Publicacions Matemàtiques, Vol **40** (1996), 21–39.

INTEGRABILITY OF A LINEAR CENTER PERTURBED BY A FOURTH DEGREE HOMOGENEOUS POLYNOMIAL*

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Abstract

In this work we study the integrability of a two-dimensional autonomous system in the plane with linear part of center type and non-linear part given by homogeneous polynomials of fourth degree. We give sufficient conditions for integrability in polar coordinates. Finally we establish a conjecture about the independence of the two classes of parameters which appear in the system; if this conjecture is true the integrable cases found will be the only possible ones.

1. Introduction

We consider the system

$$\begin{aligned} \dot{x} &= -y + X_s(x,y), \\ \dot{y} &= x + Y_s(x,y), \end{aligned} \tag{1.1}$$

where $X_s(x,y)$ and $Y_s(x,y)$ are homogeneous polynomials of degree s, with $s \geq 2$.

The aim of this paper is to find the integrable cases of system (1.1) when $s = 4$ (see Theorem 1). The integrable cases for quadratic systems, $s = 2$, and cubic homogeneous systems, $s = 3$, have been studied by several authors: Bautin [**1**], Chavarriga [**2**], Coppel [**5**], Frommer [**6**], Kapteyn [**7**], Lloyd [**8**], Lunkevich and Sibirskii [**9**], Schlomiuk [**11**] and Żoladek [**15**]. Poincaré [**10**] developed an important technique for the general solution of these problems. It consists in finding a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{1.2}$$

*Research partially supported by a University of Lleida Project/94.

where $H_2(x,y) = \frac{(x^2+y^2)}{2}$, and for each n, $H_n(x,y)$ is a homogeneous polynomial of degree n, so that the derivative of H along the solutions of system (1.1) satisfies

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k,$$

where V_{2k} are real numbers called *Lyapunov constants*. The vanishing of all Lyapunov constants is a necessary condition for the integrability of the system (1.1); in this case the series $H(x,y)$ would be a first integral of the system if it converges. It is not known if the series converges when the Lyapunov constants vanish. On the other hand, it is not possible in general to express this first integral (if it exists) by means of elementary functions.

Lyapunov constants are polynomials in the variables given by the coefficients of the polynomials $X_s(x,y)$ and $Y_s(x,y)$. Thus the ideal generated by V_{2k} has a finite number of generators by Hilbert's Theorem. We denote by $M(s)$ the minimum number of generators of such an ideal. It is well known that the number of small amplitudes limit cycles around the origin is at least $M(s)$ (see for instance [**12**]). The vanishing of these generators is a sufficient condition for the vanishing of all Lyapunov constants and for the integrability of the system.

In Section 2 we give without proof some known results which are used in the proof of Theorem 1. In Section 3 we prove Theorem 1. This theorem characterizes the integrable cases by means of polar coordinates for $s = 4$. Finally, we give an appendix on the computation of the Lyapunov constants for $s = 4$.

2. Some preliminary results

In the study of this problem we use polar coordinates. In Lemma 1 we give the expression of system (1.1) in polar coordinates. In Proposition 1 we give the expression of series (1.2) in these coordinates.

Lemma 1. *In polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ we can write system (1.1) as*

$$\begin{aligned} \dot{r} &= P_s(\varphi) r^s, \\ \dot{\varphi} &= 1 + Q_s(\varphi) r^{s-1}, \end{aligned} \tag{2.1}$$

where $P_s(\varphi)$ and $Q_s(\varphi)$ are trigonometric polynomials of the form

$$
\begin{aligned}
P_s(\varphi) &= R_{s+1}\cos\left((s+1)\varphi+\varphi_{s+1}\right)+R_{s-1}\cos\left((s-1)\varphi+\varphi_{s-1}\right)\\
&\quad+\cdots+\begin{cases} R_1\cos(\varphi+\varphi_1) & \text{if } s \text{ is even;}\\ R_0 & \text{if } s \text{ is odd;}\end{cases}\\
Q_s(\varphi) &= -R_{s+1}\sin\left((s+1)\varphi+\varphi_{s+1}\right)+r_{s-1}\sin\big((s-1)\varphi+\overline{\varphi}_{s-1}\big)\\
&\quad+\cdots+\begin{cases} r_1\sin(\varphi+\overline{\varphi}_1) & \text{if } s \text{ is even;}\\ r_0 & \text{if } s \text{ is odd;}\end{cases}
\end{aligned}
$$

being R_s, r_s, φ_s, $\overline{\varphi}_s$ arbitrary coefficients.

Proposition 1. *In polar coordinates series (1.2) for system (1.1) is $H(r,\varphi) = \sum_{m=0}^{\infty}\overline{H}_m(\varphi)r^{m(s-1)+2}$ where $\overline{H}_0(\varphi) = \frac{1}{2}$ and $\overline{H}_m(\varphi)$, $m = 0,1,\dots,$ are homogeneous trigonometric polynomials of degree $m(s-1)+2$, satisfying the differential equations*

$$
\begin{aligned}
\frac{d\overline{H}_{m+1}}{d\varphi}+(m(s-1)+2)\,\overline{H}_mP_s(\varphi)+\frac{d\overline{H}_m}{d\varphi}Q_s(\varphi)&\\
=\begin{cases} 0 & \text{if } (m+1)(s-1)+2 \text{ is odd,}\\ V_{(m+1)(s-1)+2} & \text{if } (m+1)(s-1)+2 \text{ is even,}\end{cases}&
\end{aligned}
\tag{2.2}
$$

where $V_{(m+1)(s-1)+2}$, $m = 0,1,\dots,$ are the Lyapunov constants.

Lemma 1 and Proposition 1 are proved in [**2**].

In particular, for $s = 4$ system (1.1) takes the form

$$
\begin{aligned}
\dot r &= P_4(\varphi)r^4,\\
\dot\varphi &= 1+Q_4(\varphi)r^3,
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
P_4(\varphi) &= R_5\cos(5\varphi+\varphi_5)+R_3\cos(3\varphi+\varphi_3)+R_1\cos(\varphi+\varphi_1),\\
Q_4(\varphi) &= -R_5\sin(5\varphi+\varphi_5)+r_3\sin(3\varphi+\overline{\varphi}_3)+r_1\sin(\varphi+\overline{\varphi}_1).
\end{aligned}
\tag{2.4}
$$

In this case the evaluation of $\dot H(r,\varphi)$ from system (2.3) yields

$$
\begin{aligned}
\frac{d\overline{H}_{m+1}}{d\varphi}+(3m+2)\overline{H}_mP_4(\varphi)+\frac{d\overline{H}_m}{d\varphi}Q_4(\varphi)&=\overline{V}_{m+1}\\
&=\begin{cases} 0 & \text{if } 3(m+1)+2 \text{ is odd;}\\ V_{3(m+1)+2} & \text{if } 3(m+1)+2 \text{ is even;}\end{cases}
\end{aligned}
$$

for $m = 0,1,\dots,$ with $\overline{H}_0(\varphi) = \frac{1}{2}$ and $\overline{H}_m(\varphi) = H_{3m+2}(\varphi)$.

In Proposition 2 we will give the general form of the symmetric integrable systems. In Proposition 3 we will give a class of integrable systems which have an integrant factor given by a quadratic polynomial in the variable r^{s-1} whose coefficients are functions in φ.

