - z - w = 0\}.$$

Therefore, $\{t = 0\}$ is the unique primitive degree lowering hypersurface and its height is 1. Since $[1:1:1] \notin \{t = 0\}$ and $f^2(\{t = 0\})$ is a hypersurface, f is QAS.

Example 3.6. For all integers $d \geq 2$ and $m \geq 1$, the following map is given by Bonifant-Fornæss in [**2**]

$$f\left([z:w:t]\right) := \left[zt^{d-1} : \left(wt^{d-1} + z^d\right)\cos\frac{\pi}{m} - t^d\sin\frac{\pi}{m} : \right.$$

$$\left. \left(wt^{d-1} + z^d\right)\sin\frac{\pi}{m} + t^d\cos\frac{\pi}{m}\right].$$

It can be checked that $\mathcal{I}(f) = [0:1:0]$, $\mathrm{Crit}(f) = \{t = 0\}$, and $\{t=0\}$ is the only primitive degree lowering hypersurface of f. Moreover, its height is m. Since $f^n(\{t = 0\}) = \left[0 : \cos\frac{n\pi}{m} : \sin\frac{n\pi}{m}\right] \in \{t = 0\}$ for $n = 1, \ldots, m$, f is not a QAS according to Definition 3.4 (ii). However, f satisfies conditions (i) and (iii)$_2$ of this definition.

Example 3.7. Consider the following meromorphic self-map of $\mathbb{P}^2$:

$$(3.2) \quad f\left([z:w:t]\right) := \left[(z + w + t)^2(z^3 + w^3 + t^3)z^2 - 27z^3w^4 : \right.$$

$$(z + w + t)^2(z^3 + w^3 + t^3)w^2 - 27z^3w^4 :$$

$$\left. (z + w + t)^2(z^3 + w^3 + t^3)t^2 - 27z^3w^4\right].$$

It can be checked that

$$\mathcal{I}(f) = [1:1:1] \cup \{[z:w:t],\ z + w + t = zw = 0\}$$

$$\cup \left\{[z:w:t],\ z^3 + w^3 + t^3 = zw = 0\right\},$$

and

$$f\left(\{z + w + t = 0\}\right) = f\left(\{z^3 + w^3 + t^3 = 0\}\right) = [1:1:1] \in \mathcal{I}(f).$$

Let G be an irreducible homogeneous polynomial in $\mathbb{C}^3$ such that $f(\mathcal{G})$ is a point $[a:b:c] \in \mathbb{P}^2$, where $\mathcal{G}$ is the hypersurface $\{G = 0\}$ in $\mathbb{P}^2$. We deduce from (3.2) and the equality $f(\mathcal{G}) = [a:b:c]$ that G divides both polynomials $(z + w + t)^2(z^3 + w^3 + t^3)(bz^2 - aw^2) - 27z^3w^4(b - a)$ and $(z + w + t)^2(z^3 + w^3 + t^3)(cz^2 - at^2) - 27z^3w^4(c - a)$. Hence, G divides the polynomial

$$(z + w + t)^2(z^3 + w^3 + t^3)\Big((c - a)(bz^2 - aw^2) - (b - a)(cz^2 - at^2)\Big).$$

Now it is not difficult to see that there is no hypersurface $\mathcal{G}$ which is not contained in $\mathcal{H}_0 := \left\{(z + w + t)^2(z^3 + w^3 + t^3) = 0\right\}$ and which satisfies $\mathrm{codim}(f(\mathcal{G})) > 1$. In other words, $\mathcal{H}_0$ is the unique primitive degree lowering hypersurface and its height is 1. Since $[1:1:1] \notin \mathcal{H}_0$, and $f^2(\{z + w + t = 0\})$, $f^2(\{z^3 + w^3 + t^3 = 0\})$ are hypersurfaces, it follows that f is QAS.

4. The main result

Now we are ready to formulate the main result of this article.

