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CORRIGENDA TO "UNIQUE CONTINUATION FOR SCHRÖDINGER OPERATORS" AND A REMARK ON INTERPOLATION OF MORREY SPACES

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Abstract

The purpose of this note is twofold. First it is a corrigenda of our paper [**RV1**]. And secondly we make some remarks concerning the interpolation properties of Morrey spaces.

1. Corrigenda.

In our paper "Unique continuation for Schrödinger operators with potential in Morrey spaces" [**RV1**], we claimed the following statement with the name of Theorem 1 —see (5), (6) below for the necessary definitions:

"Let $u \in H^2_{\text{loc}}(\Omega)$, $n \geq 3$, be a solution of

$$|\Delta u(x)| \leq |V(x)u(x)|, \quad x \in \Omega, \tag{1}$$

and Ω a connected, open subset of $\mathbb{R}^n$. Then there exists an $\epsilon > 0$, depending just on p and n, such that if

$$V \in F^p_{\text{loc}} = \mathcal{L}^{2,p}, \quad \|V\|_{\mathcal{L}^{2,p}} \leq \epsilon, \quad p > \frac{n-2}{2},$$

and u vanishes in an open subdomain of Ω, then u must be zero everywhere in Ω".

Unfortunately our proof happens to be incorrect. The theorem is nevertheless true, for T. Wolff obtained a closely related statement by using different arguments, see [**W**].

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Our approach to unique continuation was based upon the following Carleman estimate:

"There exists a constant $C > 0$ such that for V in F^p, $p > \frac{n-2}{2}$

$$\|e^{\tau x_n} u\|_{L^2(V)} \leq C \|V\|_{\mathcal{L}^{2,p}} \|e^{\tau x_n} \Delta u\|_{L^2(V^{-1})}, \tag{2}$$

holds for every u in C_0^∞ and τ in $\mathbb{R}$".

To obtain this inequality we took a global parametrix of the operator $e^{\tau x_n} \Delta e^{-\tau x_n}$, which later we realized can not be uniformly bounded in τ for $V \in \mathfrak{L}^{2,p}$, $p \leq (n-1)/2$ (one has to multiply the right hand side at least by $\log \tau$). In fact, the lemma in page 294 of [**RV1**] gives the following estimates for a dyadic decomposition T_δ of that parametrix :

$$\|T_\delta f\|_{L^2(V)} \leq C\delta |\log \delta| \|V\|_{\mathcal{L}^{2,p_0}} \|f\|_{L^2(V^{-1})}, \ \text{if } p_0 = \frac{n-2}{2}. \tag{3}$$

$$\|T_\delta f\|_{L^2(V)} \leq C\delta^{1+\epsilon} \|V\|_{\mathcal{L}^{2,p}} \|f\|_{L^2(V^{-1})}, \ \text{if } p > \frac{n-2}{2}. \tag{4}$$

Estimate (3) is true, but (4) holds only for $p > (n-1)/2$. In fact from (3) and if $(n-2)/2 \leq p \leq (n-1)/2$ a logarithmic growth of the type $C\delta |\log \delta|$ is easily obtained. The interesting remark is that this growth turns out to be also necessary and hence, there is no convexity for the bounds of the operator T_δ in the range $(n-1)/2 \leq p < (n-2)/2$. This fact has some consequences about the interpolation properties in Morrey spaces that we shall consider in Section 2.

If we substitute in (2) the Carleman weight τx_n by $\tau(x_n + x_n^2/2)$, we can use our approach, as we did in [**RV2**], to improve the known results on unique continuation of solutions of the inequality (1) when $V \in \mathfrak{L}^{\alpha,p}$, $\alpha < 2$ —see (5), (6) below for the definition. In any case we can not recover Wolff's result (case $\alpha = 2$) but only a weaker result for a logarithmic substitute of the space $\mathfrak{L}^{2,p}$, $p > (n-2)/2$. We do not want to get involved in these calculations in the present note. On the other hand we do not know if the inequality (2) is true or false.