Proposition 2. *In the following two cases system (2.1) is integrable (in the sense that all its Lyapunov constants vanish):*

(i) $(s+1)P_s(\varphi) + \frac{dQ_s(\varphi)}{d\varphi} = 0$.
(ii) $P_s(\varphi)$ *and* $Q_s(\varphi)$ *are of the form*

$$\begin{aligned} P_s(\varphi) &= R_{s+1}\sin(s+1)\omega + R_{s-1}\sin(s-1)\omega \\ &\quad + \cdots + \begin{cases} R_1 \sin\omega & \text{if } s \text{ is even;} \\ R_2 \sin 2\omega & \text{if } s \text{ is odd;} \end{cases} \\ Q_s(\varphi) &= R_{s+1}\cos(s+1)\omega + r_{s-1}\cos(s-1)\omega \\ &\quad + \cdots + \begin{cases} r_1 \cos\omega & \text{if } s \text{ is even;} \\ r_2 \cos 2\omega + r_0 & \text{if } s \text{ is odd;} \end{cases} \end{aligned}$$

where $\omega = \varphi + \varphi_0$ *and* φ_0 *and the coefficients* R_j *and* r_j *are arbitrary.*

Systems satisfaying (i) have null divergence and those satisfying (ii) have a certain resonance between the angular parameters φ_j, $\overline{\varphi}_j$.

Proposition 3. *For* $s \in \mathbb{N}$ *with* $s \geq 2$ *and arbitrary* k_1, k_2, $\varphi_0 \in \mathbb{R}$ *system (2.1) with*

$$\begin{aligned} P_s(\varphi) &= 2\left(-k_1\cos^{s-2}(\varphi+\varphi_0)\sin^3(\varphi+\varphi_0)\right. \\ &\qquad \left.+k_2\sin^{s-2}(\varphi+\varphi_0)\cos^3(\varphi+\varphi_0)\right), \\ Q_s(\varphi) &= \left(k_1\cos^{s-1}(\varphi+\varphi_0) - k_2\sin^{s-1}(\varphi+\varphi_0)\right)\cos 2(\varphi+\varphi_0), \end{aligned} \tag{2.5}$$

is integrable.

In cartesian coordinates $x = r\cos(\varphi+\varphi_0)$ and $y = r\sin(\varphi+\varphi_0)$ we can write system (2.5) in the form

$$\begin{aligned} \dot{x} &= -y - k_1x^{s-1}y + k_2y^{s-2}(2x^2-y^2), \\ \dot{y} &= x + k_1x^{s-2}(x^2-2y^2) + k_2xy^{s-1}, \end{aligned} \tag{2.6}$$

with $s \geq 2$. We note that the origin is a center for system (2.4).

Proposition 2 is proved in [**2**] and Proposition 3 is proved in [**3**].

In order to establish the cases of Theorem 1, we have used a certain simplification expressed in the form of a conjecture. This conjecture simplifies the great number of factors that appear in the different Lyapunov constants. This assumption, which we think to be always satisfied, is stated as Conjecture 1.

Conjecture 1. *A necessary condition for all the Lyapunov constants of system (2.3) are zero is that the angular parameters* $(\varphi_5, \varphi_3, \overline{\varphi}_3, \varphi_1, \overline{\varphi}_1)$ *and the radial ones* $(R_5, R_3, r_3, R_1, r_1)$ *must be independent.*

With reference to the number of small amplitude limit cycles around the origin that appear when we perturbed system (2.3) inside the same class of systems, we note that in the third integrable case of Theorem 1 (Case 10 in its proof) the number of relations of parameters is seven. So if Conjecture 1 is true, we think that the number of small amplitude limit cycles is at least seven.

The Lyapunov constants were obtained using the computer algebra system Mathematica.

3. The main result

Theorem 1. *System (2.3) is integrable in the following cases:*

(i) $\overline{\varphi}_1 = \varphi_1$, $\overline{\varphi}_3 = \varphi_3$, $5R_3 + 3r_3 = 0$ *and* $5R_1 + r_1 = 0$.

(ii) $\overline{\varphi}_1 = \varphi_1$, $\overline{\varphi}_3 = \varphi_3$, $\varphi_5 = 5\varphi_1$ *and* $\varphi_3 = 3\varphi_1$.

(iii) $\overline{\varphi}_1 = \varphi_1$, $\overline{\varphi}_3 = \varphi_3$, $\varphi_5 = 2\varphi_1 + \varphi_3$, $r_3 = 3R_3$, $r_1 = 2R_1$, $R_5 = R_3$, *and* $|R_1| = 2|R_3|$.

(iv) $\overline{\varphi}_1 = \varphi_1$, $\overline{\varphi}_3 = \varphi_3$, $R_5 = 0$ *and*

$$\begin{cases} \text{(iv.1)}\ R_3 r_1 - 3r_3 R_1 = 0, \\ \text{(iv.2)}\ r_1 = 3R_1 \ \textit{and}\ r_3 = -3R_3, \\ \text{(iv.3)}\ r_1 = R_1 = |3R_3| \ \textit{and}\ r_3 = -3R_3. \end{cases}$$

This theorem is independent on Conjecture 1, but if Conjecture 1 is true then system (2.3) will be integrable *only* in the cases given by Theorem 1.

Proof of Theorem 1: The first not zero Lyapunov constant is

$$V_8 = -\frac{1}{2}\left(R_1 r_1 \sin(\varphi_1 - \overline{\varphi}_1) + R_3 r_3 \sin(\varphi_3 - \overline{\varphi}_3)\right).$$

In particular the previous constant vanishes when $\overline{\varphi}_1 = \varphi_1$ and $\overline{\varphi}_3 = \varphi_3$. On the other hand, if Conjecture 1 is true, we arrive at the same condition as above. With this assumption the next not zero Lyapunov constant is V_{14} given by

$$\begin{aligned} 40V_{14} = {} & 5\left((R_3 r_1 - 3r_3 R_1)(r_1 - 3R_1)(r_1 - R_1)\right)\sin(3\varphi_1 - \varphi_3) \\ & + \big(3(5r_1^2 - 6r_1 R_1 - 11R_1^2)r_3 + (11r_1^2 - 74r_1 R_1 + 75R_1^2)R_3\big) \\ & \qquad R_5 \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\ & + \big((15r_1 - 51R_1)r_3^2 + 22(r_1 - R_1)r_3 R_3 - (21r_1 - 25R_1)R_3^2\big) \\ & \qquad R_5 \sin(\varphi_1 - 2\varphi_3 + \varphi_5). \end{aligned}$$

The simultaneous vanishing of three factors of V_{14} respect to the radials parameters with arbitrary angular parameters leads to the following cases:

1. $5R_1 + r_1 = 0$ and $5R_3 + 3r_3 = 0$;
2. $R_1 = r_1 = 0$;
3. $R_3 = r_3 = 0$;
4. $R_5 = 0$, $R_3 r_1 - 3r_3 R_1 = 0$, $R_1^2 + r_1^2 \neq 0$ and $R_3^2 + r_3^2 \neq 0$;
5. $R_5 = 0$ and $r_1 - 3R_1 = 0$;
6. $R_5 = 0$ and $r_1 - R_1 = 0$.

If we impose that the values of the angular parameters are not arbitrary, the possible dependence relations between them are the following:

7. $\varphi_3 - 3\varphi_1 = 0$ and $\varphi_5 - 5\varphi_1 = 0$;
8. $\varphi_3 - 3\varphi_1 = 0$;
9. $\varphi_1 - 2\varphi_3 + \varphi_5 = 0$;
10. $2\varphi_1 + \varphi_3 - \varphi_5 = 0$;
11. $\varphi_5 - 5\varphi_1 = 0$;
12. $\varphi_5 - 3\varphi_3 + 4\varphi_1 = 0$.