The Main Theorem. *Let f be a QAS meromorphic self-map of $\mathbb{P}^k$ which is not AS. Let $\mathcal{H}_0$ be its unique primitive degree lowering hypersurface and let n_0 be its height. We define a sequence $\{F_n : n \geq 1\}$ of maps $\mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ as follows:*

$F_1, \ldots, F_{n_0}, F_{n_0+1}$ are arbitrarily fixed liftings of $f^1(\equiv f), \ldots, f^{n_0}, f^{n_0+1}$ respectively. Let H_0 be the unique homogeneous polynomial which verifies the equality

$$(4.1) \qquad\qquad F_1 \circ F_{n_0} = H_0 \cdot F_{n_0+1}.$$

Next we define F_n for all $n > n_0 + 1$ as follows:

$$(4.2) \qquad\qquad F_n := \frac{F_1 \circ F_{n-1}}{H_0 \circ F_{n-n_0-1}}.$$

Then $\mathcal{H}_0 = \{H_0(z) = 0\}$, and for any $n \geq 0$, F_n is a lifting of f^n. Moreover, for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$,
(4.3)

$$(f^n)^* T = \begin{cases} (f^{n-1})^*(f^*T), & n = 1, \ldots, n_0, \\ (f^{n-1})^*(f^*T) - \|T\| \cdot (f^{n-n_0-1})^*[\mathcal{H}_0], & n > n_0. \end{cases}$$

Prior to the proof of the theorem we need a preparatory result.

Lemma 4.1. *We keep the above hypothesis and notation. Let $m \in \mathbb{N}$, $m \geq 1$. Then, for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic (in the Zariski sense) complex hyperplane in $\mathbb{P}^k$, the supports of $(f^p)^*[\mathcal{H}_0]$ and $(f^m)^*(f^{n_0+1})^*T$ do not contain any component of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$ for all p verifying $\max\{0, m - n_0 + 1\} \leq p \leq m - 1$.*

Proof of Lemma 4.1: To prove the assertion for $(f^p)^*[\mathcal{H}_0]$, fix an arbitrary irreducible component $\mathcal{H}$ of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$, and an integer $p : \max\{0, m - n_0 + 1\} \leq p \leq m - 1$. Suppose in order to reach a contradiction that $\mathcal{H} \subset (f^p)^{-1}(\mathcal{H}_0)$. Putting $\mathcal{G} := f^p(\mathcal{H})$, the later inclusion implies that

$$(4.4) \qquad\qquad \mathcal{G} \subset \mathcal{H}_0.$$

In virtue of Lemma 2.1, there are two cases to consider.

Case 1: $\mathcal{G}$ is an irreducible hypersurface.

In this case it follows from the inclusion $\mathcal{H} \subset (f^m)^{-1}(\mathcal{H}_0)$ and the equality $\mathcal{G} = f^p(\mathcal{H})$ that $f^{m-p}(\mathcal{G}) = f^m(\mathcal{H}) \subset \mathcal{H}_0$. Recall that $m -$

$p \leq n_0$, $\mathcal{G}$ is an irreducible component of $\mathcal{H}_0$ (by (4.4)) and f is QAS. Consequently, if follows from Definition 3.4 (ii) that $f^{m-p}(\mathcal{G}) \not\subset \mathcal{H}_0$, which contradicts the inclusion $f^{m-p}(\mathcal{G}) \subset \mathcal{H}_0$. Hence, this case cannot happen.

Case 2: $\mathcal{G}$ is an irreducible analytic set of codimension strictly greater than 1.

Let q be the greatest integer such that $0 \leq q < p$ and $f^q(\mathcal{H})$ is a hypersurface. Consider three subcases.

Case 2a: there is a smallest integer r such that $q < r < p$ and $f^r(\mathcal{H}) \subset \mathcal{I}(f)$.

In virtue of the choice of q, r, of Definition 3.4 and Lemma 2.1, we see that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$. Next, we will analyze the orbit of $f^q(\mathcal{H})$ under iterations of f.

