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EXTENSION OF DÍAZ-SAÁ'S INEQUALITY IN $\mathbb{R}^N$ AND APPLICATION TO A SYSTEM OF p-LAPLACIAN

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Abstract

The purpose of this paper is to extend the Díaz-Saá's inequality for the unbounded domains as $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} \left(-\frac{\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}}\right)(u^p - v^p)\, dx \geq 0$$

$$\text{with } \Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

The proof is based on the Picone's identity which is very useful in problems involving p-Laplacian. In a second part, we study some properties of the first eigenvalue for a system of p-Laplacian. We use Díaz-Saá's inequality to prove uniqueness and Egorov's theorem for the isolation. These results generalize J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis and F. de Thélin's work [**9**] for the first property and A. Anane's one for the isolation.

In a well-known paper [**7**] H. Brézis and L. Oswald obtain some necessary and sufficient conditions for the existence and uniqueness of a positive solution of the equation:

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

when Ω is a bounded open set in $\mathbb{R}^N$.

Later J. I. Díaz and J. E. Saá [**8**] extend these results to the case of an equation involving the p-Laplacian, $\Delta_p u = \operatorname{div}|\nabla u|^{p-2}\nabla u$ $(p > 1)$. In their proof a fundamental ingredient is the so called Díaz-Saá inequality:

$$\int_\Omega \left(-\frac{\Delta_p z_1}{z_1^{p-1}} + \frac{\Delta_p z_2}{z_2^{p-1}}\right)(z_1^p - z_2^p)\, dx \geq 0 \tag{1}$$

$$\text{where } \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

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This inequality expresses the monotony of the operator $w \mapsto \frac{-\Delta_p w^{\frac{1}{p}}}{w^{\frac{p-1}{p}}}$. It has been proved in [**8**] under the following hypotheses:

For $i = 1, 2$, $z_i \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $z_i \geq 0$ a. e. on Ω and $\Delta_p z_i \in L^\infty(\Omega)$;

For $i \neq j$, $i, j = 1, 2$, $\frac{z_i}{z_j} \in L^\infty(\Omega)$.

In the case of bounded domains, the Hopf's Maximum Principle gives them the hypothesis $\frac{z_i}{z_j} \in L^\infty(\Omega)$ which is necessary in the original proof of (1). But when we consider unbounded domains such as $\mathbb{R}^N$ this condition is not verified in general.

Here, under some adapted hypotheses for solutions of elliptic problems, we establish an inequality of Díaz-Saá type which remains true in whole $\mathbb{R}^N$. When we say the hypotheses are well adapted, it means that they naturally appear in weighted problems involving the p-Laplacian on $\mathbb{R}^N$.

In the second part, we apply this Díaz-Saá type inequality to obtain an uniqueness result of the first eigenvalue for a system of p-Laplacian in $\mathbb{R}^N$

$$(\mathcal{S}_\lambda) \begin{cases} -\Delta_p u = \lambda b(x)|u|^\alpha |v|^\beta v & x \in \mathbb{R}^N \\ -\Delta_q v = \lambda b(x)|u|^\alpha |v|^\beta u & x \in \mathbb{R}^N \\ \lim_{|x| \to +\infty} u(x) = 0 = \lim_{|x| \to +\infty} v(x). \end{cases}$$

This work follows an A. Anane's paper [**3**] where he studies the case of one equation and J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis and F. de Thélin [**9**] where a system quite different is considered. In [**9**], a local method is presented to control a possible blowing-up of the quotient $\frac{z_i}{z_j}$ in $\mathbb{R}^N$. But, in the integration by parts, some integrals on the boundary appear which make the demonstration quite boring.

Finally, following A. Anane's article [**3**], we will conclude this paper by showing the isolation of the first eigenvalue of the system $(\mathcal{S}_\lambda)$. As in this article, we will use the Egorov's theorem. This property was not proved in [**9**]; besides it seems that the method used here cannot be applied for their system because intuitively the sign of u must depend on the sign of v and vice versa.

1. Díaz-Saá inequality

Denote by $D^{1,p}(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ for the $L^p(\mathbb{R}^N)$ norm of the gradient

$$||u||^p_{D^{1,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^p \, dx.$$

The following results are the consequences of the principal theorems which will be presented later.

Proposition 1 (Díaz-Saá inequality on $\mathbb{R}^N$ in the case $1<p<N$)**.** *For $i = 1,2$, let $z_i \in D^{1,p}(\mathbb{R}^N)$ such that $z_i \geq 0$ ($\not\equiv 0$) and differentiable. Then we have*

$$\int_{\mathbb{R}^N} \left(-\frac{\Delta_p z_1}{z_1^{p-1}} + \frac{\Delta_p z_2}{z_2^{p-1}} \right) (z_1^p - z_2^p) \, dx \geq 0$$

if we assume that $\frac{\Delta_p z_i}{z_i^{p-1}} \in L^{\frac{N}{p}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ for $i = 1,2$.

If we have equality then there exists a constant C such that $z_1 = Cz_2$.

Proposition 2 (Díaz-Saá on $\mathbb{R}^N$ in the case $p \geq N$)**.** *For $i = 1,2$, let $z_i \in W^{1,p}(\mathbb{R}^N)$ such that $z_i \geq 0$ ($\not\equiv 0$) and differentiable. Then the assertions above remain true if we assume that for $N = p$, there exists some $s > 1$ such that*

$$\frac{\Delta_p z_i}{z_i^{p-1}} \in L^s(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$$

or for $N < p$,

$$\frac{\Delta_p z_i}{z_i^{p-1}} \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$$

with $i = 1,2$.