Case 1: In this case it is easy to see that $5P_4 + Q_4' = 0$, where $' = \frac{d}{d\varphi}$, and the divergence of the vector field defined by system (2.3) is zero. So the system is integrable and all Lyapunov constants are zero (see Proposition 2).

Case 2: If $r_1 = R_1 = 0$ then the first not zero Lyapunov constant after V_{14} is V_{26}, that is

$$8400V_{26} = ((5R_3 + 3r_3)(3R_3 - r_3)(3R_3 - 5r_3)(R_3 - 3r_3)(29R_3 - 89r_3)) \\ R_5^3 \sin(5\varphi_3 - 3\varphi_5).$$

The vanishing of the first factor corresponds to Case 1. The situation when $R_3 = 0$ and $r_3 = 0$ corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of V_{26} allows to express r_3 in function of R_3. If we introduce these relations in the next non-zero constant V_{32} (see Appendix 2), we can find R_5 as a function of R_3. Finally by substituting them in the next non-zero Lyapunov constant V_{38} (see Appendix 2), we can see that this constant never vanishes. Therefore there is no possible integrable cases. The vanishing of R_5 will be seen later. The vanishing of $\sin(5\varphi_3 - 3\varphi_5)$ corresponds to Case 7.

Case 3: If $r_3 = R_3 = 0$ then the first non-zero Lyapunov constant after V_{14} is V_{20}, that is

$$320V_{20} = ((r_1 + 5R_1)(5r_1 - 3R_1)(2r_1 - 3R_1)(r_1 - 3R_1)(r_1 - 6R_1)) \\ R_5 \sin(5\varphi_1 - \varphi_5).$$

The vanishing of the first factor corresponds to Case 1. The situation when $R_1 = 0$ and $r_1 = 0$ corresponds to a degenerate case of zero divergence. The vanishing of the rest of the factors of V_{20} allows to express r_1 in function of R_1. If we introduce these relations in the next non-zero constant V_{26} (see Appendix 3), we can find R_5 as a function of R_1. Finally by substituting them in the next non-zero Lyapunov constant V_{32} (see Appendix 3), we can see that this constant does not vanish. Therefore there is no possible integrable cases. The vanishing of R_5 will be seen later. The vanishing of $\sin(5\varphi_1 - \varphi_5)$ corresponds to Case 7.

We now consider the case $R_5 = 0$. Then

$$V_{14} = \frac{1}{8}(R_3 r_1 - 3r_3 R_1)(r_1 - 3R_1)(r_1 - R_1)\sin(3\varphi_1 - \varphi_3).$$

If we impose that $V_{14} = 0$ we have four possibilities. The vanishing of $\sin(3\varphi_1 - \varphi_3)$ corresponds to Case 7. The others correspond to Cases 4, 5 and 6 which we will be studied.

Case 4: In this case $R_5 = 0$ and $R_3 r_1 - 3r_3 R_1 = 0$. If $r_1 \neq 0$ and $r_3 \neq 0$, then $P_4(\varphi) = \lambda Q_4'(\varphi)$ where $\lambda = R_1/r_1 = R_3/3r_3$. System (2.3) takes the form

$$\begin{aligned} \dot{r} &= \lambda r^4 Q_4'(\varphi), \\ \dot{\varphi} &= 1 + r^3 Q(\varphi). \end{aligned}$$

The previous system is integrable and its first integral is

$$H(r,\varphi) = \left(1 + (1+3\lambda)Q(\varphi)r^3\right) r^{-\left(3+\frac{1}{\lambda}\right)}.$$

If $r_1 = 0$ or $r_3 = 0$ implies $\dot{\varphi} = 1$ and system (2.3) is trivially integrable.

Case 5: In this case $R_5 = 0$ and $r_1 - 3R_1 = 0$. We can suppose $r_1 = 3R_1 \neq 0$, because $r_1 = R_1 = 0$ corresponds to a particular case of Case 7. Then

$$\begin{aligned} 64V_{20} &= 27(R_3 - r_3)^2(r_3 + 3R_3)R_1^3 \sin(3\varphi_1 - \varphi_3), \\ 17920V_{26} &= 3\left(5283r_3^2 - 4932r_3R_3 - 28943R_3^2 - 245232R_1^2\right) \\ &\quad (R_3 - r_3)^2(r_3 + 3R_3)R_1^3 \sin(3\varphi_1 - \varphi_3). \end{aligned}$$

The simultaneous vanishing of the two previous Lyapunov constants give rise to two possible cases either $r_3 + 3R_3 = 0$ or $r_3 - R_3 = 0$. The last one corresponds to a particular case of $R_3 r_1 - 3r_3 R_1 = 0$ which has been studied in Case 4. In the first case, that is $r_1 = 3R_1$ and $r_3 = -3R_3$, system (2.3) reduces to

$$\begin{aligned}\dot r &= r^4\left(R_3\cos(3\varphi+\varphi_3)+R_1\cos(\varphi+\varphi_1)\right),\\ \dot\varphi &= 1+r^3\left(-3R_3\sin(3\varphi+\varphi_3)+3R_1\sin(\varphi+\varphi_1)\right).\end{aligned}$$

If we make the change $R = r^3$ the system takes the form

$$\begin{aligned}\dot R &= 3R^2\left(R_3\cos(3\varphi+\varphi_3)+R_1\cos(\varphi+\varphi_1)\right),\\ \dot\varphi &= 1+3R\left(-R_3\sin(3\varphi+\varphi_3)+R_1\sin(\varphi+\varphi_1)\right),\end{aligned}$$

which corresponds to an integrable case of a quadratic system with linear part of center type, see for instance [**2**].

Case 6: In this case $R_5 = 0$ and $r_1 - R_1 = 0$. We can suppose $r_1 = R_1 \neq 0$, because $r_1 = R_1 = 0$ corresponds to a particular case of Case 7. Then

$$\begin{aligned}64V_{20} &= (R_3-3r_3)(-20r_1^2+17r_3^2-10R_3r_3-3R_3^2)\sin(3\varphi_1-\varphi_3),\\ 23040V_{26} &= \big(799200r_1^4+(2238104R_3^2+421824r_3R_3-699048r_3^2)r_1^2\\ &\quad +18603r_3^4-24450r_3^3R_3-66784r_3^2R_3^2+44402r_3R_3^3+13029R_3^4\big)\\ &\quad (R_3-3r_3)r_1^3\sin(3\varphi_1-\varphi_3).\end{aligned}$$

If $R_3 = 3r_3$ the above constants are zero and Case 6 corresponds to a particular case of $R_3r_1 - 3r_3R_1 = 0$ studied in Case 4. If the second factor of V_{20} is zero, we have

$$20r_1^2-17r_3^2+10R_3r_3+3R_3^2=0. \tag{3.1}$$

By substituting the expression obtained for r_1 from (3.1) in V_{26} we get

$$\begin{aligned}38400V_{26} = -(r_3+3R_3)(3r_3-R_3)^2(1019r_3^2-326r_3R_3-273R_3^2)\\ r_1^3\sin(3\varphi_1-\varphi_3).\end{aligned}$$

From the vanishing of the previous constant we obtain three possible cases: $R_3 = -3r_3$ (already studied), $1019r_3^2 - 326r_3R_3 - 273R_3^2 = 0$ and $r_3 = -3R_3$.