Using the choice of q, we see that the following analytic sets $f^{q+1}(\mathcal{H}), \ldots, f^{p-1}(\mathcal{H})$ are of codimension strictly greater than 1. Recall from the assumption of Case 2 and Case 2a that $f^p(\mathcal{H})(= \mathcal{G})$ is also of codimension strictly greater than 1 and $f^r(\mathcal{H}) \subset \mathcal{I}(f)$. Consequently, using Definition 3.4 (iii) we see that none of the following analytic sets $f^{q+1}(\mathcal{H}), \ldots, f^{r-1}(\mathcal{H})$ and $f^{r+1}(\mathcal{H}), \ldots, f^p(\mathcal{H})$ is contained in $\mathcal{I}(f)$. Hence,

$$f^{p+1}(\mathcal{H}) = f(f^p(\mathcal{H})) = f(\mathcal{G}) \subset f(\mathcal{H}_0),$$

where the inclusion follows from (4.4). Since f is QAS, we deduce from the later inclusion (i.e. $f^{p+1}(\mathcal{H}) \subset f(\mathcal{H}_0)$) and Definition 3.4 (ii) that there is a smallest integer s such that $p < s \leq p + n_0$, and $\mathrm{codim}(f^{p+1}(\mathcal{H})), \ldots, \mathrm{codim}(f^s(\mathcal{H})) > 1$, and none of the following sets $f^t(\mathcal{H})$ $(p+1 \leq t \leq s-1)$ is contained in $\mathcal{I}(f)$, but $f^s(\mathcal{H}) \subset \mathcal{I}(f)$.

In summary, we have shown that for some $q < r < p < s$,

- $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$;
- all the analytic sets $f^t(\mathcal{H})$ $(q+1 \leq t \leq r-1)$ are of codimension strictly greater than 1, and none of them is contained in $\mathcal{I}(f)$;
- $f^r(\mathcal{H}) \subset \mathcal{I}(f)$;
- all the analytic sets $f^t(\mathcal{H})$ $(r+1 \leq t \leq s-1)$ are of codimension strictly greater than 1, and none of them is contained in $\mathcal{I}(f)$;
- $f^s(\mathcal{H}) \subset \mathcal{I}(f)$.

Since $f^t(\mathcal{H}) = f^{t-q}(f^q(\mathcal{H}))$ for $q+1 \leq t \leq s$, Definition 3.4 (iii) says that $f^q(\mathcal{H})$ cannot satisfy all the above items. Hence, Case 2a cannot happen.

Case 2b: $f^r(\mathcal{H}) \not\subset \mathcal{I}(f)$, $r = q+1, \ldots, p-1$, but $f^p(\mathcal{H}) \subset \mathcal{I}(f)$.

Under this assumption and using the choice of q and Definition 3.4, we see that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$ and $p = q + n_0$. Therefore, it follows from (4.4) that $f^{n_0}(f^q(\mathcal{H})) = f^p(\mathcal{H}) \subset \mathcal{H}_0$. On the other hand, Definition 3.4 (ii) says that $f^{n_0}(f^q(\mathcal{H})) \not\subset \mathcal{H}_0$. Hence, Case 2b is impossible.

Case 2c: $f^r(\mathcal{H}) \not\subset \mathcal{I}(f)$, $r = q+1, \ldots, p$.

First we claim that there is a smallest integer s such that $s > q$ and $f^s(\mathcal{H}) \subset \mathcal{I}(f)$. Otherwise, using the fact that $f^q(\mathcal{H})$ is a hypersurface (by the choice of q), one would deduce that $f^{n+t}(\mathcal{H}) = f^n(f^t(\mathcal{H}))$ for all $t \geq q$ and $n \geq 0$. In particular, using (4.4) and the identity $\mathcal{G} = f^p(\mathcal{H})$ we would have

$$f^{p+n_0}(\mathcal{H}) = f^{n_0}(\mathcal{G}) \subset f^{n_0}(\mathcal{H}_0) \subset \mathcal{I}(f),$$

which is a contradiction. Hence, the above claim has been proved.

Using this claim, the assumption of Case 2c and the choice of q, we see that $p < s$, and all the analytic sets $f^t(\mathcal{H})$ ($q+1 \leq t \leq s-1$) are of codimension strictly greater than 1, and none of them is contained in $\mathcal{I}(f)$, but $f^s(\mathcal{H}) \subset \mathcal{I}(f)$. This implies that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$, and $s = q + n_0$. Since $f^{p-q}(f^q(\mathcal{H})) = f^p(\mathcal{H}) \subset \mathcal{H}_0$ (by (4.4)) and $0 < p - q < s - q = n_0$, we obtain a contradiction with Definition 3.4 (ii). Hence, Case 2c cannot happen.

