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A NOTE ON NONEXISTENCE OF RADIAL SOLUTIONS TO SEMILINEAR ELLIPTIC INEQUATIONS

Mohammed Guedda

Abstract

We study the nonexistence result of radial solutions to $-\Delta u + c\dfrac{u}{|x|^2} + |x|^\sigma |u|^q u \leq 0$ posed in B or in $B \setminus \{0\}$ where B is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$. Moreover, we give a complete classification of radial solutions to the problem $-\Delta u + c\dfrac{u}{|x|^2} + |x|^\sigma |u|^q u = 0$. In particular we prove that the latter has exactly one family of radial solutions.

1. Introduction

The aim of the present note is to discuss the nonexistence result of solutions, in the class of radial function, to the problem

$$-\Delta u + c\frac{u}{|x|^2} + |x|^\sigma |u|^q u \leq 0, \tag{1.1}$$

posed in $B \subset \mathbb{R}^N$, $N \geq 3$, or in $B \setminus \{0\}$ where B is the unit ball centered at the origin, σ, $c \in \mathbb{R}$ and q is a nonnegative parameter larger than $(2+\sigma)/(N-2)$.

This paper is motivated by the results of [**1**]. The authors have proved that any solution, $u \in C^2(B \setminus \{0\})$, to

$$-\Delta u + |u|^q u = 0, \quad \text{in } B \setminus \{0\}, \tag{1.2}$$

can be extended to all B when $q \geq \frac{2}{N-2}$. We obtain here sufficient conditions which guaranty that (1.1) and then (1.2) have no radial solution, and we complete the results obtained in [**5**].

The main ingredients used is to derive properties of solutions to

$$v''(t) + lv'(t) \geq bv(t) + |v|^q v(t), \quad \text{for } t \geq 0. \tag{1.3}$$

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As a consequence of the main result concerning (1.3) we give quite a complete classification of radial solutions to

$$-u'' - \frac{N-1}{r}u' + c\frac{u}{r^2} + r^\sigma |u|^q u = 0, \quad \text{in } r_0 < r < 1, \tag{1.4}$$

subject to the condition

$$u(1) = \alpha, \quad u'(1) = \gamma,$$

where $\alpha, \gamma \in \mathbb{R}$ and $r_0 \geq 0$. In particular we prove that

$$\begin{aligned} -\Delta u + c\frac{u}{|x|^2} + |u|^q u = 0, &\quad \text{in } B, \\ u = \alpha, &\quad \text{in } \partial B, \end{aligned}$$

has no nontrivial radial solution for any $\gamma \neq f(\alpha)$.

2. Properties of radial subsolutions

Let B denotes the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$, and $q > 0$. We are concerned with the existence and the nonexistence of radial solutions to

$$-\Delta u + c\frac{u}{|x|^2} + |x|^\sigma |u|^q u = 0, \tag{2.1}$$

posed in the domain B or $B \setminus \{0\}$. To begin with, we give in this section some results concerning radial solutions of the following problem

$$-\Delta u + c\frac{u}{|x|^2} + |x|^\sigma |u|^q u \leq 0, \quad 0 < r_0 < |x| < 1. \tag{2.2}$$

We capture our conclusions in the following.

Theorem 2.1. *Let $q > 0$, assume that*

$$\frac{(2+\sigma)(N-2)}{q^2}\left\{q - \frac{2+\sigma}{N-2}\right\} + c \geq 0.$$

Then the problem

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + c\dfrac{u}{r^2} + r^\sigma |u|^q u \leq 0, & \textit{in } 0 < r < 1, \\ -u'(1) \geq \dfrac{2+\sigma}{q}u(1),\ u(1) > 0 \end{cases}$$

has no solution.

Since the function $v(x) = \lambda^{\frac{2+\sigma}{q}} u(\lambda x)$ satisfies the same equation it is natural to consider the function

$$v(t) := r^{\frac{2+\sigma}{q}} u(r),$$

where

$$t := -\log(r).$$

In view of (2.2) v satisfies

$$v'' + lv' \geq bv + |v|^q v, \quad t \in (0, T_0), \tag{2.3}$$

where

$$l := \frac{N-2}{q} \left\{ \frac{2(2+\sigma)}{N-2} - q \right\}, \quad T_0 \in (0, +\infty],$$

and

$$b := \frac{(2+\sigma)(N-2)}{q^2} \left\{ q - \frac{2+\sigma}{N-2} \right\} + c.$$

For the proof of Theorem 2.1 we shall show.

