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ASYMPTOTIC EQUIVALENCE OF VOLTERRA DIFFERENCE SYSTEMS

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Abstract

The purpose of this paper is to give some results on the asymptotic relationship between the solutions of a linear difference equation and its perturbed nonlinear equation.

1. Introduction

The problem of the asymptotic equivalence for systems of ordinary differential equations has been studied by many authors, as e.g. Brauer [**3**], Brauer and Wong [**4**], Boundorides and Georgiou [**2**], Lowell Lovelady [**11**], Morchało [**13**], Švec [**17**], Szufla [**18**], and others.

The problem of the asymptotic equivalence for integrodifferential equations has been studied by Morchało [**14**], Razapov [**16**], Talpalaru [**19**].

The problem of the asymptotic behavior of solutions of ordinary difference equations has been studied by Benzaid [**1**], Conffman [**5**], Drozdowicz, Popenda [**6**], Elaydi, Gyori [**8**], Li [**10**] and Pinto [**15**].

In this paper, we shall consider some results on the asymptotic relationship between the solutions of a linear Volterra difference equation and its perturbed nonlinear equation. The author knows only the works of Talpalaru [**19**], Ved and Gołovina [**21**], Ved and Kaptagaev [**20**], dealing with the above problem for special case.

2. Notations and Definitions

Here $N(n_0) = \{n_0, n_0 + 1, \dots\}$, where n_0 is a natural number or zero; R^k - the k - dimensional real euclidean space with the norm

$$|x| = \sum_{i=1}^{k} |x_i|, \quad x = (x_1, \dots, x_k);$$

M^k - the space of all $k \times k$ matrices $D = (d_{ij})$ with the norm $|D| = \max_j \sum_{i=1}^{k} |d_{ij}|$, I - identity matrix.

We denote by $\Phi(N, R^k)$ the space of all functions from $N(n_0)$ into R^k.

Let $\Phi_1 = \Phi_1(N, R^k)$ be the Banach space in Φ of all bounded functions $u : N(n_0) \to R^k$ with norm $\|x\| = |x(n)|_{\Phi_1} = \sup\{|x(n)| : n \in N(n_0)\}$.

In this paper we consider the following systems of difference equations

$$x(n+1) = [A + B(n)]x(n) + \sum_{r=0}^{n} [K(n-r) + Q(n,r)]x(r), \tag{2.1}$$

$$x(n+1) = Ax(n) + \sum_{r=0}^{n} K(n-r)x(r) + f(n) + F(n, x(n)) \tag{2.2}$$

thought et as perturbations of

$$y(n+1) = Ay(n) + \sum_{r=0}^{n} K(n-r)y(r), \tag{2.1'}$$

$$y(n+1) = Ay(n) + \sum_{r=0}^{n} K(n-r)y(r) + f(n) \tag{2.2'}$$

where x, y, f are k-dimensional vectors,

A - is a constant matrix $k \times k$,
B, $K : N(n_0) \to M^k$
$Q : N(n_0) \times N(n_0) \to M^k$,
$F : N(n_0) \times U \to R^k$ is for any $n \in N(n_0)$ continuous as a function of $x \in U$
(U - a region in R^k).

We define the resolvent matrix $R(n, m)$ of the equation

$$x(n+1) = [A + B(n)]x(n) + \sum_{r=0}^{n} [K(n-r) + Q(n,r)]x(r) + f(n) \tag{2.3}$$

as the unique solution of the matrix difference equation [**7**]

$$\begin{aligned} R(n+1, m) = {} & [A + B(n)]R(n, m) \\ & + \sum_{r=m}^{n} [K(n-r) + Q(n,r)]R(r, m), \quad n \geq m, \end{aligned} \tag{2.4}$$

with $R(m,m) = I$.

Using the resolvent matrix $R(n,m)$ we can establish the following relation [**7**] (Variation of Constants Formula)

$$x(n,0,x_0) = R(n,0)x_0 + \sum_{r=0}^{n-1} R(n,r+1)f(r), \tag{2.5}$$

where $x(n,0,x_0)$ is the unique solution of the equation (2.3) satisfying $x(0,0,x_0) = x_0$.

Let $Y(n)$ denote the fundamental matrix of the system (2.1') [**7**]. Notice that $Y(0) = I$ and $y(n,0,y_0) = Y(n)y_0$ is the unique solution of (2.1') with $y(0,0,y_0) = y_0$.

Moreover,

$$Y(n+1) = AY(n) + \sum_{r=0}^{n} K(n-r)Y(r). \tag{2.6}$$

Remark [**7**]. We remark here that the resolvent matrix $R(n,m)$ for equations of nonconvolution type is closely related to the fundamental matrix $Y(n)$. By uniqueness of solutions, it is easy to see that for equations of convolutions type such as (2.1'), $R(n,0) = Y(n)$ and $R(n,m) = Y(n-m)$.