Hence, the proof of the assertion for $(f^p)^*[\mathcal{H}_0]$ is complete.

To prove the assertion for $(f^m)^*(f^{n_0+1})^*T$, fix an arbitrary irreducible component $\mathcal{H}$ of the hypersurface $(f^m)^{-1}(\mathcal{H}_0)$, and a current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane of $\mathbb{P}^k$. Suppose in order to reach a contradiction that $\nu\big((f^m)^*(f^{n_0+1})^*T, z\big) > 0$ for any generic point $z \in \mathcal{H}$. Putting $\mathcal{G} := f^m(\mathcal{H})$, the later inequality and the choice of $\mathcal{H}$ imply that

$$(4.5) \qquad\qquad \mathcal{G} \subset \mathcal{H}_0 \cap \mathcal{I}(f^{n_0+1}).$$

Let q be the greatest integer such that $0 \leq q \leq m$ and $f^q(\mathcal{H})$ is a hypersurface. Clearly, $q < m$ because of (4.5): $\mathrm{codim}(\mathcal{I}(f^{n_0+1})) > 1$. In virtue of the choice of q, and of Definition 3.4, and of the inclusion $\mathcal{G} = f^m(\mathcal{H}) \subset \mathcal{I}(f^{n_0+1})$ (see (4.5)), we conclude that $f^q(\mathcal{H})$ is an irreducible component of $\mathcal{H}_0$. Since $f^{m-q}(f^q(\mathcal{H})) = f^m(\mathcal{H}) \subset \mathcal{H}_0$ (see (4.5)), we obtain a contradiction with Definition 3.4 (ii)–(iii).

Hence, the proof of the assertion for $(f^m)^*(f^{n_0+1})^*T$ is finished. This completes the proof of the lemma. $\qquad\square$

Now we arrive at

Proof of the Main Theorem: The assertion $\mathcal{H}_0 = \{H_0(z) = 0\}$ follows immediately from (4.1) and the hypothesis on $\mathcal{H}_0$ and n_0. Moreover, the hypothesis of the theorem implies that F_n is a lifting of f^n and (4.3) is valid for $n = 1, \ldots, n_0$.

We will prove (4.3) and the fact that F_n is a lifting of f^n by induction on $n \geq n_0 + 1$. For $n = n_0 + 1$, these assertions are immediate consequences of (4.1)–(4.2). Suppose them true for $n - 1$, we like to show them for n.

To this end let G be the homogeneous polynomial given by

$$(4.6) \qquad\qquad G \cdot F_n := F_1 \circ F_{n-1},$$

and let $\mathcal{G}$ be the hypersurface $\{G(z) = 0\}$. We may rewrite (4.6) as

$$(4.7) \qquad\qquad (f^{n-1})^*(f^*T) = (f^n)^*T + [\mathcal{G}],$$

for any current $T \in \mathcal{C}_1^+(\mathbb{P}^k)$ of mass 1. In virtue of (4.2) and (4.7), we only need to show that

$$(4.8) \qquad\qquad [\mathcal{G}] = (f^{n-n_0-1})^*[\mathcal{H}_0].$$

One breaks the proof of this identity into two steps.

Step I: Proof of the inclusion $\mathcal{G} \subset (f^{n-n_0-1})^{-1}(\mathcal{H}_0)$.

Consider an arbitrary irreducible component $\mathcal{F}$ of $\mathcal{G}$. Then we deduce from (4.7) that

$$\nu\Big((f^{n-1})^*(f^*T), z\Big) > 0$$

for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane in $\mathbb{P}^k$, and for a generic point $z \in \mathcal{F}$. Since for any point z outside $\mathcal{I}(f)$, we can choose T so that f^*T vanishes in a neighborhood of z, it follows from the later inequality that $(f^{n-1})(\mathcal{F}) \subset \mathcal{I}(f)$.