Lemma 2.1. *Let v be a solution to* (2.3) *such that $b \geq 0$, and suppose that*

$$v(0) > 0, \quad v'(0) \geq 0.$$

Then v blows up at finite point $T^\star \geq T_0$, i.e.,

$$\lim_{t \to T^{\star -}} v(t) = +\infty.$$

Proof: By continuity of v we have $v > 0$ on $(0, \varepsilon)$, and then $v' > 0$ on $(0, \varepsilon)$, thanks to (2.3). Assume that v has a positive local maximum at t_0. Using (2.3) we arrive at $v''(t_0) \geq bv(t_0) + v^{q+1}(t_0) > 0$. This is impossible, then $v'(t) \geq 0$, for any t. Suppose now $v'(t_0) = 0$ and $v' > 0$ on $(t_0 - \varepsilon, t_0)$. Using again (2.3) one sees $v''(t_0) > 0$ and then $v' < 0$ on $(t_0 - \varepsilon, t_0)$, a contradiction. This shows in particular that if $v(t)$ exists on $(0, T_0)$ then necessarily

$$v > 0, \quad v' > 0,$$

on $(0, T_0)$. From now on we assume that $T_0 = +\infty$. It is clear that v goes to infinity with t. Otherwise there exists t_n converging to infinity (since v has a finite limit at infinity) such that $v'(t_n)$ tends to 0. Integrating the inequality of v over $(0, t_n)$ and passing to the limit yield

$$-v'(0) + l(v(\infty) - v(0)) \geq \int_0^\infty v(b + v^q)\, ds,$$

which implies immediately that v is integrable and then $v(\infty) = 0$. This is impossible. Now as in [**8**] the function w defined by

$$w = \frac{v^2}{2},$$

satisfies

$$w'' \geq (v')^2 + bv^2 + \frac{2+q}{2} w^{\frac{2+q}{2}} - lvv'.$$

Hence, we have

$$w'' \geq 2^{\frac{2+q}{2}} w^{\frac{2+q}{2}},$$

if $l \leq 0$, and if $l > 0$,

$$w'' \geq C w^{\frac{2+q}{2}},$$

for t large enough, thanks to Young's inequality. Therefore w develops a singularity in a finite point, a contradiction. This ends the proof. $\square$

Note that, as inequation (2.3) is autonomous, if there exists $t_0 \in (0, T_0)$ such that $v(t_0) > 0$ and $v'(t_0) \geq 0$ the conclusion of the precedent lemma remains true. The following result shows that solutions to (2.3) may blow up at a finite point in the case where $v'(0) < 0$. This shows in particular that the condition, $v'(0) \geq 0$, is not essential as it seems to be asserted in [**8**, Remark 1.2, p. 295] (see Figure 3.1). To be more precise we have.

Lemma 2.2. *Assume that l and b are positive. Let v be a solution to* (2.3) *such that*

$$v(0) > 0, \quad 0 < -v'(0) \leq \sqrt{bv^2(0) + \frac{2}{q+2} v^{2+q}(0)}. \tag{2.4}$$

Then v cannot be global.

Proof: In view of hypothesis (2.4) v is nonnegative decreasing function for small t. Suppose that there exists $t_0 \in (0, T_0)$ such that $v > 0$, $v' < 0$ on $[0, t_0)$, $v'(t_0) = 0$ and $v(t_0) > 0$. It follows from above that v blows up at a finite point. Now assume that $v(t_0) = 0$ (and then $v'(t_0) \leq 0$).

Define

$$H(t) = (v')^2(t) - bv^2(t) - \frac{2}{q+2} v^{q+2}(t).$$

Using inequation (2.3) and the fact that $v' \le 0$ on $(0, t_0)$ we deduce that the function H is decreasing on $[0, t_0]$. Hence $H(0) \ge H(t) \ge 0$ for any $t \in [0, t_0]$. Therefore $H(t) = 0$, in $(0, t_0)$ which implies

$$-v' = \sqrt{bv^2 + \frac{2}{q+2} v^{q+2}}, \quad \forall\, 0 \le t \le t_0.$$

Using this and (2.3) we get a contradiction. Now assume that v is global. This shows immediately that $v > 0$ on $(0, \infty)$ and $v'(t) < 0$ for any $t \ge 0$. Arguing as above we deduce that v and then H tend to 0 as $t \to \infty$. Hence $H(t) = 0$ for all t, a contradiction. □