In this paper we consider the notion of asymptotic equivalence given by,

Definition. We say that the equations (2.1) and (2.1') or (2.2), (2.2') are asymptotically equivalent if, corresponding to each solution $x = x(n)$ of (2.1), ((2.2)), there exists a solution $y = y(n)$ of (2.1'), ((2.2')) with the property

$$\lim[x(n) - y(n)] = 0 \text{ as } n \to \infty \text{ and conversely.} \tag{2.7}$$

3. Asymptotic equivalence

We state the following lemma.

Lemma 3.1. *If*

1. $\varphi(n)$ *is bounded on* $N(n_0)$ *and* $\lim_{n\to\infty} \varphi(n) = \varphi(\infty)$ *exists,*
2. $\sum_{k=n_0}^{\infty} |g(k)| < \infty,$

then

$$\lim_{n\to\infty} \sum_{k=n_0}^{n} \varphi(n-k)g(k) = \varphi(\infty) \sum_{k=n_0}^{\infty} g(k).$$

Theorem 3.2. *Assume that*

1. *all solutions of the system (2.1') tend to finite limits as* $n \to \infty$,
2. $\sum_{r=0}^{\infty} \left[|B(r)| + \sum_{s=0}^{r} |Q(r,s)| \right] < \infty$,
3. $\det P \neq 0$, *where* $P = \lim Y(n)$ *as* $n \to \infty$, P *is a constant matrix,*
4. $q = \sum_{r=0}^{\infty} \left[|B(r)|\,|R(r,0)| + \sum_{s=0}^{r} |Q(r,s)|\,|R(s,0)| \right] < 1$.

Then,

a) *corresponding to each solution* $x = x(n) \in \Phi_1$ *of (2.1), there exists a solution* $y = y(n) \in \Phi_1$ *of (2.1') such that (2.7) is satisfied provided that Conditions 1, 2 hold,*

b) *in Relation (2.7) the solution* $y = y(n)$ *of (2.1') is unique if Conditions 1, 2 and 3 are satisfied,*

c) *to each non-zero solution* $x = x(n) \in \Phi_1$ *of (2.1) there corresponds in Relation (2.7) a non-zero solution* $y = y(n) \in \Phi_1$ *of (2.1'), if Conditions 1, 2 and 4 hold and conversely,*

d) *in Relation (2.7) the solution* $x = x(n)$ *of (2.1) is unique if Conditions 1, 2, 3 and 4 are satisfied.*

Proof: By Formula (2.5) the solutions $x(n)$ of (2.1) and $y(n)$ of (2.1') can be written as

$$x(n) = Y(n)x_0 + \sum_{r=0}^{n-1} Y(n-r-1) \left[B(r)x(r) + \sum_{s=0}^{r} Q(r,s)x(s) \right] \tag{3.1}$$

and

$$y(n) = Y(n)y_0, \quad n \in N. \tag{3.2}$$

Furthermore, from the Relations (3.1) and (3.2) we obtain

$$\begin{aligned} x(n) - y(n) &= Y(n)[x_0 - y_0] \\ &\quad + \sum_{r=0}^{n-1} Y(n-r-1) \left[B(r)x(r) + \sum_{s=0}^{r} Q(r,s)x(s) \right]. \end{aligned} \tag{3.3}$$

From Assumptions 1, 2 and (3.1) we obtain

$$|x(n)| \leq |Y(n)|\,|x_0| + \sum_{r=0}^{n-1} |Y(n-r-1)| \left[|B(r)|\,|x(r)| + \sum_{s=0}^{r} |Q(r,s)|\,|x(s)| \right].$$

Hence and difference inequality [**9**] we can easily obtain that all solutions of (2.1) are bounded.

Thus

$$\sum_{n=0}^{\infty} \left| B(r)x(r) + \sum_{s=0}^{r} Q(r,s)x(s) \right| < \infty. \tag{3.4}$$

By Assumption 1, Lemma 3.1 and Relations (3.3) (3.4) we get

$$\lim_{n\to\infty} [x(n) - y(n)] = P\left\{ x_0 - y_0 + \sum_{n=0}^{\infty} \left[B(n)x(n) + \sum_{s=0}^{n} Q(n,s)x(s) \right] \right\}. \tag{3.5}$$

This shows that for arbitrary solutions $x(n)$ and $y(n)$ of (2.1), (2.1') respectively Relation (2.7) hold iff

$$P\left\{ x_0 - y_0 + \sum_{n=0}^{\infty} \left[B(n)x(n) + \sum_{s=0}^{n} Q(n,s)x(s) \right] \right\} = 0. \tag{3.6}$$

Equality (3.6) defines a relation between all solutions $x(n)$, $y(n)$ of (2.1), (2.1'), respectively, for which (2.7) holds.