Now let m be the greatest integer such that $0 \leq m < n-1$ and $f^m(\mathcal{F})$ is a hypersurface. Put $\mathcal{H} := f^m(\mathcal{F})$. Therefore, $f^{m+1}(\mathcal{F}), \ldots, f^{n-1}(\mathcal{F})$ are analytic sets of codimension strictly greater than 1. Since we have shown that $f^{n-1}(\mathcal{F}) \subset \mathcal{I}(f)$, there is a smallest integer p such that $m + 1 \leq p \leq n - 1$ and $f^p(\mathcal{F}) \subset \mathcal{I}(f)$. Using the hypothesis that f is QAS, one concludes that $\mathcal{H} := f^m(\mathcal{F})$ is an irreducible component of $\mathcal{H}_0$. Moreover, one has $p = n_0 + m$.

Recall from the previous paragraph that all the analytic sets $f^t(\mathcal{H})$ $(1 \leq t \leq n-1-m)$ are of codimension strictly greater than 1. In addition, we have that $f^{p-m}(\mathcal{H}) = f^p(\mathcal{F}) \subset \mathcal{I}(f)$, and $f^{n-1-m}(\mathcal{H}) = f^{n-1}(\mathcal{F}) \subset \mathcal{I}(f)$ with $n - 1 - m \geq p - m = n_0$. Invoking Definition 3.4 (iii) and the choice of p, it follows that $p - m = n - 1 - m$. This, combined with the equality $p = n_0 + m$, implies that $m = n - n_0 - 1$. In summary, we have

shown that $f^{n-n_0-1}(\mathcal{F}) = \mathcal{H} \subset \mathcal{H}_0$. Since $\mathcal{F}$ is an arbitrary component of $\mathcal{G}$, we deduce that $\mathcal{G} \subset (f^{n-n_0-1})^{-1}(\mathcal{H}_0)$. This completes Step I.

Step II: Proof of identity (4.8).[1]

In what follows T is a current in $\mathcal{C}_1^+(\mathbb{P}^k)$ of mass 1, and $d := \mathrm{d}(f)$. Moreover, we make the following convention $(f^m)^*[\mathcal{H}_0] := 0$ for all $m < 0$. Next, we apply the hypothesis of induction (i.e. identity (4.3)) for $n - 1, \ldots, n - n_0$ repeatedly by taking into account the identity $\|(f^m)^* T\| = d^m$, $m = 1, \ldots, n_0$, (see (2.1)). Consequently, one gets

$$
\begin{aligned}
(f^{n-1})^*(f^*)T &= (f^{n-2})^*(f^*f^*)T - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= \cdots \\
&= (f^{n-n_0})^*(f^{n_0})^*T \\
&\quad - d^{n_0-1}(f^{n-2n_0})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= (f^{n-n_0-1})^*f^*(f^{n_0})^*T \\
&\quad - d^{n_0}(f^{n-2n_0+1})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0] \\
&= (f^{n-n_0-1})^*(f^{n_0+1})^*T + (f^{n-n_0-1})^*[\mathcal{H}_0] \\
&\quad - d^{n_0}(f^{n-2n_0+1})^*[\mathcal{H}_0] - \cdots - d(f^{n-n_0-2})^*[\mathcal{H}_0].
\end{aligned}
$$

(4.9)

On the one hand, applying Lemma 4.1 to the right-hand side of (4.9), we deduce that

$$
\nu\Big((f^{n-1})^*(f^*)T, z\Big) = \nu\Big((f^{n-n_0-1})^*[\mathcal{H}_0], z\Big)
$$

for any current $T := [l(z) = 0]$, where $\{l(z) = 0\}$ is a generic complex hyperplane in $\mathbb{P}^k$, and for a generic point z in any irreducible component of $(f^{n-n_0-1})^{-1}(\mathcal{H}_0)$. On the other hand, under the same condition,

$$
\nu\Big((f^n)^*T, z\Big) = 0.
$$

We combine the later two equalities with (4.7) and taking into account the result of Step I. Consequently, (4.8) follows. This completes Step II. The proof of the theorem is thereby finished. □

Theorem 4.2. *Let f be a QAS meromorphic self-map of $\mathbb{P}^k$. Then its first dynamical degree $\lambda_1(f)$ is an algebraic integer. Moreover,*

$$
\lim_{n \to \infty} \frac{\mathrm{d}(f^{n+1})}{\mathrm{d}(f^n)} = \lambda_1(f). \tag{4.10}
$$

---

[1]The author thanks Ch. Favre for suggesting him the use of currents in this step. This helps to clarify the author's proof.