Corollary 2.1. *Let $l \ge 0$, $\alpha > 0$ and β fixed, set $b_c := \left[\frac{\beta^2}{\alpha^2} - \frac{2}{q+2}\alpha^q\right]_+$, where $[\cdot]_+ = \max\{\cdot, 0\}$.*

Then any solution to the following differential inequality

$$v'' + lv' \ge bv + |v|^q v, \quad v(0) = \alpha,\ v'(0) = \beta, \tag{2.5}$$

blows up at a finite point for any $b \ge b_c$.

Remark 2.1. Consider

$$v_s(t) := \alpha e^{\frac{\beta}{\alpha} t},$$

where $\alpha > 0$ and $\beta < 0$. The function v_s will be a solution to (2.1) provided

$$\frac{\beta^2}{\alpha^2} + l\frac{\beta}{\alpha} - \alpha^q \ge b. \tag{2.6}$$

Consequently if we return to the original variables, x and u, the function

$$u_s(x) = \alpha |x|^{-\frac{2+\sigma}{q} - \frac{\beta}{\alpha}},$$

satisfies (2.3).

Remark 2.2. We point out that condition (2.4) reads

$$\beta < 0, \quad \frac{\beta^2}{\alpha^2} - \frac{2}{q+2}\alpha^q \le b,$$

and the last inequality prevents v to be zero. This property is the key of the proof of Lemma 2.2. Now if we impose the following

$$\beta < 0, \quad \frac{\beta^2}{\alpha^2} - \frac{2}{q+2}\alpha^q > b,$$

where $\alpha > 0$, can one prove that v is global?

In the case of equality in (2.3) instead of inequality, we shall see that if there exists a finite time t_0 such that $v(t_0) = 0$ and $v'(t_0) \neq 0$ then v is not global.

3. Classification of radial solutions

In this section we are interested in radial solutions to the equation

$$-\Delta u + c\frac{u}{|x|^2} + |x|^\sigma |u|^q u = 0, \tag{3.1}$$

posed in the domain $\{r_0 < |x| < 1\}$, $r_0 \geq 0$.

By using the shooting argument we give a complete classification of solutions according to their boundary data $u(1)$. As in the preceding section it is enough to study the problem

$$v'' + lv' = bv + |v|^q v, \quad t \in (0, T_{\alpha,\beta}), \quad T_{\alpha,\beta} \leq \infty, \tag{3.2}$$

subject to the initial condition

$$v(0) = \alpha, \quad v'(0) = \beta, \tag{3.3}$$

where

$$l := \frac{N-2}{q}\left\{\frac{2(2+\sigma)}{N-2} - q\right\},$$

$$b := \frac{(2+\sigma)(N-2)}{q^2}\left\{q - \frac{2+\sigma}{N-2}\right\} + c,$$

and α, β are real parameters. Since the function $-v$ is also a solution of (3.2) we restrict our considerations to solutions of (3.2)–(3.3) where $\alpha \geq 0$. This problem has a unique local solution, $v_{\alpha,\beta} \in C^2([0,\varepsilon))$, we shall investigate whether it admits an entire extension, and its properties. We assume that $b \geq 0$. Let us note that if there exists $t_0 \in [0, T_{\alpha,\beta})$ such that $v(t_0) > 0$ and $v'(t_0) \geq 0$ then $v_{\alpha,\beta}$ is not global; that is $T_{\alpha,\beta} < \infty$. And then $\lim_{t \to T^-_{\alpha,\beta}} v_{\alpha,\beta}(t) = +\infty$. This proves immediately.

Proposition 3.1. *Let $v_{0,\beta}$ be the solution to* (3.2)–(3.3) *where $\alpha = 0$. If $T_{0,\beta} = \infty$ then $\beta = 0$ and therefore $v_{0,\beta} \equiv 0$.*

Proof: Assume $\beta > 0$. Then $v_{0,\beta}(\varepsilon) > 0$ and $v'_{0,\beta}(\varepsilon) > 0$ for $\varepsilon > 0$ small enough. And then $v_{0,\beta}$ is not global. Now if $\beta < 0$, we consider $-v_{0,\beta}$ instead of $v_{0,\beta}$. □

The following result gives the complete classification of solutions to (3.2)–(3.3) in terms of α and β (see Figure 3.1). In particular we prove, by using the phase plane method that the problem has exactly two non trivial global solutions, up to translations in t, in the case where $b > 0$.