The proof of the theorems that we will present here needs Picone's identity for the p-Laplacian described for example in [**2**] by W. Allegretto and Y. X. Huang. Picone's result is an equality true almost everywhere which avoids some integration problems when we work in unbounded open sets. Now, we present a Picone's identity for the p-Laplacian quite more general than the W. Allegretto and Y. X. Huang's one because we only assume a sign condition for one of the two equations.

Proposition 3 (Generalized Picone's identity for the p-Laplacian)**.** *Let u, v differentiable and $v > 0$ a. e. on Ω a subset of $\mathbb{R}^N$. Note*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{|u|^p}{v^p}|\nabla v|^p - p\frac{|u|^{p-2}u}{v^{p-1}}\nabla u \cdot \nabla v |\nabla v|^{p-2} \quad \textit{a. e. on } \Omega$$

$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{|u|^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \qquad \textit{a. e. on } \Omega.$$

Then $L(u,v) = R(u,v) \geq 0$.

Moreover, $L(u,v) = 0$ a. e. on Ω if and only if $\nabla\left(\frac{u}{v}\right) = 0$ a. e. on Ω.

Proof of Proposition 3: By a simple calculation, we show that $R(u,v) = L(u,v)$, because

$$\nabla\left(\frac{|u|^p}{v^{p-1}}\right) = p\frac{|u|^{p-2}u\nabla u}{v^{p-1}} - (p-1)\frac{|u|^p \nabla v}{v^p}.$$

To prove positivity, we observe that

$$\frac{|u|^{p-2}u}{v^{p-1}}\nabla u \cdot \nabla v|\nabla v|^{p-2} \leq \frac{|u|^{p-1}}{v^{p-1}}|\nabla v|^{p-1}|\nabla u| \tag{2}$$

and by Young's inequality

$$p\frac{|u|^{p-1}}{v^{p-1}}|\nabla v|^{p-1}|\nabla u| \leq (p-1)\frac{|u|^p}{v^p}|\nabla v|^p + |\nabla u|^p. \tag{3}$$

Hence by (2) and (3)

$$|\nabla u|^p + (p-1)\frac{|u|^p}{v^p}|\nabla v|^p - p\frac{|u|^{p-2}u}{v^{p-1}}\nabla u \cdot \nabla v|\nabla v|^{p-2} = L(u,v) \geq 0.$$

Moreover, if $L(u,v) = 0$ then by (2) and (3)

$$|\nabla u|^p + (p-1)\frac{|u|^p}{v^p}|\nabla v|^p - p\frac{|u|^{p-1}}{v^{p-1}}|\nabla v|^{p-1}|\nabla u| = 0 \quad \text{a. e. on } \Omega. \tag{4}$$

We define $N := \left\{x \in \Omega \text{ such that } \frac{|u|}{v}|\nabla v| = 0\right\} \subset \Omega.$

On N, by the equation (4) we have

$$\frac{|u|}{v}|\nabla v| = |\nabla u| = 0 \quad \text{a e. on } N$$

and hence

$$\frac{u}{v}\nabla v = \nabla u = 0 \quad \text{a. e. on } N. \tag{5}$$

On N^c, we note $Q := \dfrac{|\nabla u|}{|\nabla v| \frac{|u|}{v}}$ and substituting in (4) we obtain

$$Q^p - pQ + p - 1 = 0 \quad \text{iff } Q = 1 \quad \text{because } p > 1,$$

$$\text{i. e. } |\nabla u| = \frac{|u|}{v} |\nabla v| \quad \text{a. e. on } N^c. \tag{6}$$

Using (6) in $L(u,v) = 0$, it follows that

$$|\nabla u|^p + (p-1)|\nabla u|^{p-2}|\nabla u|^2 - p\nabla u \cdot \nabla v \frac{u}{v} |\nabla u|^{p-2} = 0 \quad \text{a. e. on } N^c,$$

$$\text{so } \nabla u \cdot \left(\nabla u - \nabla v \frac{u}{v} \right) = 0 \quad \text{a. e. on } N^c,$$

$$\text{and } \frac{u}{v} \nabla v = \nabla u \quad \text{a e. on } N^c \quad \text{because } |\nabla u| = \frac{|u|}{v} |\nabla v|. \tag{7}$$

Indeed ∇v cannot be perpendicular to $\frac{u}{v}\nabla v - \nabla u$ because it would signify that ∇u is the hypotenuse of a right triangle with edges $\frac{u}{v}\nabla v$ and $\frac{u}{v}\nabla v - \nabla u$, that is not possible because $\frac{u}{v}|\nabla v| = |\nabla u|$.

By (5) and (7) we have $\dfrac{u}{v}\nabla v = \nabla u$ a. e. on Ω and finally,

$$\nabla \left(\frac{u}{v} \right) = 0 \quad \text{a. e. on } \Omega. \qquad \square$$

We can easily remark that in Picone's identity, we can take any set Ω, for example non connected and unbounded. Now, we can establish the following theorem whose principal argument of the proof is the above identity.