Let $1019r_3^2 - 326r_3R_3 - 273R_3^2 = 0$. From (3.1) we obtain $6520r_1^2 + 4648r_3^2 + 1752R_3^2 = 0$, which implies $r_1 = r_3 = R_3 = 0$. Finally let $r_3 = -3R_3$. By replacing the value of r_3 in (3.1) we obtain $r_1^2 = 9R_3^2$. So $r_1 = 3|R_3|$, and system (2.3) reduces to

$$\begin{aligned}\dot r &= r^4\left(R_3\cos(3\varphi+\varphi_3)+3|R_3|\cos(\varphi+\varphi_1)\right),\\ \dot\varphi &= 1+r^3\left(-3R_3\sin(3\varphi+\varphi_3)+3|R_3|\sin(\varphi+\varphi_1)\right).\end{aligned}$$

If we make the same change $R = r^3$, then we can write the system as

$$\begin{aligned}\dot{R} &= 3R^2 \left(R_3 \cos(3\varphi + \varphi_3) + 3|R_3| \cos(\varphi + \varphi_1)\right), \\ \dot{\varphi} &= 1 + 3R \left(-R_3 \sin(3\varphi + \varphi_3) + |R_3| \sin(\varphi + \varphi_1)\right),\end{aligned}$$

which corresponds to an integrable case of a quadratic system with linear part of center type, see for instance [**2**].

Case 7: We call this case *resonance angles*. Then system (2.3) is integrable (see Proposition 2).

Case 8: Let $\varphi_3 - 3\varphi_1 = 0$. We also assume that $5\varphi_1 - \varphi_5 \neq 0$ (the opposite case has been studied in Case 7). Then a term that appears in V_{26} and which must be zero independently on the rest is

$$\frac{1}{8400}((5R_3 + 3r_3)(3R_3 - r_3)(3R_3 - 5r_3)(R_3 - 3r_3)(29R_3 - 89r_3)) \\ R_5^3 \sin(15\varphi_1 - 3\varphi_5).$$

From the vanishing of the factors of the previous expression and their substitution in the constants V_{14}, V_{20} and the remaining terms of V_{26}, see Appendix 1, we obtain cases already studied.

Case 9: Let $\varphi_1 - 2\varphi_3 + \varphi_5 = 0$. We also assume $3\varphi_1 - \varphi_3 \neq 0$ (the opposite case has been studied in Case 7). Then a term that appears in V_{20} and which must be zero independently on the rest is

$$\frac{1}{320}((r_1 + 5R_1)(5r_1 - 3R_1)(2r_1 - 3R_1)(r_1 - 3R_1)(r_1 - 6R_1)) \\ R_5 \sin(6\varphi_1 - 2\varphi_3).$$

From the vanishing of the factors of the previous expression and their substitution into the constants V_{14}, V_{26} and the remaining terms of V_{20}, see Appendix 1, we obtain cases already studied.

Case 10: Let $2\varphi_1 + \varphi_3 - \varphi_5 = 0$. Then

$$\begin{aligned}40V_{14} = \big(&(5(R_3 r_1 - 3r_3 R_1)(r_1 - 3R_1)(r_1 - R_1)) \\ &+ \left((15r_1 - 51R_1)r_3^2 + 22(r_1 - R_1)r_3 R_3 - (21r_1 - 25R_1)R_3^2\right) R_5\big) \\ &\sin(3\varphi_1 - \varphi_3).\end{aligned}$$

From the vanishing of the previous constant (we can assume that $3\varphi_1 - \varphi_3 \neq 0$, the opposite case has been studied in Case 7) we obtain R_5 as a function of the remaining radial parameters. By substituting this value in V_{20} and V_{26} and imposing that these Lyapunov constants to be zero, we obtain a system of three equations since V_{26} has two terms with independent trigonometric part, see Appendix 1. By solving this system we obtain some cases already studied, and a new solution given by $R_1 = \frac{r_1}{2}$, $R_5 = R_3$, $r_3 = 3R_3$ and $R_3 = \left|\frac{r_1}{4}\right|$. These relations give rise to a new integrable case (see Proposition 3). System (2.3) takes the form

$$\begin{aligned}\dot{r} &= r^4 R_5 \left(\cos(5\varphi + 2\varphi_1 + \varphi_3) + \cos(3\varphi + \varphi_3) \pm 2\cos(\varphi + \varphi_1)\right), \\ \dot{\varphi} &= 1 + r^3 R_5 \left(-\sin(5\varphi + 2\varphi_1 + \varphi_3) + 3\sin(3\varphi + \varphi_3) \pm 4\sin(\varphi + \varphi_1)\right).\end{aligned} \tag{3.2}$$

For $R_3 = -\frac{r_1}{4}$ system (3.2) becomes

$$\begin{aligned}\dot{r} &= 2r^4 \left(-k_1 \sin(\varphi + \varphi_0) + k_2 \cos(\varphi + \varphi_0)\right) \sin^2(\varphi + \varphi_0) \cos^2(\varphi + \varphi_0), \\ \dot{\varphi} &= 1 + r^3 \left(k_1 \cos^3(\varphi + \varphi_0) - k_2 \sin^3(\varphi + \varphi_0)\right) \cos(2(\varphi + \varphi_0)),\end{aligned}$$

where $\varphi_0 = \frac{\varphi_1 + \varphi_3}{4}$, $k_1 = -8\sin\left[\frac{3\varphi_1 - \varphi_3}{4}\right]$ and $k_2 = -8\cos\left[\frac{3\varphi_1 - \varphi_3}{4}\right]$.

For $R_3 = \frac{r_1}{4}$ we make the change $\varphi = \omega + \pi/4$ and system (3.2) becomes

$$\begin{aligned}\dot{r} &= 2r^4 \left(-k_1 \sin(\omega + \omega_0) + k_2 \cos(\omega + \omega_0)\right) \sin^2(\omega + \omega_0) \cos^2(\omega + \omega_0), \\ \dot{\omega} &= 1 + r^3 \left(k_1 \cos^3(\omega + \omega_0) - k_2 \sin^3(\omega + \omega_0)\right) \cos(2(\omega + \omega_0)),\end{aligned}$$

where $\omega_0 = \frac{\varphi_1 + \varphi_3}{4}$, $k_1 = 8\sin\left[\frac{3\varphi_1 - \varphi_3}{4} + \frac{\pi}{4}\right]$ and $k_2 = 8\cos\left[\frac{3\varphi_1 - \varphi_3}{4} + \frac{\pi}{4}\right]$.

Cases 11 and 12: In these cases we do not find anything, except the integrable cases mentioned before.

Appendix 1

Lyapunov constants V_{14}, V_{20} and V_{26} if $\varphi_1 = \overline{\varphi}_1$ and $\varphi_3 = \overline{\varphi}_3$:

$$
\begin{gathered}
40V_{14} = 5\left((r_1 - 3R_1)(r_1 - R_1)(-3R_1 r_3 + r_1 R_3)\right)\sin(3\varphi_1 - \varphi_3) \\
+\left(15r_1^2 r_3 - 18r_1 R_1 r_3 - 33R_1^2 r_3 + 11r_1^2 R_3 - 74r_1 R_1 R_3 + 75R_1^2 R_3\right) \\
R_5 \sin(2\varphi_1 + \varphi_3 - \varphi_5) \\
+\left(15r_1 r_3^2 - 51R_1 r_3^2 + 22r_1 r_3 R_3 - 22R_1 r_3 R_3 - 21r_1 R_3^2 + 25R_1 R_3^2\right) \\
R_5 \sin(\varphi_1 - 2\varphi_3 + \varphi_5).
\end{gathered}
$$