Theorem 3.1. *Assume $b > 0$. Let $v_{\alpha,\beta}$ be the unique local solution to (3.2)–(3.3) and $(0, T_{\alpha,\beta})$ be the maximal interval of existence. There exists exactly two nontrivial global solutions ($T_{\alpha_c,\beta_c} = \infty$) $\pm v_{\alpha_c,\beta_c}$ where $\alpha_c > 0$, $\beta_c < 0$, and v_{α_c,β_c} satisfies*

$$v_{\alpha_c,\beta_c}(t) = (A + o(1))e^{-\frac{l+\sqrt{l^2+4b}}{2}t},$$

as $t \to +\infty$, moreover there exists a unique continuous function, f, such that $f(\alpha_c) = \beta_c$, $f(0) = 0$ and for any $(\alpha, \beta) \notin \{\pm(\gamma, f(\gamma)), \gamma \geq 0\}$ we have

$$T_{\alpha,\beta} < \infty.$$

Proof: It is obvious that $T_{\alpha,\beta} < \infty$ if $\alpha \cdot \beta > 0$. Suppose that $\alpha > 0$ and $\beta < 0$. One of the following three possibilities holds:

i) $v_{\alpha,\beta}$ is nonnegative global function and goes to 0,
ii) $v_{\alpha,\beta}$ is nonnegative and $T_{\alpha,\beta} < \infty$,
iii) $v_{\alpha,\beta}$ changes sign.

Assume that item iii) holds. Set $w = -v_{\alpha,\beta}$. Clearly we have

$$w(t_0) = 0, \text{ and } w'(t_0) > 0.$$

Since w satisfies equation (3.1) we deduce that w blows-up; that is $T_{\alpha,\beta} < \infty$.

For the first case we investigate in more detail how $(v_{\alpha,\beta}, v'_{\alpha,\beta})$ behaves in the phase plane as t increases. Equation (3.2) is reduced to the first order system

$$\begin{cases} v' = w, \\ w' = bv - lw + |v|^q v, \end{cases} \tag{3.4}$$

with the initial condition

$$v(0) = \alpha, \quad w(0) = \beta. \tag{3.5}$$

This system has only one critical point $(0, 0)$. The linearization at the equilibrium gives

$$\begin{cases} v' = w, \\ w' = bv - lw. \end{cases} \tag{3.6}$$

The corresponding eigenvalues are

$$\lambda_1 = \frac{-l+\sqrt{l^2+4b}}{2}, \quad \lambda_2 = -\frac{l+\sqrt{l^2+4b}}{2}.$$

It follows from the standard theory of dynamical systems that $(0,0)$ is a saddle point. This means that (3.4)–(3.5) has exactly two nontrivial solutions, $\pm v_{\alpha_c,\beta_c}$, which converge to 0 as t goes to infinity. Moreover

$$v_{\alpha_c,\beta_c}(t) = (A+o(1))e^{-\frac{l+\sqrt{l^2+4b}}{2}t},$$

for sufficiently large t. $\square$

As a consequence of this theorem we have

Theorem 3.2. *Let $\Omega = B\setminus\{0\}$ or B. There exists a unique continuous function F such that for any $u \in C^2(\Omega)$ radial solution to*

$$-\Delta u + c\frac{u}{|x|^2} + |x|^\sigma |u|^q u = 0, \quad \text{in } \Omega,$$

where

$$q > 0, \quad \frac{(2+\sigma)(N-2)}{q^2}\left\{q - \frac{2+\sigma}{N-2}\right\} + c > 0,$$

we have

$$\left(u(x), \frac{\partial u}{\partial \nu}(x)\right) = \pm(\alpha, F(\alpha))$$

on ∂B, for some real α, where ν is the unit outward normal to $\partial\Omega$.