Theorem 1. *For $1 < p < N$. Let Φ in $D^{1,p}(\mathbb{R}^N)$ and $z \geq 0$ ($\not\equiv 0$) in $D^{1,p}(\mathbb{R}^N)$ both differentiable.*

Then we have

$$\int_{\mathbb{R}^N} |\nabla \Phi|^p \, dx \geq \int_{\mathbb{R}^N} \left(\frac{-\Delta_p z}{z^{p-1}} \right) |\Phi|^p \, dx,$$

if we assume that $\dfrac{\Delta_p z}{z^{p-1}} \in L^{\frac{N}{p}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$.

Moreover, in the equality case there exists C such that $z = C\Phi$ on $\mathbb{R}^N$.

Proof of Theorem 1: First we remark that $z \in D^{1,p}(\mathbb{R}^N)$ is a non trivial solution of the problem

$$\begin{cases} -\Delta_p v = \left(\frac{-\Delta_p z}{z^{p-1}} \right) v^{p-1} \\ v \geq 0 \end{cases} \qquad \text{in } \mathbb{R}^N.$$

Seeing that $\frac{\Delta_p z}{z^{p-1}} \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, we can apply Vázquez's Strong Maximum Principle [**16**] to prove that $z > 0$ on $\mathbb{R}^N$. Moreover, we use P. Tolksdorf's regularity theorem [**15**] to show that for all $r > 0$, there exists $\alpha(r) > 0$ such that $z \in C^{1,\alpha}(B_r)$. In particular for Ω_0 a bounded domain of $\mathbb{R}^N$, there exists $\alpha_0 > 0$ such that $z \in C^{1,\alpha_0}(\Omega_0)$.

Let $(\Phi_n)_{n\in\mathbb{N}}$ a sequence of functions in $C_0^\infty(\mathbb{R}^N)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ in $D^{1,p}(\mathbb{R}^N)$. We apply Picone's identity to the functions Φ_n and z

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} L(\Phi_n, z)\,dx \le \int_{\mathbb{R}^N} R(\Phi_n, z)\,dx \\ &\le \int_{\mathbb{R}^N} |\nabla\Phi_n|^p\,dx - \int_{\mathbb{R}^N} \nabla\left(\frac{|\Phi_n|^p}{z^{p-1}}\right)|\nabla z|^{p-2}\nabla z\,dx. \end{aligned}$$

But $\Phi_n \in C_0^\infty(\mathbb{R}^N)$ and $z > 0$ then $\frac{|\Phi_n|^p}{z^{p-1}}$ is an admissible function test, integrating by parts we obtain

$$0 \le \int_{\mathbb{R}^N} |\nabla\Phi_n|^p\,dx + \int_{\mathbb{R}^N} \frac{\Delta_p z}{z^{p-1}}|\Phi_n|^p\,dx.$$

$(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ in $D^{1,p}(\mathbb{R}^N)$, so we have that $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ in $L^{p^*}(\mathbb{R}^N)$ and $(|\Phi_n|^p)_{n\in\mathbb{N}}$ converges to $|\Phi|^p$ in $L^{\frac{p^*}{p}}(\mathbb{R}^N)$. Consequently,

$$\int_{\mathbb{R}^N} \frac{\Delta_p z}{z^{p-1}}\left(|\Phi_n|^p - |\Phi|^p\right)dx \le \underbrace{\left\|\frac{\Delta_p z}{z^{p-1}}\right\|_{L^{\frac{N}{p}}(\mathbb{R}^N)} \left\||\Phi_n|^p - |\Phi|^p\right\|_{L^{\frac{p^*}{p}}(\mathbb{R}^N)}}_{\text{tends to } 0}. \tag{8}$$

And the result follows:

$$\int_{\mathbb{R}^N} |\nabla\Phi|^p\,dx \ge \int_{\mathbb{R}^N}\left(-\frac{\Delta_p z}{z^{p-1}}\right)|\Phi|^p\,dx. \tag{9}$$

We now consider the equality case

$$\int_{\mathbb{R}^N} |\nabla\Phi|^p\,dx = \int_{\mathbb{R}^N}\left(-\frac{\Delta_p z}{z^{p-1}}\right)|\Phi|^p\,dx.$$

Let Ω_0 a bounded domain in $\mathbb{R}^N$ and $(\Phi_n)_{n\in\mathbb{N}}$ defined as before.

$$\begin{aligned} 0 &\le \int_{\Omega_0} L(\Phi_n, z)\,dx \le \int_{\mathbb{R}^N} L(\Phi_n, z)\,dx \\ &\le \int_{\mathbb{R}^N} |\nabla\Phi_n|^p\,dx + \int_{\mathbb{R}^N} \frac{\Delta_p z}{z^{p-1}}|\Phi_n|^p\,dx \quad \text{tends to } 0 \quad \text{when } n \to \infty. \end{aligned}$$

But $\int_{\Omega_0} L(\Phi_n, z)\,dx$ converges to $\int_{\Omega_0} L(\Phi, z)\,dx$ because Ω_0 is bounded and $z \in C^{1,\alpha_0}(\Omega_0)$. So $L(\Phi, z) = 0$ a. e. on Ω_0, the set Ω_0 being taken arbitrary in $\mathbb{R}^N$ we can conclude that $L(\Phi, z) = 0$ a. e. on $\mathbb{R}^N$. And, by Picone's identity (Proposition 3), there exists $C > 0$ such that $\Phi = Cz$. □

Theorem 2. *For $p \geq N$. Let Φ in $W^{1,p}(\mathbb{R}^N)$ and $z \geq 0$ ($\not\equiv 0$) in $W^{1,p}(\mathbb{R}^N)$ both differentiable.*

Then we have

$$\int_{\mathbb{R}^N} |\nabla\Phi|^p\,dx \geq \int_{\mathbb{R}^N} \left(\frac{-\Delta_p z}{z^{p-1}}\right) |\Phi|^p\,dx,$$

if we assume that for $p = N$, there exists some $s > 1$ such that

$$\frac{\Delta_p z}{z^{p-1}} \in L^s(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$$

or for $p > N$,

$$\frac{\Delta_p z}{z^{p-1}} \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N).$$

Moreover, if the above integral is zero then there exists C such that $z = C\Phi$ on $\mathbb{R}^N$.