Proof: If f is AS, then the theorem is trivial since $d(f^n) = d(f)^n$ and $\lambda_1(f) = d(f)$. Suppose now that f is non AS. Then in virtue of identities (4.1)–(4.2), we have that

(4.11) $$d(f^n) = \begin{cases} d(f)^n, & n = 0, \ldots, n_0, \\ d(f) \cdot d(f^{n-1}) - \deg(H_0) \cdot d(f^{n-n_0-1}), & n > n_0. \end{cases}$$

Let $h := \deg(H_0)$. Then the characteristic polynomial associated to the sequence $(d(f^n))_{n=0}^\infty$ in (4.11) is

(4.12) $$P(\sigma) := \sigma^{n_0+1} - d\sigma^{n_0} + h.$$

Let $\sigma_1, \ldots, \sigma_s$ be all the roots of P with multiplicities $m_1, \ldots, m_s$ respectively $(\sum_{j=1}^s m_j = n_0 + 1)$. Since $P \in \mathbb{R}[\sigma]$, we remark that for every $p : 1 \le p \le s$, there is a unique q such that $\sigma_p = \overline{\sigma}_q$ and $m_p = m_q$. Then the Newton formula yields

(4.13) $$d(f^n) = \sum_{j=1}^s P_j(n)\sigma_j^n, \quad n \in \mathbb{N},$$

where P_j is a complex polynomial of one variable and $\deg(P_j) \le m_j - 1$, $j = 1, \ldots, s$.

There are two cases to consider.

Case 1: all roots of P are distinct.

In this case we know from (4.13) that all polynomials P_j appearing in this formula are constant. Namely, $P_j \equiv c_j$, $j = 1, \ldots, n_0 + 1$. In virtue of (4.11) and (4.13), we obtain the following system of linear equations with unknowns $c_1, \ldots, c_{n_0+1}$:

(4.14) $$\begin{cases} c_1 + c_2 + \cdots + c_{n_0+1} = 1 \\ \sigma_1 c_1 + \sigma_2 c_2 + \cdots + \sigma_{n_0+1} c_{n_0+1} = d \\ \vdots \\ \sigma_1^{n_0} c_1 + \sigma_2^{n_0} c_2 + \cdots + \sigma_{n_0+1}^{n_0} c_{n_0+1} = d^{n_0}. \end{cases}$$

Since all roots of P are distinct, this system is nonsingular.

We like to prove the following

Claim 1. *If $\sigma_q = \overline{\sigma_p}$ for some $1 \le p < q \le n_0 + 1$, then $c_q = \overline{c_p}$.*

Indeed, using the remark made just after (4.12), the assumption of Case 1 and solving system (4.14), Claim 1 follows.

Claim 2. *If $|\sigma_p| = |\sigma_q|$ for some $1 \le p < q \le n_0 + 1$, then $\sigma_q = \overline{\sigma_p}$.*

Indeed, it follows from (4.12) and the identities $P(\sigma_p) = P(\sigma_q) = 0$ that

$$h = |\sigma_p|^{n_0}|d - \sigma_p| = |\sigma_q|^{n_0}|d - \sigma_q|.$$

This, combined with the equality $|\sigma_p| = |\sigma_q|$, implies that $|d - \sigma_p| = |d - \sigma_q|$. In summary, we have that $|d - \sigma_p| = |d - \sigma_q|$ and $|\sigma_p| = |\sigma_q|$. Hence, $\sigma_q = \overline{\sigma_p}$. This completes the proof of Claim 2.