$$
\begin{gathered}
144000V_{20} = 25\left(-1620r_1^4 R_1 r_3 + 4050r_1^3 R_1^2 r_3 + 137700r_1^2 R_1^3 r_3\right. \\
-541350r_1 R_1^4 r_3 + 406620R_1^5 r_3 - 4320r_1^2 R_1 r_3^3 + 20790r_1 R_1^2 r_3^3 \\
-21060R_1^3 r_3^3 + 540r_1^5 R_3 - 1350r_1^4 R_1 R_3 - 45900r_1^3 R_1^2 R_3 \\
+180450r_1^2 R_1^3 R_3 - 135540r_1 R_1^4 R_3 + 1440r_1^3 r_3^2 R_3 - 2520r_1^2 R_1 r_3^2 R_3 \\
-9540r_1 R_1^2 r_3^2 R_3 + 14850R_1^3 r_3^2 R_3 - 1470r_1^3 r_3 R_3^2 + 17340r_1^2 R_1 r_3 R_3^2 \\
-56280r_1 R_1^2 r_3 R_3^2 + 40320R_1^3 r_3 R_3^2 - 3940r_1^3 R_3^3 + 17110r_1^2 R_1 R_3^3 \\
-13440r_1 R_1^2 R_3^3 - 135r_1^3 r_3 R_5^2 - 16659r_1^2 R_1 r_3 R_5^2 + 73359r_1 R_1^2 r_3 R_5^2 \\
-69957R_1^3 r_3 R_5^2 + 5283r_1^3 R_3 R_5^2 - 23913r_1^2 R_1 R_3 R_5^2 \\
\left.+\, 26469r_1 R_1^2 R_3 R_5^2 - 3375R_1^3 R_3 R_5^2\right)\sin(3\varphi_1 - \varphi_3) \\
+90\left(-2005r_1 r_3^3 + 2872R_1 r_3^3 - 414r_1 r_3^2 R_3 + 4539R_1 r_3^2 R_3\right. \\
+2947r_1 r_3 R_3^2 - 6646R_1 r_3 R_3^2 - 848r_1 R_3^3 \\
\left.+\, 1475R_1 R_3^3\right) R_5^2 \sin(\varphi_1 + 3\varphi_3 - 2\varphi_5) \\
+450\left((r_1 - 6R_1)(2r_1 - 3R_1)(5r_1 - 3R_1)(-r_1 + 3R_1)\right. \\
\left.(r_1 + 5R_1)\right) R_5 \sin(5\varphi_1 - \varphi_5) \\
+R_5\left(63000r_1^4 r_3 - 133200r_1^3 R_1 r_3 - 3105000r_1^2 R_1^2 r_3\right. \\
+3596400r_1 R_1^3 r_3 + 7268400R_1^4 r_3 + 108000r_1^2 r_3^3 - 145125r_1 R_1 r_3^3 \\
-313200R_1^2 r_3^3 + 32400r_1^4 R_3 - 414000r_1^3 R_1 R_3 - 1351800r_1^2 R_1^2 R_3 \\
+15145200r_1 R_1^3 R_3 - 16335000R_1^4 R_3 - 55125r_1^2 r_3^2 R_3 \\
-420975r_1 R_1 r_3^2 R_3 + 941175R_1^2 r_3^2 R_3 - 396375r_1^2 r_3 R_3^2 \\
+919875r_1 R_1 r_3 R_3^2 + 150825R_1^2 r_3 R_3^2 - 189700r_1^2 R_3^3 \\
+1546225r_1 R_1 R_3^3 - 1680000R_1^2 R_3^3 + 68715r_1^2 r_3 R_5^2 \\
-111618r_1 R_1 r_3 R_5^2 - 109053R_1^2 r_3 R_5^2 + 50661r_1^2 R_3 R_5^2 \\
\left.-\, 343134r_1 R_1 R_3 R_5^2 + 629325R_1^2 R_3 R_5^2\right)\sin(2\varphi_1 + \varphi_3 - \varphi_5) \\
+R_5\left(108000r_1^3 r_3^2 - 661500r_1^2 R_1 r_3^2 - 2166750r_1 R_1^2 r_3^2\right.
\end{gathered}
$$

$$
\begin{gathered}
+10277550R_1^3r_3^2+63000r_1r_3^4-267525R_1r_3^4+117000r_1^3r_3R_3\\
-213300r_1^2R_1r_3R_3-4360500r_1R_1^2r_3R_3+4516200R_1^3r_3R_3\\
+77175r_1r_3^3R_3-48375R_1r_3^3R_3-259650r_1^3R_3^2+976050r_1^2R_1R_3^2\\
+3284100r_1R_1^2R_3^2-5040000R_1^3R_3^2-466875r_1r_3^2R_3^2\\
+1394325R_1r_3^2R_3^2-481775r_1r_3R_3^3+457775R_1r_3R_3^3\\
+498075r_1R_3^4-605000R_1R_3^4+73440r_1r_3^2R_5^2\\
-304776R_1r_3^2R_5^2+127872r_1r_3R_3R_5^2-155952R_1r_3R_3R_5^2\\
-71136r_1R_3^2R_5^2+185400R_1R_3^2R_5^2\big)\sin(\varphi_1-2\varphi_3+\varphi_5).
\end{gathered}
$$

$$
\begin{gathered}
1995840000V_{26}=2475R_5\big(31500r_1^7-229320r_1^6R_1-3226860r_1^5R_1^2\\
+22267980r_1^4R_1^3+38922660r_1^3R_1^4-393996960r_1^2R_1^5+601983900r_1R_1^6\\
-232186500R_1^7+176400r_1^5r_3^2-1408950r_1^4R_1r_3^2+34316631r_1^3R_1^2r_3^2\\
-145310598r_1^2R_1^3r_3^2+193311819r_1R_1^4r_3^2-71737362R_1^5r_3^2\\
+223110r_1^5r_3R_3-20756259r_1^4R_1r_3R_3+120925167r_1^3R_1^2r_3R_3\\
-243611151r_1^2R_1^3r_3R_3+12120243r_1R_1^4r_3R_3+294793050R_1^5r_3R_3\\
-725446r_1^5R_3^2-4407647r_1^4R_1R_3^2+95407266r_1^3R_1^2R_3^2\\
-448139203r_1^2R_1^3R_3^2+926142250r_1R_1^4R_3^2-620151000R_1^5R_3^2\\
+217545r_1^5R_5^2-1023621r_1^4R_1R_5^2-3370902r_1^3R_1^2R_5^2\\
+25845846r_1^2R_1^3R_5^2-45661203r_1R_1^4R_5^2+23776335R_1^5R_5^2\big)\sin(5\varphi_1-\varphi_5)\\
+900\big(1376595r_1^2r_3^4-22590630r_1R_1r_3^4+58656123R_1^2r_3^4\\
+17168520r_1^2r_3^3R_3-177991152r_1R_1r_3^3R_3+367268520R_1^2r_3^3R_3\\
+15459842r_1^2r_3^2R_3^2-53348860r_1R_1r_3^2R_3^2-123298102R_1^2r_3^2R_3^2\\
-27940120r_1^2r_3R_3^3+275319488r_1R_1r_3R_3^3-274457320R_1^2r_3R_3^3\\
+6854907r_1^2R_3^4-107758510r_1R_1R_3^4\\
+119730875R_1^2R_3^4\big)R_5^2\sin(2\varphi_1-4\varphi_3+2\varphi_5)\\
+9900\big(36225r_1^4r_3^3-414900r_1^3R_1r_3^3+5560362r_1^2R_1^2r_3^3\\
-20524968r_1R_1^3r_3^3+18080001R_1^4r_3^3+95775r_1^4r_3^2R_3\\
-5749548r_1^3R_1r_3^2R_3+48599826r_1^2R_1^2r_3^2R_3-135196320r_1R_1^3r_3^2R_3\\
+134036907R_1^4r_3^2R_3+430523r_1^4r_3R_3^2-9397688r_1^3R_1r_3R_3^2\\
+29371338r_1^2R_1^2r_3R_3^2\stackrel{-}{-}38087228r_1R_1^3r_3R_3^2\stackrel{+}{+}17244575R_1^4r_3R_3^2\\
\stackrel{-}{-}67159r_1^4R_3^3\stackrel{+}{+}7174896r_1^3R_1R_3^3-36221622r_1^2R_1^2R_3^3\\
+74045100r_1R_1^3R_3^3-49529375R_1^4R_3^3\big)R_5\sin(4\varphi_1-3\varphi_3+\varphi_5)
\end{gathered}
$$