The proof of this theorem is quite similar to Theorem 1. We solve the problem of convergence in (8) by the particular embeddings of the space $W^{1,p}(\mathbb{R}^N)$ in the case $p \geq N$ (see [**6**]):

If $p = N$, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in any $L^q(\mathbb{R}^N)$ where $q \in [p, +\infty)$;

If $p > N$, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$.

The existence result of a solution in $W^{1,p}(\mathbb{R}^N)$ for the p-Laplacian where $p \geq N$ has been established by W. Allegretto and Y. X. Huang in [**1**].

Proofs of Proposition 1 and Proposition 2 (Díaz-Saá inequality): We only have to apply the preceding results for the couple of the functions (z_1, z_2) and we have

$$\int_{\mathbb{R}^N} |\nabla z_1|^p\,dx \geq \int_{\mathbb{R}^N} \left(\frac{-\Delta_p z_2}{z_2^{p-1}}\right) z_1^p\,dx$$

$$\text{and} \quad \int_{\mathbb{R}^N} \left(-\frac{\Delta_p z_1}{z_1^{p-1}} + \frac{\Delta_p z_2}{z_2^{p-1}}\right) z_1^p\,dx \geq 0.$$

Doing the same work for the couple of the functions (z_2, z_1) and adding the two inequalities obtained we arrive to the expected result. We note that in Díaz-Saá inequality, we assume functions are positives so we needn't the absolute values as in the theorems. □

Following a discussion with P. Takáč, it appears that Díaz-Saá inequality in $\mathbb{R}^N$ and both theorems can be proved using very carefully J. I. Díaz and J. E. Saá's method. For this, we have to remark first that if this inequality is true for positive functions Φ it remains true for functions Φ changing sign [**11**, "Chain rule", Lemma 7.6]. After that, we just have to prove the convexity of the application $w \longmapsto |\nabla w^{\frac{1}{p}}|^p$ and compute the directional derivative of $J(w) = \int_\Omega |\nabla w^{\frac{1}{p}}|^p \, dx$ which is formally:

$$J'(w)v = \frac{1}{p} \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} \nabla w^{\frac{1}{p}} \nabla (w^{\frac{1}{p}-1} v) \, dx.$$

To have more details on this method we can see J. Fleckinger, J. Hernández, P. Takáč and F. de Thélin's article [**10**]. But that proof needs a good attention in the computation because some terms, in particular the derivative of J, have to be defined correctly, and we think that our proof using Picone's identity is easier.

2. Property of first eigenvalue for a system of p-Laplacian

In this second part, we study some properties of the first eigenvalue for a potential system of p-Laplacian:

$$(\mathcal{S}_\lambda) \begin{cases} -\Delta_p u = \lambda b(x) |u|^\alpha |v|^\beta v & x \in \mathbb{R}^N \\ -\Delta_q v = \lambda b(x) |u|^\alpha |v|^\beta u & x \in \mathbb{R}^N \\ \lim_{|x| \to +\infty} u(x) = 0 = \lim_{|x| \to +\infty} v(x). \end{cases}$$

And we assume:

($\mathcal{H}1$) $N > p > 1, \quad N > q > 1, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \dfrac{\alpha + 1}{p} + \dfrac{\beta + 1}{q} = 1$ and $\alpha + \beta + 2 < N$.

($\mathcal{H}2$) $b \in C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\gamma \in (0, 1)$,

$b \in L^{\frac{N}{\alpha+\beta+2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $b \geq 0$ ($\not\equiv 0$).

For a start, we establish existence of a first eigenvalue for $(\mathcal{S}_\lambda)$ and the regularity of the associated eigenfunctions.

Theorem 3. *We suppose that Hypotheses ($\mathcal{H}1$) and ($\mathcal{H}2$) are satisfied.*

(i) *System ($\mathcal{S}_\lambda$) admits a first eigenvalue λ_1 which is positive and defined by*

$$\lambda_1 = \inf_\Gamma \left\{ \frac{\alpha+1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{\beta+1}{q} \int_{\mathbb{R}^N} |\nabla v|^q \, dx \right\}$$

$$\text{where} \quad \Gamma = \left\{ x \in \mathbb{R}^N \text{ such that } \int_{\mathbb{R}^N} b(x)|u|^\alpha |v|^\beta uv \, dx = 1 \right\};$$

(ii) *If (u, v) is a couple of eigenfunctions solution of ($\mathcal{S}_{\lambda_1}$) then for all $r > 0$, $u \in C^{1,\rho}(B_r)$ and $v \in C^{1,\gamma}(B_r)$ where $\rho = \rho(r) > 0$ and $\gamma = \gamma(r) > 0$;*

(iii) *There exists a couple of eigenfunctions solution of ($\mathcal{S}_{\lambda_1}$) which are positive on $\mathbb{R}^N$.*

The proof of this theorem is more or less the same as J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis and F. de Thélin's one in [**9**] for the system:

$$\begin{cases} -\Delta_p u = \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1} & x \in \mathbb{R}^N \\ -\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v & x \in \mathbb{R}^N. \end{cases}$$

The approach done in [**9**] is standard because their problem as ($\mathcal{S}_\lambda$) is variational.