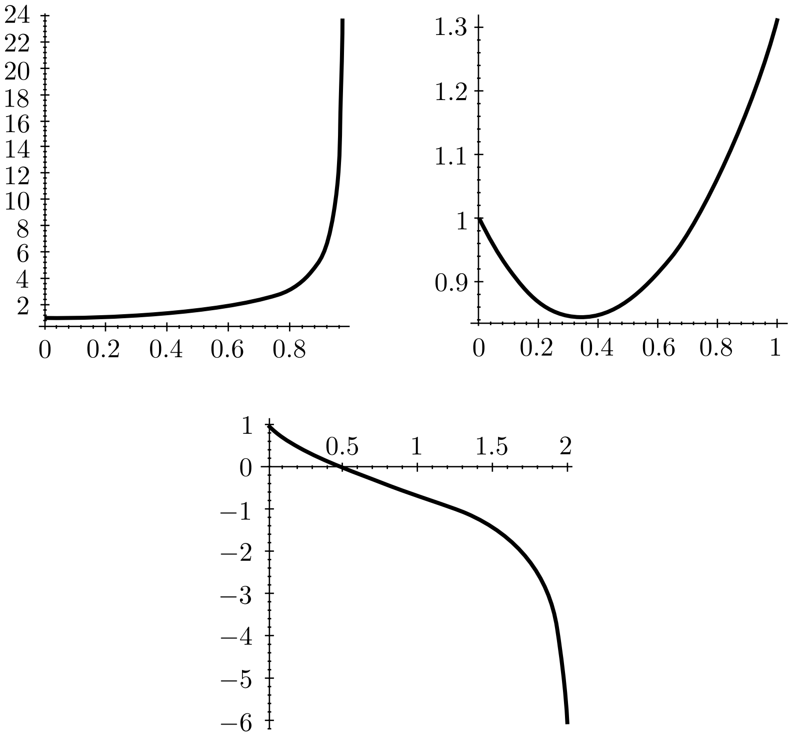

FIGURE 3.1. Classification of solutions according to $v(0)$ and $v'(0)(b > 0)$.

4. Properties of radial solutions to $-\Delta u + c\frac{x}{|x|}\nabla u + |u|^q u = 0$

We end this paper by another application of the results of Section 2. Let us consider the problem

$$-\Delta u + c\frac{x}{|x|}\nabla u + |u|^q u = 0, \quad \text{in } B \setminus \{0\}, \tag{4.1}$$

where $q > 0$ and $c \in \mathbb{R}$. This equation appears in the study of solutions to $-\Delta u + |u|^q u = 0$ which are singular on some submanifold [**3**]. Here we shall complete the properties of radial solutions obtained in [**3**], so we consider the following problem

$$-u_{rr} - \frac{N-1}{r}u_r + cu_r + |u|^q u = 0, \quad 0 < r < 1. \tag{4.2}$$

Using the function w defined by

$$w(t) = r^{\frac{2}{q}} u(r), \quad t = -\log(r),$$

we give some properties of global solutions to the following ordinary differential equation

$$\begin{aligned} (4.3) \quad w'' + \left[\frac{4}{q} + 2 - N - ce^{-t}\right] w' \\ = \frac{2}{q}\left[N - 2 - \frac{2}{q} + \frac{2c}{q} e^{-t}\right] w + |w|^q w, \quad t > 0, \end{aligned}$$

where c is real parameter.

Proposition 4.1. *Let $c \in \mathbb{R}$. Assume $q > \frac{2}{N-2}$. Then any global non trivial solution, w, to* (4.3)*, is asymptotically monotone, goes to* 0 *and*

$$w(t) = A\left(e^{-\frac{2}{q}t} + o(1)\right), \tag{4.4}$$

as t approaches infinity.

Proof: Let w be a global solution to (4.3). Assume that there exists a sequence $\{t_n\}$ converging to infinity with n such that $w'(t_n) = 0$, $w(t_n)$ is local minimum and (from the equation)

$$w''(t_n) = \left[\frac{2}{q}\left(N - 2 - \frac{2}{q} + \frac{2c}{q} e^{-t_n}\right) + |w(t_n)|^q\right] w(t_n).$$

Thus $w(t_n) > 0$. For fixed n large, the function $v(t) = w(t+t_n)$, satisfies (4.3) with ce^{-t_n} instead of c, moreover, $v(0) > 0$ and $v'(0) = 0$. Using the result of Section 2 we deduce that v is not global which is impossible. Therefore w is asymptotically monotone. Now since $-w$ satisfies (4.3), we may assume that

$$w > 0, \quad w' < 0,$$

on $(t_0, +\infty)$, t_0 for some t_0 large. A simple integration of equation (4.3) implies that w approaches 0 as t tends infinity and then $w' \to 0$ as $t \to \infty$, by using the function $H = \frac{1}{2}(w')^2 - \frac{1}{q+2} w^{q+2}$.