$$
\begin{gathered}
+45R_5^2\,\big(188010900r_1^3r_3^3+277148475r_1^2R_1r_3^3-11275089480r_1R_1^2r_3^3\\
+16481281185R_1^3r_3^3+127378350r_1r_3^5-224994825R_1r_3^5\\
+406138275r_1^3r_3^2R_3-1976826630r_1^2R_1r_3^2R_3+1650294105r_1R_1^2r_3^2R_3\\
+11618192850R_1^3r_3^2R_3-106129650r_1r_3^4R_3-144600225R_1r_3^4R_3\\
-40820670r_1^3r_3R_3^2-2528927865r_1^2R_1r_3R_3^2+26140686450r_1R_1^2r_3R_3^2\\
-40014039075R_1^3r_3R_3^2-833488975r_1r_3^3R_3^2+1699300075R_1r_3^3R_3^2\\
-119309545r_1^3R_3^3+1977099700r_1^2R_1R_3^3-13674395475r_1R_1^2R_3^3\\
+15412320000R_1^3R_3^3+129133125r_1r_3^2R_3^3+881613975R_1r_3^2R_3^3\\
+866625925r_1r_3R_3^4-2044436350R_1r_3R_3^4-267013175r_1R_3^5\\
+478403750R_1R_3^5+141583815r_1r_3^3R_5^2-213503121R_1r_3^3R_5^2\\
+61169427r_1r_3^2R_3R_5^2-290758437R_1r_3^2R_3R_5^2-167905851r_1r_3R_3^2R_5^2\\
+513742653R_1r_3R_3^2R_5^2+45109089r_1R_3^3R_5^2\\
-127607175R_1R_3^3R_5^2\big)\sin(\varphi_1+3\varphi_3-2\varphi_5)\\
+R_5\big(-2058210000r_1^5r_3^2+13831884000r_1^4R_1r_3^2+210252588975r_1^3R_1^2r_3^2\\
-1372000268925r_1^2R_1^3r_3^2-1923091386975r_1R_1^4r_3^2\\
+12382675328925R_1^5r_3^2-4241160000r_1^3r_3^4+29529299250r_1^2R_1r_3^4\\
+113916777975r_1R_1^2r_3^4-615399800400R_1^3r_3^4-919957500r_1r_3^6\\
+4478206500R_1r_3^6-1564893000r_1^5r_3R_3+19971928350r_1^4R_1r_3R_3\\
+108674779950r_1^3R_1^2r_3R_3+167108074650r_1^2R_1^3r_3R_3\\
-6084308542950r_1R_1^4r_3R_3+5474234205000R_1^5r_3R_3\\
-2452254750r_1^3r_3^3R_3+14151396600r_1^2R_1r_3^3R_3\\
+90500362425r_1R_1^2r_3^3R_3+72915274575R_1^3r_3^3R_3-759888000r_1r_3^5R_3\\
-1032793875R_1r_3^5R_3+4201631775r_1^5R_3^2+5173227675r_1^4R_1R_3^2\\
-728892620775r_1^3R_1^2R_3^2+2780678258325r_1^2R_1^3R_3^2\\
+1510872165000r_1R_1^4R_3^2-4778061750000R_1^5R_3^2+48134738025r_1^3r_3^2R_3^2\\
-297515651400r_1^2R_1r_3^2R_3^2-400254737850r_1R_1^2r_3^2R_3^2\\
+2437146129825R_1^3r_3^2R_3^2+17095807125r_1r_3^4R_3^2-63625541625R_1r_3^4R_3^2\\
+38843573175r_1^3r_3R_3^3-220177744875r_1^2R_1r_3R_3^3\\
-514855484550r_1R_1^2r_3R_3^3+649385055000R_1^3r_3R_3^3\\
+18862131375r_1r_3^3R_3^3-14658559875R_1r_3^3R_3^3-67676355450r_1^3R_3^4\\
+344964715425r_1^2R_1R_3^4+460610865000r_1R_1^2R_3^4-972798750000R_1^3R_3^4\\
-73138105875r_1r_3^2R_3^4+206310721125R_1r_3^2R_3^4-67495697875r_1r_3R_3^5
\end{gathered}
$$

$$+63496446250R_1r_3R_3^5+69710238750r_1R_3^6-85032062500R_1R_3^6$$
$$-4606085925r_1^3r_3^2R_5^2+8490640320r_1^2R_1r_3^2R_5^2$$
$$+225636825015r_1R_1^2r_3^2R_5^2-705982117950R_1^3r_3^2R_5^2$$
$$+4002884325r_1r_3^4R_5^2-15080254380R_1r_3^4R_5^2-8093798865r_1^3r_3R_3R_5^2$$
$$-14054083065r_1^2R_1r_3R_3R_5^2+312683482305r_1R_1^2r_3R_3R_5^2$$
$$-330863491575R_1^3r_3R_3R_5^2-5443565715r_1r_3^3R_3R_5^2$$
$$+32781602745R_1r_3^3R_3R_5^2+6281712270r_1^3R_3^2R_5^2+7011577665r_1^2R_1R_3^2R_5^2$$
$$-213859335600r_1R_1^2R_3^2R_5^2+390057418125R_1^3R_3^2R_5^2$$
$$-4308252165r_1r_3^2R_3^2R_5^2-47037249315R_1r_3^2R_3^2R_5^2$$
$$+46035049455r_1r_3R_3^3R_5^2-56178436725R_1r_3R_3^3R_5^2$$
$$-20091786300r_1R_3^4R_5^2+41674516875R_1R_3^4R_5^2-1148827860r_1r_3^2R_5^4$$
$$+4807068444R_1r_3^2R_5^4-1761106968r_1r_3R_3R_5^4+3028199688R_1r_3R_3R_5^4$$
$$+\,1552507884r_1R_3^2R_5^4-1823480100R_1R_3^2R_5^4\big)\sin(\varphi_1-2\varphi_3+\varphi_5)$$
$$+346500\,\big((r_1-3R_1)(r_1-R_1)(3R_1r_3-r_1R_3)(-135r_1^3r_3$$
$$+402r_1^2R_1r_3-25893r_1R_1^2r_3+36390R_1^3r_3-750r_1^3R_3$$
$$+36349r_1^2R_1R_3-155106r_1R_1^2R_3+285455R_1^3R_3)\big)\sin(6\varphi_1-2\varphi_3)$$
$$+237600\,((3r_3-R_3)(-5r_3+3R_3)(-r_3+3R_3)(3r_3+5R_3)$$
$$(-89r_3+29R_3))\,R_5^3\sin(5\varphi_3-3\varphi_5)$$
$$+25\,\big(16839900r_1^6R_1r_3+7858620r_1^5R_1^2r_3-4663529640r_1^4R_1^3r_3$$
$$+13708052820r_1^3R_1^4r_3+172100784240r_1^2R_1^5r_3$$
$$-691301096640r_1R_1^6r_3+518438774700R_1^7r_3+110394900r_1^4R_1r_3^3$$
$$-424258560r_1^3R_1^2r_3^3-9581368500r_1^2R_1^3r_3^3+44697104100r_1R_1^4r_3^3$$
$$-42068475900R_1^5r_3^3+89812800r_1^2R_1r_3^5-491243940r_1R_1^2r_3^5$$
$$+594809325R_1^3r_3^5-5613300r_1^7R_3-2619540r_1^6R_1R_3$$
$$+1554509880r_1^5R_1^2R_3-4569350940r_1^4R_1^3R_3-57366928080r_1^3R_1^4R_3$$
$$+230433698880r_1^2R_1^5R_3-172812924900r_1R_1^6R_3-36798300r_1^5r_3^2R_3$$
$$+62672940r_1^4R_1r_3^2R_3+2442498300r_1^3R_1^2r_3^2R_3-791671320r_1^2R_1^3r_3^2R_3$$
$$-28404883980r_1R_1^4r_3^2R_3+33535244160R_1^5r_3^2R_3-29937600r_1^3r_3^4R_3$$
$$-37992240r_1^2R_1r_3^4R_3+698120775r_1R_1^2r_3^4R_3-948808080R_1^3r_3^4R_3$$
$$+26248860r_1^5r_3R_3^2-722690100r_1^4R_1r_3R_3^2+3366982080r_1^3R_1^2r_3R_3^2$$
$$+15849244620r_1^2R_1^3r_3R_3^2-89956761840r_1R_1^4r_3R_3^2$$
$$+72090364500R_1^5r_3R_3^2+67246740r_1^3r_3^3R_3^2-973454625r_1^2R_1r_3^3R_3^2$$
$$+3727249020r_1R_1^2r_3^3R_3^2-3430541565R_1^3r_3^3R_3^2+324373500r_1^5R_3^3$$