2. Interpolation and Morrey-Campanato spaces.

Morrey-Campanato classes form a two parameter family of spaces $\mathcal{L}^{\alpha,p}$, $\alpha \in (-1, n/p]$, $p \in [1, \infty)$. We say that $f \in \mathcal{L}^{\alpha,p}$, if f is in L^p_{loc} and there exists a constant $C > 0$, which depends on f, such that for every $x \in \mathbb{R}^n$ and every $r > 0$, we can find a number $\sigma \in \mathbb{R}$, which depends on f, x, and r such that

$$r^\alpha \left(r^{-n} \int_{Q(x,r)} |f(y) - \sigma|^p \, dy \right)^{1/p} < C, \tag{3}$$

where $Q(x,r)$ is a cube centered at x and volume r^n.

The case $\alpha > 0$ was introduced by Morrey in the study of the regularity problem of the Calculus of Variations. It was proved by Campanato that σ can be taken zero without loss of generality —see [**C1**]. Then for $\alpha > 0$ and $f \in \mathcal{L}^{\alpha,p}$ we define

$$\|f\|_{\mathcal{L}^{\alpha,p}} = \sup_x \sup_r r^\alpha \left(r^{-n} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{1/p}. \tag{4}$$

Notice that if $\alpha > 0$ and $p = n/\alpha$, $p \geq 1$ we obtain the Lebesgue space L^p.

The class was extended by Campanato to $\alpha \leq 0$ and he [**C2**] and Meyers [**M**] proved to be the space of $(-\alpha)$-Holder continuous functions; this theorem is known as the integral characterization of Holder-continuous functions. When $\alpha = 0$ we have John-Niremberg space BMO. Notice that in this case, i.e. $\alpha \leq 0$, the space does not change with p if α is fixed.

Recently Morrey classes, i.e. $\alpha > 0$, have been the object of some works on weighted Sobolev estimates, unique continuation properties —see [**FP**], [**CS**], [**ChR**], [**W**], [**RV2**], and some other problems in PDE —see [**RV3**], and [**T**] for example.

In this note we are concerned with interpolation properties of Morrey spaces for $\alpha > 0$. In particular we prove the lack of the convexity which characterizes interpolation functors of exponent θ —see [**BL**, p. 27]. In particular the complex and real methods have this property.

The interpolation properties of Morrey-Campanato spaces have been studied in several works during the 60's —see [**S**], [**P**] and the references there in. In particular, Stampacchia, [**S**], and Campanato and Murthy, [**CM**] proved that for T a linear operator, and $\|T\|_{L^{q_i} \to \mathcal{L}^{\alpha_i,p_i}} = K_i$, with $1 \leq p_i$, $q_i \leq \infty$, $\alpha_i > 0$, $i = 1,2$, then T is bounded from L^{q_θ} to $\mathcal{L}^{\alpha_\theta,p_\theta}$ with norm at most $KK_1^{1-\theta}K_2^\theta$, where $\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_1} + \theta\frac{1}{p_2}$, $\frac{1}{q_\theta} = (1-\theta)\frac{1}{q_1} + \theta\frac{1}{q_2}$, and $\alpha_\theta = (1-\theta)\alpha_1 + \theta\alpha_2$, and K depending just on θ, α_i, p_i, q_i, $i = 1,2$.

In the more general setting of Morrey-Campanato classes, Stein and Zygmund, [**StZ**], constructed a linear operator bounded from $\mathcal{L}^{\alpha,p}$ to $\mathcal{L}^{\alpha,p}$ for some $\alpha < 0$ and from L^2 to L^2, which is not bounded from BMO to BMO. Let us remark that BMO $= \mathcal{L}^{0,p}$ and $L^2 = \mathcal{L}^{n/2,2}$. Hence interpolation through the line $\alpha = 0$ does not hold.

We have the following result.

Theorem.

Set $n > 1$ and $0 < \alpha < n$. Then, given any p_1, p_2, p_3, and $C > 0$, such that $1 \leq p_2 < p_3 < \frac{n-1}{\alpha} < p_1 < \infty$, and $C > 0$, there exists a continuous linear operator $T : \mathcal{L}^{\alpha,p_i} \to L^1$, $i = 1, 2, 3$, such that $\|T\|_{\mathcal{L}^{\alpha,p_i}\to L^1} \leq K_i$, $i = 1, 2$ and

$$\|T\|_{\mathcal{L}^{\alpha,p_3}\to L^1} > C K_1^{1-\theta} K_2^{\theta}, \tag{4}$$

for $\frac{1}{p_3} = (1-\theta)\frac{1}{p_1} + \theta\frac{1}{p_2}$.