We present now an uniqueness and isolation result for the first eigenvalue. The proof is an interesting application of Theorem 1 and is simpler than in [**9**].

Theorem 4. *We suppose that Hypotheses ($\mathcal{H}1$) and ($\mathcal{H}2$) are satisfied.*

(i) *In the set of continuous functions, the dimension of eigenspace corresponding to principal eigenvalue λ_1 is 1;*

(ii) *λ_1 is the only one eigenvalue of ($\mathcal{S}_{\lambda_1}$) which corresponds to a constant sign eigenvector;*

(iii) *λ_1 is isolated i. e. there exists $\epsilon > 0$ such that for all $\lambda \in (\lambda_1, \lambda_1 + \epsilon]$ the system ($\mathcal{S}_\lambda$) has no solution.*

Proof of Theorem 4: (i)–(ii) Let (u,v) a positive solution of $(\mathcal{S}_\lambda)$ and (ϕ,ψ) a solution of $(\mathcal{S}_{\lambda_1})$. By the definition of λ_1, it is clear that $\lambda_1 \leq \lambda$. We apply Theorem 1 to the functions (ϕ,u) and (ψ,v),

$$\int_{\mathbb{R}^N} |\nabla\phi|^p\,dx \geq \int_{\mathbb{R}^N} \left(-\frac{\Delta_p u}{u^{p-1}}\right)|\phi|^p\,dx \tag{10}$$

$$\int_{\mathbb{R}^N} |\nabla\psi|^q\,dx \geq \int_{\mathbb{R}^N} \left(-\frac{\Delta_q v}{v^{q-1}}\right)|\psi|^q\,dx. \tag{11}$$

On the other hand, by the definition of (ϕ,ψ) we have

$$\begin{aligned} J &:= \frac{\alpha+1}{p}\int_{\mathbb{R}^N} |\nabla\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N} |\nabla\psi|^q\,dx \\ &= \lambda_1 \int_{\mathbb{R}^N} b(x)|\phi|^\alpha|\psi|^\beta\phi\psi\,dx \leq \lambda\int_{\mathbb{R}^N} b(x)\frac{|\phi|^{\alpha+1}|\psi|^{\beta+1}}{u^{\alpha+1}v^{\beta+1}}u^{\alpha+1}v^{\beta+1}\,dx. \end{aligned}$$

By Young's inequality, we obtain

$$\begin{aligned} J &\leq \lambda\int_{\mathbb{R}^N} b(x)u^{\alpha+1}v^{\beta+1}\left(\frac{\alpha+1}{p}\frac{|\phi|^p}{u^p} + \frac{\beta+1}{q}\frac{|\psi|^q}{v^q}\right)dx \\ &\leq \lambda\frac{\alpha+1}{p}\int_{\mathbb{R}^N} b(x)\frac{u^\alpha v^{\beta+1}}{u^{p-1}}|\phi|^p\,dx + \lambda\frac{\beta+1}{q}\int_{\mathbb{R}^N} b(x)\frac{u^{\alpha+1}v^\beta}{v^{q-1}}|\psi|^q\,dx \\ &\leq \frac{\alpha+1}{p}\int_{\mathbb{R}^N}\left(\frac{-\Delta_p u}{u^{p-1}}\right)|\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N}\left(\frac{-\Delta_q v}{v^{q-1}}\right)|\psi|^q\,dx. \end{aligned}$$

So, by (10) and (11) we arrive to

$$\begin{aligned} &\frac{\alpha+1}{p}\int_{\mathbb{R}^N} |\nabla\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N} |\nabla\psi|^q\,dx \\ &= \frac{\alpha+1}{p}\int_{\mathbb{R}^N}\left(\frac{-\Delta_p u}{u^{p-1}}\right)|\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N}\left(\frac{-\Delta_q v}{v^{q-1}}\right)|\psi|^q\,dx. \end{aligned}$$

Again with (10) and (11) we conclude that

$$\int_{\mathbb{R}^N} |\nabla\phi|^p\,dx = \int_{\mathbb{R}^N}\left(\frac{-\Delta_p u}{u^{p-1}}\right)|\phi|^p\,dx$$

$$\text{and}\quad \int_{\mathbb{R}^N} |\nabla\psi|^q\,dx = \int_{\mathbb{R}^N}\left(\frac{-\Delta_q v}{v^{q-1}}\right)|\psi|^q\,dx.$$

Then applying the second result of Theorem 1 we conclude the existence of two constants c and $\hat{c}$ such that $u = c\phi$ and $v = \hat{c}\psi$. By the particular form of the problem and Maximum Principle we can say that if $c > 0$ then $\hat{c} > 0$ and $c^p = \hat{c}^q$. That means that the solutions u and v have the same sign.