Next to get (4.4) we put

$$x(t) = e^{-t}, \quad v = w',$$

then the function (w, v, x) satisfies

$$
\begin{cases}
w' = v, \\
v' = \dfrac{2}{q}\left[N - 2 - \dfrac{2}{q} + \dfrac{2c}{q}e^{-t}\right] w \\
\qquad - \left[\dfrac{4}{q} + 2 - N - ce^{-t}\right] v + |w|^q w, \\
x' = -x,
\end{cases}
\tag{4.5}
$$

and

$$
\lim_{t\to\infty} (w, v, x) = (0, 0, 0). \tag{4.6}
$$

The characteristic roots of the linearized system of (4.5) are

$$
\lambda_1 = -\frac{2}{q}, \quad \lambda_2 = N - 2 - \frac{2}{q}, \quad \lambda_3 = -1.
$$

Then any solution to (4.5)–(4.6) satisfies

$$
\frac{v}{w} = -\frac{2}{q} + o(1),
$$

as $t \to \infty$, and this implies (4.4). □

Remark 4.1. If we return to the function u we get that the limit

$$
\lim_{r\to 0} u(r) = A
$$

exists. This means that the singularity is removable [**3**], [**9**].

Remark 4.2. In [**4**] a similar result is obtained for

$$
\gamma^2 w'' + \left(\gamma^2 + \frac{1}{2}\gamma e^{-2t/\gamma}\right) w' + \lambda w + w^q = 0,
$$

where $\gamma > 0$ and $\lambda < 0$. This equation appears in the study of positive radial solutions to

$$
\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{1}{q-1} u + u^q = 0,
$$

in $B \setminus \{0\}$. It is shown that if $\frac{2}{N-2} < q < \frac{4}{N-2}$ then w has a finite limit at infinity.

The following is an extension of Proposition 4.1.

Proposition 4.2. *Assume $q > \frac{2}{N-2}$. Let $u \in C^2(B \setminus \{0\})$ be a radial solution to*

$$-\Delta u + c\frac{x}{|x|}u + |u|^q u = 0,$$

in $B \setminus \{0\}$, vanishing on ∂B. Then $u \equiv 0$.

The situation is different in the case where $q < \frac{2}{N-2}$. In [**3**] it is shown the existence of radial positive function $u_\gamma \in L^{q+1}(B) \cap C^2(B \setminus \{0\})$ such that

$$-\Delta u_\gamma + c\frac{x}{|x|}u_\gamma + |u_\gamma|^q u_\gamma = \gamma(N-2)|S^{N-1}|\delta_0, \quad \text{in } D'(B \setminus \{0\}),$$

and vanishing on ∂B for all $\gamma > 0$.

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References

[1] H. Brézis and L. Véron, Removable singularities for some nonlinear elliptic equations, *Arch. Rational Mech. Anal.* **75(1)** (1980/81), 1–6.

[2] B. Gidas, Wei-Ming Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68(3)** (1979), 209–243.

[3] M. Grillot, Solutions singulières sur une sous-variété d'équations elliptiques non linéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **322(1)** (1996), 49–54.

[4] M. Guedda and M. Kirane, A note on singularities in semilinear problems, *Trans. Amer. Math. Soc.* **347(9)** (1995), 3595–3603.

[5] B. Guerch and L. Véron, Local properties of stationary solutions of some nonlinear singular Schrödinger equations, *Rev. Mat. Iberoamericana* **7(1)** (1991), 65–114.

[6] M. Marcus and L. Véron, Trace au bord des solutions positives d'équations elliptiques non linéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **321(2)** (1995), 179–184.

[7] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, *Arch. Rational Mech. Anal.* **114(3)** (1998), 201–231.

[8] Ph. Souplet, Nonexistence of global solutions to some differential inequalities of the second order and applications, *Portugal. Math.* **52(3)** (1995), 289–299.

[9] L. Véron, Singular solutions of some nonlinear elliptic equations, *Nonlinear Anal.* **5(3)** (1981), 225–242.

LAMFA, CNRS FRE 2270
Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33 Rue Saint-Leu
80039 Amiens
France
E-mail address: guedda@u-picardie.fr

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