We may assume without loss of generality that

$$(4.15) \qquad\qquad |\sigma_1| = \max_{c_j \neq 0,\, j=1,\dots,n_0+1} |\sigma_j|.$$

Hence, $c_1 \neq 0$. Next, we show that σ_1 is a real number. Suppose in order to get a contradiction that this is not the case. Without loss of generality we assume that $\sigma_2 = \overline{\sigma_1}$. Then, by Claim 1, $c_2 = \overline{c_1}$. Write $\sigma_1 := |\sigma_1|e^{i\phi}$ and $c_1 = |c_1|e^{i\psi}$ for $0 \leq \phi,\psi < 2\pi$, $\phi > 0$ and $\phi \neq \pi$. Then formula (4.13) yields that

$$\frac{d(f^n)}{|\sigma_1|^n} = \frac{2\operatorname{Re}(c_1\sigma_1^n)}{|\sigma_1|^n} + \sum_{j>2} c_j\left(\frac{\sigma_j}{|\sigma_1|}\right)^n = 2|c_1|\cos(n\phi + \psi) + \sum_{j>2} c_j\left(\frac{\sigma_j}{|\sigma_1|}\right)^n.$$

Recall from Claim 2 and (4.15) that $|\sigma_1| > |\sigma_j|$ for all $j > 2$ with $c_j \neq 0$. Letting n to infinity in the above formula, we see that any accumulation point of the left side is a positive number, on the other hand the sum in the right side tends to zero. Thus any accumulation point of the set $\{\cos(n\phi + \psi) : n \in \mathbb{N}\}$ is also a positive number. We consider two subcases

(a) $\frac{\phi}{2\pi}$ is rational. Since $0 < \phi < 2\pi$ and $\phi \neq \pi$, there is a subsequence $(n_k)_{k=1}^{\infty}$ such that $\cos(n_k\phi + \psi) = c < 0$ for all k. This contradicts our observation above.

(b) $\frac{\phi}{2\pi}$ is irrational. It is clear that the set $\{\cos(n\phi + \psi) : n \in \mathbb{N}\}$ is dense on the interval $[-1, 1]$. Therefore we obtain a contradiction.

In summary, we have shown that the algebraic integer σ_1 is a real number. Applying Claim 2 we see that $|\sigma_1| > |\sigma_j|$ for any $j \geq 2$ with $c_j \neq 0$. Then it follows from (1.1) and (4.13) that $\sigma_1 > 0$, $\lambda_1(f) = \sigma_1$ and that (4.10) holds. This completes the proof of Case 1.

Case 2: the polynomial P has a multiple root.

Since $h \geq 1$, it can be checked that $P(\sigma) = P'(\sigma) = 0$ if and only if $\sigma = \sigma_{n_0} := \frac{dn_0}{n_0+1}$ and $h = h_0 := \left(\frac{d}{n_0+1}\right)^{n_0+1} n_0^{n_0}$. Moreover, $P''(\sigma_{n_0}) \neq 0$ for $h = h_0$. Consequently, we conclude that $h = h_0$ and the only multiple root of P is σ_{n_0}, and it is a double root. In the remaining part of the proof we may assume without loss of generality that $\sigma_{n_0+1} = \sigma_{n_0}$.

Consider the system of linear equations with unknowns $c_1, \ldots, c_{n_0+1}$:

$$(4.16) \quad \begin{cases} c_1 + \cdots + c_{n_0-1} + c_{n_0+1} = 1 \\ \sigma_1 c_1 + \sigma_2 c_2 + \cdots + \sigma_{n_0-1} c_{n_0-1} + \sigma_{n_0} c_{n_0} + \sigma_{n_0} c_{n_0+1} = d \\ \vdots \\ \sigma_1^{n_0} c_1 + \sigma_2^{n_0} c_2 + \cdots + \sigma_{n_0-1}^{n_0} c_{n_0-1} + n_0 \sigma_{n_0}^{n_0} c_{n_0} + \sigma_{n_0}^{n_0} c_{n_0+1} = d^{n_0} \end{cases}$$

and the formula

$$(4.17) \qquad d(f^n) = \sum_{j=1}^{n_0-1} c_j \sigma_j^n + (nc_{n_0} + c_{n_0+1}) \sigma_{n_0}^n, \quad n \in \mathbb{N}.$$

Using (4.16)–(4.17) instead of (4.13) and (4.14), and taking into account that $\sigma_{n_0+1} = \sigma_{n_0}$ is a positive number, we can proceed as in Case 1. There is only a small caution. Namely, the corresponding Claim 1 should be read as follows:

If $\sigma_q = \overline{\sigma_p}$ for some $1 \le p < q \le n_0 - 1$, then $c_q = \overline{c_p}$.