$$
\begin{gathered}
-2689812180r_1^4R_1R_3^3 - 568891620r_1^3R_1^2R_3^3 + 26259449040r_1^2R_1^3R_3^3 \\
-24030121500r_1R_1^4R_3^3 + 224885925r_1^3r_3^2R_3^3 - 534330720r_1^2R_1r_3^2R_3^3 \\
-1209966615r_1R_1^2r_3^2R_3^3 + 2212376760R_1^3r_3^2R_3^3 - 200887500r_1^3r_3R_3^4 \\
+2384299665r_1^2R_1r_3R_3^4 - 7784636640r_1R_1^2r_3R_3^4 \\
+5582808000R_1^3r_3R_3^4 - 533268725r_1^3R_3^5 + 2349059240r_1^2R_1R_3^5 \\
-1860936000r_1R_1^2R_3^5 + 7543800r_1^5r_3R_5^2 + 361375047r_1^4R_1r_3R_5^2 \\
-940143699r_1^3R_1^2r_3R_5^2 - 31309275195r_1^2R_1^3r_3R_5^2 \\
+129987690327r_1R_1^4r_3R_5^2 - 119364276240R_1^5r_3R_5^2 + 39313350r_1^3r_3^3R_5^2 \\
-2210262228r_1^2R_1r_3^3R_5^2 + 8944718139r_1R_1^2r_3^3R_5^2 - 8152816131R_1^3r_3^3R_5^2 \\
-113416974r_1^5R_3R_5^2 + 170430183r_1^4R_1R_3R_5^2 + 9968595615r_1^3R_1^2R_3R_5^2 \\
-41819430609r_1^2R_1^3R_3R_5^2 + 43642223955r_1R_1^4R_3R_5^2 \\
-4894931250R_1^5R_3R_5^2 + 239127876r_1^3r_3^2R_3R_5^2 \\
-6348916944r_1^2R_1r_3^2R_3R_5^2 + 8965782621r_1R_1^2r_3^2R_3R_5^2 \\
+11323304793R_1^3r_3^2R_3R_5^2 + 148481577r_1^3r_3R_3^2R_5^2 \\
-1670198787r_1^2R_1r_3R_3^2R_5^2 + 15377567166r_1R_1^2r_3R_3^2R_5^2 \\
-23931769950R_1^3r_3R_3^2R_5^2 + 995250213r_1^3R_3^3R_5^2 - 2676696849r_1^2R_1R_3^3R_5^2 \\
-2146582350r_1R_1^2R_3^3R_5^2 + 5665140000R_1^3R_3^3R_5^2 + 8690355r_1^3r_3R_5^4 \\
+473339457r_1^2R_1r_3R_5^4 - 2427142887r_1R_1^2r_3R_5^4 \\
+2868219531R_1^3r_3R_5^4 - 142118739r_1^3R_3R_5^4 + 762823359r_1^2R_1R_3R_5^4 \\
-1144504377r_1R_1^2R_3R_5^4 + 360595125R_1^3R_3R_5^4\big)\sin(3\varphi_1 - \varphi_3) \\
+R_5\big(-919957500r_1^6r_3 + 2884167000r_1^5R_1r_3 + 130419679500r_1^4R_1^2r_3 \\
-209395692000r_1^3R_1^3r_3 - 4046142820500r_1^2R_1^4r_3 \\
+4456353429000r_1R_1^5r_3 + 9210309322500R_1^6r_3 - 4054050000r_1^4r_3^3 \\
+8518182750r_1^3R_1r_3^3 + 227511882675r_1^2R_1^2r_3^3 - 238260928950r_1R_1^3r_3^3 \\
-725514324975R_1^4r_3^3 - 2245320000r_1^2r_3^5 + 3523074750r_1R_1r_3^5 \\
+7655265450R_1^2r_3^5 - 228541500r_1^6R_3 + 6781401000r_1^5R_1R_3 \\
+48681121500r_1^4R_1^2R_3 - 757941624000r_1^3R_1^3R_3 - 1382213200500r_1^2R_1^4R_3 \\
+19012697055000r_1R_1^5R_3 - 20662791187500R_1^6R_3 + 2202180750r_1^4r_3^2R_3 \\
+50964502050r_1^3R_1r_3^2R_3 - 331994225475r_1^2R_1^2r_3^2R_3 \\
-425366295750r_1R_1^3r_3^2R_3 + 1216448894925R_1^4r_3^2R_3 \\
+4220646750r_1^2r_3^4R_3 + 4866982200r_1R_1r_3^4R_3 - 31276991700R_1^2r_3^4R_3 \\
+34489414575r_1^4r_3R_3^2 - 151947259200r_1^3R_1r_3R_3^2 \\
-864582278175r_1^2R_1^2r_3R_3^2 + 2890713493800r_1R_1^3r_3R_3^2
\end{gathered}
$$