Proof of the theorem:

Let ϕ be a C_0^∞ non negative function such that $\phi \leq 1$, $\phi(x) = 1$ if $|x| < 1/4$, and $\phi(x) = 0$ if $|x| > 1/2$. For $0 < \delta < 1$ consider the operator T given by multiplication by $\Phi_\delta(x) = \phi(|x'|)\phi(\delta x_n)$, with $x' = (x_1, \dots , x_{n-1})$:

$$Tf(x) = \Phi_\delta(x) f(x).$$

Then,

$$\begin{aligned}\int |Tf| &= \int \Phi_\delta |f| \\ &\leq \delta^{-(1-1/p)} \left(\int |\Phi_\delta f|^p \right)^{1/p} \\ &\leq \delta^{-(1-1/p)} \delta^{\alpha - n/p} \|f\|_{\mathcal{L}^{\alpha,p}}.\end{aligned}$$

And also

$$\int |Tf| = \sum_\nu \int_{Q_\nu} \Phi_\delta |f|,$$

where Q_ν is a collection of δ^{-1} cubes of volume one. Hence,

$$\int |Tf| \leq \delta^{-1} \|f\|_{\mathfrak{L}^{\alpha,p}}.$$

Therefore

$$\|T\|_{\mathfrak{L}^{\alpha,p}\to L^1} \leq \begin{cases} \delta^{-1+\alpha-(n-1)/p}, & \text{if } p \geq (n-1)/\alpha \\ \delta^{-1}, & \text{if } 1 \leq p \leq (n-1)/\alpha. \end{cases}$$

Now take $f = \Phi_\delta(x)$. Then there exists a dimensional constant c_n such that

$$\int |Tf| \geq c_n \delta^{-1},$$

and

$$\|f\|_{\mathfrak{L}^{\alpha,p}} = \begin{cases} c_n^{-1}\delta^{-\alpha+(n-1)/p}, & \text{if } p \ge (n-1)/\alpha \\ c_n^{-1}, & \text{if } 1 \le p \le (n-1)/\alpha. \end{cases}$$

Therefore

$$\|T\|_{\mathfrak{L}^{\alpha,p}\to L^1} \ge c_n^2 \delta^{-1}, \text{ if } 1 \le p \le (n-1)/\alpha.$$

Fix $\alpha > 0$ and take p_1 and p_2 such that $(n-1)/\alpha \le p_1 \le n/\alpha$, and $1 \le p_2 \le (n-1)/\alpha$. Then, on one hand, for $\theta \in (0,1)$, we have

$$\|T\|^{\theta}_{\mathfrak{L}^{\alpha,p_1}\to L^1} \|T\|^{1-\theta}_{\mathfrak{L}^{\alpha,p_2}\to L^1} \le \delta^{-1+\epsilon},$$

where $\epsilon = (\alpha - (n-1)/p_1)\theta > 0$.

On the other hand for $p_2 < p_3 < (n-1)/\alpha$, we have

$$\|T\|_{\mathfrak{L}^{\alpha,p_3}\to L^1} \ge c_n^2 \delta^{-1}.$$

Taking δ small enough we have proved the theorem. ■

Final remarks.

The above theorem can be extended to a more general situation. In particular the restriction on the dimension, α, and $p's$ can be avoided. In fact we have an example of a bounded linear operator which is unbounded in a given intermediate space. Therefore Morrey spaces are not closed by interpolation in a strong form, and not just by the lack of convexity.

Writing this example would have made this note too large and getting us too far from the initial purpose which is the corrigenda of our previous paper. On the other hand the counterexample given in the theorem, is a small variation of the one needed for the inequality (4). We have preferred to write it in this way, to illustrate that the lack of convexity is due to the particular structure of Morrey spaces and their bad behaviour with respect to interpolation. Details about the above questions will appear elsewhere.

Finally we would like to thank J. Peetre for sharing with us his belief that Morrey spaces are not closed under interpolation.

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