Let

$$(4.18) \qquad\qquad \lambda := \max_{c_j \neq 0, \, j=1,\ldots,n_0+1} |\sigma_j|.$$

There are two subcases to consider.

Case 2a: $\lambda \neq \sigma_{n_0}$.

Then $n_0 > 1$. We may suppose without loss of generality that $|\sigma_1| = \lambda$. Using (4.17)–(4.18), we proceed as in Case 1. Hence, the desired conclusion follows.

Case 2b: $\lambda = \sigma_{n_0}$.

Using (4.17)–(4.18), we see that

$$\frac{d(f^n)}{|\sigma_{n_0}|^n} = (nc_{n_0} + c_{n_0+1}) + \sum_{j<n_0} c_j \left(\frac{\sigma_j}{|\sigma_{n_0}|} \right)^n.$$

Since the sum in the right side tends to zero as $n \to \infty$, it follows that $\lambda_1(f) = \sigma_{n_0}$ and that (4.10) holds. Hence, the proof of Case 2 follows.

This finishes the proof. □

Applications. We apply the Main Theorem to the examples given in Section 3.

First consider Example 3.5. Put $H(z, w, t) := t$ and let $F \colon \mathbb{C}^3 \to \mathbb{C}^3$ be the lifting of f given by the right-hand side of (3.1). In the light of

the Main Theorem, a sequence of liftings F_n of f^n may be defined as follows

$$F_0 := \mathrm{Id}, \quad F_1 := F, \quad F_n := \frac{F \circ F_{n-1}}{H(F_{n-2})}, \quad n \geq 2.$$

As a consequence, one obtains the equation $\mathrm{d}(f^n) - 2\,\mathrm{d}(f^{n-1}) + \mathrm{d}(f^{n-2}) = 0$. Therefore, a straightforward computation shows that $\mathrm{d}(f^n) = n + 1$ and $\lambda_1(f) = 1$.

Next consider Example 3.7. Put $H(z, w, t) := (z+w+t)^2(z^3+w^3+t^3)$ and let $F: \mathbb{C}^3 \to \mathbb{C}^3$ be the lifting of f given by the right-hand side of (3.2). Using the Main Theorem, a sequence of liftings F_n of f^n may be defined as follows

$$F_0 := \mathrm{Id}, \quad F_1 := F, \quad F_n := \frac{F \circ F_{n-1}}{H(F_{n-2})}, \quad n \geq 2.$$

As a consequence, one obtains the equation $\mathrm{d}(f^n) - 7\,\mathrm{d}(f^{n-1}) + 5\,\mathrm{d}(f^{n-2}) = 0$. Therefore, a straightforward computation shows that

$$\mathrm{d}(f^n) = \frac{1}{\sqrt{29}} \cdot \left(\frac{7 + \sqrt{29}}{2} \right)^{n+1} - \frac{1}{\sqrt{29}} \cdot \left(\frac{7 - \sqrt{29}}{2} \right)^{n+1},$$

and $\lambda_1(f) = \frac{7+\sqrt{29}}{2}$.

Concluding remarks. One may widen the class of QAS self-maps by weakening considerably the conditions in Definition 3.4. Of course the recurrent law would be then more complicating. One might also seek to

- for any given $k, d \geq 2$, find many families of QAS (but non AS) self-maps of $\mathbb{P}^k$ with the algebraic degree d;

- generalize the Main Theorem to meromorphic self-maps in compact Kähler manifolds;

- construct an *appropriate* Green current for every QAS self-map f with $\lambda_1(f) > 1$.

We hope to come back these issues in a future work.

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