To prove the uniqueness of the eigenvalue which corresponds to a constant sign eigenvector, we suppose $\phi > 0$, $\psi > 0$. (u, v) is a solution of $(\mathcal{S}_\lambda)$ and satisfies $u = c\phi$ and $v = \hat{c}\psi$ so we can write

$$\frac{\alpha+1}{p}\int_{\mathbb{R}^N} |\nabla c\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N} |\nabla \hat{c}\psi|^q\,dx = \lambda\int_{\mathbb{R}^N} b(x)c^{\alpha+1}\hat{c}^{\beta+1}|\phi|^\alpha|\psi|^\beta\phi\psi\,dx.$$

But $c^{\alpha+1}\hat{c}^{\beta+1} = c^p = \hat{c}^q$ because $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$ so

$$\frac{\alpha+1}{p}\int_{\mathbb{R}^N} |\nabla\phi|^p\,dx + \frac{\beta+1}{q}\int_{\mathbb{R}^N} |\nabla\psi|^q\,dx = \lambda\int_{\mathbb{R}^N} b(x)|\phi|^\alpha|\psi|^\beta\phi\psi\,dx.$$

Moreover (ϕ, ψ) is an eigenvector corresponding to λ_1

$$\lambda_1\int_{\mathbb{R}^N} b(x)|\phi|^\alpha|\psi|^\beta\phi\psi\,dx = \lambda\int_{\mathbb{R}^N} b(x)|\phi|^\alpha|\psi|^\beta\phi\psi\,dx.$$

Finally, we obtain $\lambda = \lambda_1$ because $\phi > 0$, $\psi > 0$ and $b \not\equiv 0$.

(iii) Let (u_0, v_0) an eigenvector corresponding to λ_0 ($\lambda_0 > \lambda_1 > 0$) solution of $(\mathcal{S}_{\lambda_0})$ such that $\int b|u_0|^\alpha|v_0|^\beta u_0v_0 = 1$. We have seen above that the solutions u_0 and v_0 change sign because $\lambda_0 \neq \lambda_1$, so we note:

$$U_0^- = \{x \in \mathbb{R}^N \text{ such that } u_0(x) < 0\}$$

$$\text{and}\quad V_0^- = \{x \in \mathbb{R}^N \text{ such that } v_0(x) < 0\}.$$

Multiplying the first equation by $u_0^- = \max(0, -u_0)$, the second by $v_0^- = \max(0, -v_0)$ and integrating over $\mathbb{R}^N$ we obtain

$$\int_{\mathbb{R}^N} |\nabla u_0^-|^p\,dx = \lambda_0\int_{\mathbb{R}^N} b(x)|u_0^-|^\alpha|v_0^-|^\beta u_0^-v_0^-\,dx - \lambda_0\int_{\mathbb{R}^N} b(x)|u_0^-|^\alpha|v_0^+|^\beta u_0^-v_0^+\,dx$$

$$\int_{\mathbb{R}^N} |\nabla v_0^-|^q\,dx = \lambda_0\int_{\mathbb{R}^N} b(x)|u_0^-|^\alpha|v_0^-|^\beta u_0^-v_0^-\,dx - \lambda_0\int_{\mathbb{R}^N} b(x)|u_0^+|^\alpha|v_0^-|^\beta u_0^+v_0^-\,dx,$$

where $u_0 = u_0^+ - u_0^-$, $v_0 = v_0^+ - v_0^-$, so

$$\int_{\mathbb{R}^N} |\nabla u_0^-|^p \, dx \le \lambda_0 \int_{\mathbb{R}^N} b(x)|u_0^-|^\alpha |v_0^-|^\beta u_0^- v_0^- \, dx$$

$$\int_{\mathbb{R}^N} |\nabla v_0^-|^q \, dx \le \lambda_0 \int_{\mathbb{R}^N} b(x)|u_0^-|^\alpha |v_0^-|^\beta u_0^- v_0^- \, dx.$$

Using Hölder and Sobolev's inequalities, we have:

$$\|u_0^-\|^p_{D^{1,p}(\mathbb{R}^N)} \le \lambda_0 \|b\|_{L^{\frac{N}{\alpha+\beta+2}}(U_0^- \cap V_0^-)} \|u_0^-\|^{\alpha+1}_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \|v_0^-\|^{\beta+1}_{L^{\frac{Nq}{N-q}}(\mathbb{R}^N)}$$

so

$$\|u_0^-\|_{D^{1,p}(\mathbb{R}^N)} \le (\lambda_0 C_S)^{\frac{1}{p}} \|b\|^{\frac{1}{p}}_{L^{\frac{N}{\alpha+\beta+2}}(U_0^- \cap V_0^-)} \|u_0^-\|^{\frac{\alpha+1}{p}}_{D^{1,p}(\mathbb{R}^N)} \|v_0^-\|^{\frac{\beta+1}{p}}_{D^{1,q}(\mathbb{R}^N)}$$

and

$$\|u_0^-\|^{\frac{1}{q}}_{D^{1,p}(\mathbb{R}^N)} \le (\lambda_0 C_S)^{\frac{1}{p(\beta+1)}} \|b\|^{\frac{1}{p(\beta+1)}}_{L^{\frac{N}{\alpha+\beta+2}}(U_0^- \cap V_0^-)} \|v_0^-\|^{\frac{1}{p}}_{D^{1,q}(\mathbb{R}^N)} \tag{12}$$

because $\frac{1}{q} = \frac{1}{\beta+1} - \frac{\alpha+1}{p(\beta+1)}$. And in a same way:

$$\|v_0^-\|^{\frac{1}{p}}_{D^{1,q}(\mathbb{R}^N)} \le (\lambda_0 C_S)^{\frac{1}{q(\alpha+1)}} \|b\|^{\frac{1}{q(\alpha+1)}}_{L^{\frac{N}{\alpha+\beta+2}}(U_0^- \cap V_0^-)} \|u_0^-\|^{\frac{1}{q}}_{D^{1,p}(\mathbb{R}^N)}. \tag{13}$$