If $P = 0$, then (3.6) means that (2.7) holds for arbitrary solutions $x(n)$, $y(n)$ of (2.1), (2.1') respectively.

On the other hand if $P \neq 0$, then for arbitrary solution $x(n)$ of (2.1) we have

$$y_0 = x_0 + \sum_{n=0}^{\infty} \left[B(n)x(n) + \sum_{s=0}^{n} Q(n,s)x(s) \right]. \tag{3.7}$$

Hence for suitable solution $y(n)$ of (2.1') we conclude that (2.7) holds. Since Condition 3 holds, we claim that the solution $y(n)$ with the initial condition y_0 defined by (3.7) is unique in (2.7).

From (2.5) for $f(n) = 0$ and (3.7) we have

$$y_0 = x_0 \left\{ I + \sum_{n=0}^{\infty} \left[B(n)R(n,0) + \sum_{s=0}^{n} Q(n,s)R(s,0) \right] \right\}$$

or

$$(I + P_0)x_0 = y_0 \tag{3.8}$$

where

$$P_0 = \sum_{n=0}^{\infty} \left[B(n)R(n,0) + \sum_{s=0}^{n} Q(n,s)R(s,0) \right].$$

Assume that $(I+P_0)^{-1}$ exists and $x(n) \neq 0$ for $n \in N$ $(x_0 \neq 0)$. Then, by (3.8), we have $y_0 \neq 0$ $(y(n) \neq 0$ for $n \in N)$. Such a matrix exists if, for example, $|P_0| < 1$ [**12**] (Banach Theorem's). From Assumption 4 it follows that $|P_0| < 1$.

Let the initial condition y_0 of solution $y(n) = y(n, 0, y_0)$ be arbitrary.

From (3.8) we obtain

$$x_0 = (I + P_0)^{-1} y_0.$$

Hence for every solution $y(n) \neq 0$ on N there exists a unique solution $x(n) \neq 0$ on N such that (2.7) holds and conversely. ■

Remark. If all solutions of (2.1') tends to zero as $n \to \infty$ and Condition 2 hold, then all solutions of (2.1) tend to zero as $n \to \infty$.

Now, we consider asymptotic equivalence between Equations (2.2) and (2.2').

Lemma 3.3. *Suppose that the following conditions hold:*

1. *every solution of (2.2') is bounded on* N,
2. $|F(n, x_1) - F(n, x_2)| \leq g(n)\, |x_1 - x_2|$ *for* $n \in N$, x_1, $x_2 \in U$,
3. $\sum_{n=0}^{\infty} g(n) < \infty$ *and* $\sum_{n=0}^{\infty} |F_0(n)| < \infty$ *where* $F_0(n) \equiv F(n, 0)$.

Then every solution of (2.2) is bounded on N *and*

$$|x(n)| \leq L\, M(n), \quad n \in N \tag{3.9}$$

where

$$Y_0 = \sup_N |Y(n)|,$$

$$L = \sup_N \left| y_0(n) + \sum_{r=0}^{n-1} Y(n-r-1)F_0(r) \right| < \infty,$$

$$M(n) = \exp\left(Y_0 \sum_{r=0}^{n-1} g(r) \right),$$

$y_0(n)$ *is a solution of (2.2').*

Proof: By the formula (2.5) the solution of (2.2) can be written as

$$x(n,0,x_0) = y_0(n) + \sum_{r=0}^{n-1} Y(n-r-1)F(r,x(r)) \tag{3.10}$$

where

$$y_0(n) = Y(n)x_0 + \sum_{r=0}^{n-1} Y(n-r-1)f(r), \quad n \in N.$$

Furthermore, it follows from (2.5) and in view 1 that all solutions of (2.1') are bounded on N. Now, using the Relation (3.10) and the Condition 2, we get

$$|x(n)| \le L + Y_0 \sum_{r=0}^{n-1} g(r)|x(r)|, \tag{3.11}$$

which implies, by Gronwall inequality

$$|x(n)| \le L \exp Y_0 \sum_{r=0}^{n-1} g(r) = L\, M(n).$$

Because the function $M_1(n)$ is bounded on N, we can conclude that the solution $x(n)$ of (2.2) is also bounded on N. ■