$$-572024722500R_1^4r_3R_3^2 + 20500037550r_1^2r_3^3R_3^2$$
$$-49479662700r_1R_1r_3^3R_3^2 - 19379780550R_1^2r_3^3R_3^2 + 14229762525r_1^4R_3^3$$
$$-244978014600r_1^3R_1R_3^3 + 632738742075r_1^2R_1^2R_3^3$$
$$+1386330907500r_1R_1^3R_3^3 - 2417530500000R_1^4R_3^3 - 7980247800r_1^2r_3^2R_3^3$$
$$-70150101300r_1R_1r_3^2R_3^3+151108935300R_1^2r_3^2R_3^3-53409883950r_1^2r_3R_3^4$$
$$+127889399550r_1R_1r_3R_3^4 + 15480487500R_1^2r_3R_3^4 - 24927916550r_1^2R_3^5$$
$$+211221387500r_1R_1R_3^5 - 232617000000R_1^2R_3^5 + 1453308075r_1^4r_3R_5^2$$
$$-42985052190r_1^3R_1r_3R_5^2 + 242789114880r_1^2R_1^2r_3R_5^2$$
$$-171680012730r_1R_1^3r_3R_5^2 - 404427233475R_1^4r_3R_5^2$$
$$-1421290800r_1^2r_3^3R_5^2 + 3652068060r_1R_1r_3^3R_5^2 - 3585857040R_1^2r_3^3R_5^2$$
$$+1543409055r_1^4R_3R_5^2 - 36386475210r_1^3R_1R_3R_5^2$$
$$+338814678060r_1^2R_1^2R_3R_5^2 - 1182414401550r_1R_1^3R_3R_5^2$$
$$+1446537268125R_1^4R_3R_5^2 - 2616459570r_1^2r_3^2R_3R_5^2$$
$$+14166828720r_1R_1r_3^2R_3R_5^2 - 26265773610R_1^2r_3^2R_3R_5^2$$
$$+12524885820r_1^2r_3R_3^2R_5^2 - 5917042980r_1R_1r_3R_3^2R_5^2$$
$$-30720899700R_1^2r_3R_3^2R_5^2 + 9427256550r_1^2R_3^3R_5^2$$
$$-70208058600r_1R_1R_3^3R_5^2+119035248750R_1^2R_3^3R_5^2-1130948460r_1^2r_3R_5^4$$
$$+1929611592r_1R_1r_3R_5^4 + 2916332532R_1^2r_3R_5^4 - 804688884r_1^2R_3R_5^4$$
$$+ 6239285496r_1R_1R_3R_5^4 - 7028745300R_1^2R_3R_5^4\big) \sin(2\varphi_1 + \varphi_3 - \varphi_5)$$
$$+2475\big(-104580r_1^4r_3^2 - 20131965r_1^3R_1r_3^2 + 59842161r_1^2R_1^2r_3^2$$
$$+6015393r_1R_1^3r_3^2 - 86403753R_1^4r_3^2 - 1872885r_1^4r_3R_3$$
$$-3885144r_1^3R_1r_3R_3 + 77226918r_1^2R_1^2r_3R_3 - 246205968r_1R_1^3r_3R_3$$
$$+273837735R_1^4r_3R_3 - 1461101r_1^4R_3^2 + 14956237r_1^3R_1R_3^2$$
$$-75810827r_1^2R_1^2R_3^2 + 235432455r_1R_1^3R_3^2$$
$$- 201316500R_1^4R_3^2\big) R_5^2 \sin(4\varphi_1 + 2\varphi_3 - 2\varphi_5).$$

Appendix 2

Lyapunov constants V_{32} and V_{38} if $\varphi_1 = \overline{\varphi}_1$, $\varphi_3 = \overline{\varphi}_3$ and $R_1 = r_1 = 0$:

$$3386880000V_{32} = (3r_3 + 5R_3)R_5^3\big(-1459292625r_3^6$$
$$+8614578420r_3^5R_3 - 3561439455r_3^4R_3^2 - 34418277320r_3^3R_3^3$$
$$+41410717145r_3^2R_3^4 - 16050420540r_3R_3^5 + 2034246375R_3^6$$
$$-2170849005r_3^4R_5^2 + 6255604188r_3^3R_3R_5^2 - 6059843478r_3^2R_3^2R_5^2$$

$$+ 2315564748 r_3 R_3^3 R_5^2 - 303704613 R_3^4 R_5^2\big) \sin(5\varphi_3 - 3\varphi_5).$$

$$\begin{gathered}
1482030950400000 V_{38} = (3r_3 + 5R_3) R_5^3 \big(-1149145404553125 r_3^8 \\
+8848194134872050 r_3^7 R_3 + 1185862616782200 r_3^6 R_3^2 \\
-96387033484384650 r_3^5 R_3^3 + 58538665263406650 r_3^4 R_3^4 \\
+332980469705167550 r_3^3 R_3^5 - 413020341318321800 r_3^2 R_3^6 \\
+161016094351964250 r_3 R_3^7 - 20445404262373125 R_3^8 \\
+2891909356127550 r_3^6 R_5^2 - 42487871210003790 r_3^5 R_3 R_5^2 \\
+152493752666073780 r_3^4 R_3^2 R_5^2 - 176828521900465980 r_3^3 R_3^3 R_5^2 \\
+96197776031891430 r_3^2 R_3^4 R_5^2 - 25839946288107990 r_3 R_3^5 R_5^2 \\
+2766024379211400 R_3^6 R_5^2 - 3621290735420280 r_3^4 R_5^4 \\
+14619453338344608 r_3^3 R_3 R_5^4 - 13917796242684048 r_3^2 R_3^2 R_5^4 \\
+ 5015027246100768 r_3 R_3^3 R_5^4 - 622727965831608 R_3^4 R_5^4\big) \sin(5\varphi_3 - 3\varphi_5).
\end{gathered}$$

Appendix 3

Lyapunov constants V_{26} and V_{32} if $\varphi_1 = \overline{\varphi}_1$, $\varphi_3 = \overline{\varphi}_3$ and $R_3 = r_3 = 0$:

$$\begin{gathered}
268800 V_{26} = \big((r_1 + 5R_1) R_5 (10500 r_1^6 - 128940 r_1^5 R_1 - 430920 r_1^4 R_1^2 \\
+9577260 r_1^3 R_1^3 - 34912080 r_1^2 R_1^4 + 43228080 r_1 R_1^5 - 15479100 R_1^6 \\
+72515 r_1^4 R_5^2 - 703782 r_1^3 R_1 R_5^2 + 2395276 r_1^2 R_1^2 R_5^2 \\
- 3361098 r_1 R_1^3 R_5^2 + 1585089 R_1^4 R_5^2\big) \sin(5\varphi_1 - \varphi_5).
\end{gathered}$$

$$\begin{gathered}
2483712000 V_{32} = \big((r_1 + 5R_1) R_5 (97020000 r_1^8 - 1217046600 r_1^7 R_1 \\
-19884803400 r_1^6 R_1^2 + 277869853800 r_1^5 R_1^3 + 222079611600 r_1^4 R_1^4 \\
-12326128630200 r_1^3 R_1^5 + 46086997987800 r_1^2 R_1^6 \\
-57154152825000 r_1 R_1^7 + 20495280960000 R_1^8 + 1020766985 r_1^6 R_5^2 \\
+1114175216 r_1^5 R_1 R_5^2 - 223447537033 r_1^4 R_1^2 R_5^2 \\
+1715463721944 r_1^3 R_1^3 R_5^2 - 5101631600457 r_1^2 R_1^4 R_5^2 \\
+6576435520200 r_1 R_1^5 R_5^2 - 2941989081255 R_1^6 R_5^2 + 1712647422 R_1^4 R_5^4 \\
-17633067984 r_1^3 R_1 R_5^4 + 64359273276 r_1^2 R_1^2 R_5^4 \\
- 97959123360 r_1 R_1^3 R_5^4 + 53197740630 R_1^4 R_5^4\big) \sin(5\varphi_1 - \varphi_5).
\end{gathered}$$

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Keywords. center-focus problem, integrable systems in the plane.

1991 *Mathematics subject classifications*: Primary 34A05; Secondary 34C05.

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Primera versió rebuda el 16 de Desembre de 1994,
darrera versió rebuda el 24 de Novembre de 1995