Combining Inequalities (12) and (13), we deduce that there exists a constant $C > 0$ (independent of $\|u_0^-\|_{D^{1,p}(\mathbb{R}^N)}$ and $\|v_0^-\|_{D^{1,q}(\mathbb{R}^N)}$) such that:

$$\|b\|_{L^{\frac{N}{\alpha+\beta+2}}(U_0^- \cap V_0^-)} \ge \frac{C}{\lambda_0} > 0. \tag{14}$$

Claim. λ_1 *is isolated.*

If the claim is not true there exists a sequence $(\lambda_n, u_n, v_n)_{n \in N}$ where (u_n, v_n) is the couple of eigenfunctions associated to the eigenvalue λ_n, such that $(\lambda_n)_{n\in\mathbb{N}}$ converges to λ_1. We can assume that $\int_{\mathbb{R}^N} b(x)|u_n|^\alpha |v_n|^\beta u_n v_n \, dx = 1$, $(u_n, v_n)_{n\in\mathbb{N}}$ weakly converges to $(\tilde{u}, \tilde{v})$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ and $2\lambda_1 > \lambda_n > \lambda_1$ for all $n \in \mathbb{N}$. We note

$$U_n^- = \left\{x \in \mathbb{R}^N \text{ such that } u_n(x) < 0\right\},$$

$$V_n^- = \left\{x \in \mathbb{R}^N \text{ such that } v_n(x) < 0\right\},$$

$$U_n^+ = \left\{x \in \mathbb{R}^N \text{ such that } u_n(x) > 0\right\}$$

$$\text{and} \quad V_n^+ = \left\{x \in \mathbb{R}^N \text{ such that } v_n(x) > 0\right\}.$$

By (14), there exists a constant C independent of n such that

$$||b||_{L^{\frac{N}{\alpha+\beta+2}}(U_n^-\cap V_n^-)} \geq \frac{C}{\lambda_n} \geq \frac{C}{2\lambda_1} > 0. \tag{15}$$

Let R be such that $||b||_{L^{\frac{N}{\alpha+\beta+2}}(B_R^c)} < \frac{C}{4\lambda_1}$. Then (15) implies that $||b||_{L^{\frac{N}{\alpha+\beta+2}}(B_R\cap U_n^-\cap V_n^-)} \geq \frac{C}{4\lambda_1}$.

But

$$\left(\int_{B_R\cap U_n^-\cap V_n^-} b(x)^{\frac{N}{\alpha+\beta+2}}\,dx\right)^{\frac{\alpha+\beta+2}{N}} \leq ||b||_{L^\infty(\mathbb{R}^N)} \left|B_R\cap U_n^-\cap V_n^-\right|^{\frac{\alpha+\beta+2}{N}}.$$

And therefore there is some C_0 independent of n such that:

$$\left|B_R\cap U_n^-\cap V_n^-\right| \geq \left(\frac{C}{4\lambda_1||b||_{L^\infty(\mathbb{R}^N)}}\right)^{\frac{N}{\alpha+\beta+2}} = C_0. \tag{16}$$

In the same way we can prove that

$$\left|B_R\cap U_n^+\cap V_n^+\right| \geq C_0. \tag{17}$$

In the remainder of the proof, we choose R such that (16) is satisfied and intentionally we will forget to extract sub-sequences.

First, we observe that for all $(\Phi,\Psi)\in\left(L^p(\mathbb{R}^N)\right)^N\times\left(L^q(\mathbb{R}^N)\right)^N$

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\Phi|^{p-2}\Phi-|\Psi|^{p-2}\Psi)(\Phi-\Psi)\,dx\\
&\qquad=\int_{\mathbb{R}^N}(|\Phi|^p+|\Psi|^p-|\Phi|^{p-2}\Phi\Psi-|\Psi|^{p-2}\Psi\Phi)\,dx\\
&\qquad\geq\left[\left(\int_{\mathbb{R}^N}|\Phi|^p\,dx\right)^{\frac{p-1}{p}}-\left(\int_{\mathbb{R}^N}|\Psi|^p\,dx\right)^{\frac{p-1}{p}}\right]\\
&\qquad\quad\times\left[\left(\int_{\mathbb{R}^N}|\Phi|^p\,dx\right)^{\frac{1}{p}}-\left(\int_{\mathbb{R}^N}|\Psi|^p\,dx\right)^{\frac{1}{p}}\right]\geq 0.
\end{aligned}\tag{18}$$

By (18) we obtain

$$
\begin{aligned}
0 &\le \left[\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx\right)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \, dx\right)^{\frac{p-1}{p}}\right] \\
&\quad \times \left[\left(\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \, dx\right)^{\frac{1}{p}}\right] \\
&\le \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n - \nabla u_m) \, dx \\
&\le \int_{B_R} b(x)(\lambda_n |u_n|^\alpha |v_n|^\beta v_n - \lambda_m |u_m|^\alpha |v_m|^\beta v_m)(u_n - u_m) \, dx \\
&\quad + \lambda_n \int_{B_R^c} (b(x)|u_n|^\alpha |v_n|^\beta v_n)(u_n - u_m) \, dx \\
&\quad + \lambda_m \int_{B_R^c} (b(x)|u_m|^\alpha |v_m|^\beta v_m)(u_m - u_n) \, dx \\
&\le \|b\|_{L^\infty(\mathbb{R}^N)} ||\lambda_n |u_n|^\alpha |v_n|^\beta v_n \\
&\quad - \lambda_m |u_m|^\alpha |v_m|^\beta v_m||_{L^{p'}(B_R)} ||u_n - u_m||_{L^p(B_R)} \\
&\quad + \lambda_n \|b\|_{L^{\frac{N}{\alpha+\beta+2}}(B_R^c)} ||u_n||^\alpha_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} ||v_n||^{\beta+1}_{L^{\frac{Nq}{N-q}}(\mathbb{R}^N)} \\
&\qquad\qquad \times (||u_n||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} + ||u_m||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)}) \\
&\quad + \lambda_m \|b\|_{L^{\frac{N}{\alpha+\beta+2}}(B_R^c)} ||u_m||^\alpha_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} ||v_m||^{\beta+1}_{L^{\frac{Nq}{N-q}}(\mathbb{R}^N)} \\
&\qquad\qquad \times (||u_m||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} + ||u_n||_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)}).
\end{aligned}
$$

The second and the third term can be taken as small as we wish and independently of n by choosing R great enough ($b \in L^{\frac{N}{\alpha+\beta+2}}(\mathbb{R}^N)$ and $(u_n, v_n)_{n\in\mathbb{N}}$ are bounded in $L^{\frac{Np}{N-p}}(\mathbb{R}^N) \times L^{\frac{Nq}{N-q}}(\mathbb{R}^N)$). For R fixed like that, the first term tends to 0 because $(u_n)_{n\in\mathbb{N}}$ converges in $L^p(B_R)$ and $(|u_n|^\alpha |v_n|^\beta v_n)_{n\in\mathbb{N}}$ converges in $L^{p'}(B_R)$ ($\frac{\alpha}{p} + \frac{\beta+1}{q} = \frac{p-1}{p}$) under the compact embedding of $D^{1,p}(B_R)$ into $L^p(B_R)$. It follows that $(\nabla u_n)_{n\in\mathbb{N}}$ and $(\nabla v_n)_{n\in\mathbb{N}}$ are Cauchy sequences of $\left(L^p(\mathbb{R}^N)\right)^N$ and $\left(L^q(\mathbb{R}^N)\right)^N$ respectively.

Doing the same work for $(v_n)_{n\in N}$, we arrive to conclude: $(\nabla u_n, \nabla v_n)_{n\in\mathbb{N}}$ converges to some limit for the norm of $\big(L^p(\mathbb{R}^N)\big)^N \times \big(L^q(\mathbb{R}^N)\big)^N$; and because $(\nabla u_n, \nabla v_n)_{n\in\mathbb{N}}$ converges weakly to $(\nabla\tilde{u}, \nabla\tilde{v})$ in $\big(L^p(\mathbb{R}^N)\big)^N \times \big(L^q(\mathbb{R}^N)\big)^N$, we obtain that $(\nabla u_n, \nabla v_n)_{n\in\mathbb{N}}$ (or a sub-sequence) converges strongly to $(\nabla\tilde{u}, \nabla\tilde{v})$ in $\big(L^p(\mathbb{R}^N)\big)^N \times \big(L^q(\mathbb{R}^N)\big)^N$.

We can take the limit in the system $(\mathcal{S}_{\lambda_n})$ and we obtain that $(\tilde{u}, \tilde{v})$ is a solution of $(\mathcal{S}_{\lambda_1})$.

Because λ_1 is simple and $(\tilde{u}, \tilde{v})$ is an eigenvector associated to the eigenvalue λ_1 with $\int_{\mathbb{R}^N} b|\tilde{u}|^\alpha|\tilde{v}|^\beta\tilde{u}\tilde{v}\,dx = 1$ (indeed $(u_n, v_n)_{n\in\mathbb{N}} \to (\tilde{u}, \tilde{v})$ in $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ and $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ embeds continuously into $L^{\frac{Np}{N-p}}(\mathbb{R}^N) \times L^{\frac{Nq}{N-q}}(\mathbb{R}^N)$). It implies that $(\tilde{u}, \tilde{v}) = (-u_1, -v_1)$ or $(\tilde{u}, \tilde{v}) = (u_1, v_1)$, for example $(\tilde{u}, \tilde{v}) = (u_1, v_1)$.

Consequently, considering the convergence of $(u_n, v_n)_{n\in\mathbb{N}}$ to (u_1, v_1) in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, we have also $(u_n, v_n)_{n\in\mathbb{N}}$ (or a sub-sequence) which converges strongly to (u_1, v_1) in $L^p(B_R) \times L^q(B_R)$ where R is fixed such that (16) is satisfied. And by Egorov's theorem $(u_n, v_n)_{n\in\mathbb{N}}$ converges uniformly to (u_1, v_1) on B_R except on a set with arbitrary small measure. It signifies that for n great enough, u_n and v_n are positive on B_R except on a set with arbitrary small measure: this fact contradicts the assertion (16). Identically, if we put $(\tilde{u}, \tilde{v}) = (-u_1, -v_1)$, we follow the same way, replacing the sets U_n^- and V_n^- by U_n^+ and V_n^+.

Finally, we conclude that the first principal eigenvalue λ_1 is isolated and the claim is proved. □

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