Remark. From (3.9), we have

$$|x(n)| \le [Y_0|x_0| + f_0 + Y_0F_1]M(n) \tag{3.12}$$

where

$$f_0 = \sup_N \left| \sum_{r=0}^{n-1} Y(n-r-1)f(r) \right| < \infty,$$

$$F_1 = \sum_{n=0}^{\infty} |F_0(n)| < \infty.$$

Theorem 3.4. *Let*

1. *all solutions of the system (2.1') tend to finite limits as* $n \to \infty$*,*
2. *Conditions 2 and 3 of lemma 3.3 hold,*
3. $\sup_{n \in N} \left\{ \sum_{r=0}^{n-1} |Z_1(n,r)|\,|g(r)| + \sum_{r=n}^{\infty} |Z_2(n,r)|\,|g(r)| < 1 \right\}$ *where*

$$Z_1(n,r) = Y(n-r-1) - Y(n), \quad Z_2(n,r) = -Y(n).$$

Then for each solution $x(n)$ *of (2.2) there corresponds a solution* $y(n)$ *of (2.2') such that (2.7) holds.*

Moreover, suppose that Condition 3 of Theorem 3.2 holds. Then the solution $y(n)$ *of (2.2') in Relation (2.7) is unique.*

Let Conditions 1-3 hold and

4. $q_1 = Y_0 \sum_{n=0}^{\infty} g(n) M_1(n) < 1.$

Then for each solution of (2.2) with $x_0 \neq 0$ *and*

$$|x_0| > (1-q_1)^{-1}[F_2 + q_1(F_1 + Y_0^{-1} f_0)]$$

where

$$F_2 = \left| \sum_{n=0}^{\infty} F_0(n) \right| < \infty$$

there corresponds a solution $y(n)$ *of (2.2') with* $y_0 \neq 0$ *such that (2.7) holds and conversely.*

If, in addition the Condition 3 of Theorem 3.2 holds, then the solution $x(n)$ *of (2.2) in Relation (2.7) is unique.*

Proof: By 1 it follows that solutions of (2.2') are bounded on N. Then, the bounded properties of solutions of (2.2') imply that solutions of (2.1') are bounded too. The first two parts of Theorem are easily verified (see Theorem 3.2 and Lemma 3.3). Since $P \neq 0$, then we can find a initial condition y_0 of the solution $y(n)$ of (2.2') such that

$$y_0 = x_0 + \sum_{n=0}^{\infty} F(n, x(n)), \tag{3.13}$$

where $x(n)$ is a given solution of (2.2).

Let $x_0 \neq 0$, then from (3.13), (3.12) we have

$$
\begin{aligned}
|y_0| &\geq |x_0| - \left|\sum_{n=0}^{\infty} F(n, x(n))\right| \\
&= |x_0| - \left|\sum_{n=0}^{\infty} F(n, x(n)) - F(n, 0)] + \sum_{n=0}^{\infty} F(n, 0)\right| \\
&\geq |x_0| - \sum_{n=0}^{\infty} |F(n, x(n)) - F_0(n)| - F_2 \\
&\geq |x_0| - \sum_{n=0}^{\infty} g(n) M_1(n)[Y_0|x_0| + f_0 + Y_0 F_1] - F_2 \\
&\geq |x_0|(1 - q_1) - [(F_1 + Y_0^{-1} f_0) q_1 + F_2] > 0.
\end{aligned}
\tag{3.14}
$$

Hence $y_0 \neq 0$.

Let the initial condition y_0 of the solution $y(n)$ of (2.2') be arbitrary selection. Then by Conditions 1, 2 and (3.13) the solution $x(n)$ of (2.2) be defined for all $n \in N$ and (2.7) be hold. By this means we give some conditions for existence and uniqueness of the solution $x(n)$ of (2.2) in Φ_1 which satisfied (3.13).

Since the equation (2.2) with initial condition x_0 is equivalent to the equation (3.10), then substituting for x_0 from (3.13) into (3.10)

$$
x(n) = y(n) + \sum_{r=0}^{n-1} Z_1(n, r) F(r, x(r)) + \sum_{r=n}^{\infty} Z_2(n, r) F(r, x(r)) \tag{3.14}
$$

where

$$
y(n) = Y(n) y_0 + \sum_{r=0}^{n-1} Y(n - r - 1) f(r)
$$

is arbitrary solution of (2.2'),

$$
\begin{aligned}
Z_1(n, r) &= Y(n - r - 1) - Y(n) \\
Z_2(n, r) &= -Y(n).
\end{aligned}
$$

Let T be the operator defined for each $x \in \Phi_1$ by the equation

$$
Tx(n) = \sum_{r=0}^{n-1} Z_1(n, r) F(r, x(r)) + \sum_{r=0}^{\infty} Z_2(n, r) F(r, x(r)).
$$

It is obvious that

$$|F(n,x(n))| \leq |F(n,x(n)) - F(n,0)| + |F(n,0)| \leq g(n)|x(n)| + |F(n,0)|.$$

The above inequality gives

$$|Fx(n)| \leq \sum_{r=0}^{n-1} |Z_1(n,r)|[g(r)|x(r)| + |F(r,0)|] \\ + \sum_{r=n}^{\infty} |Z_2(n,r)|[g(r)|x(r)| + |F(r,0)|].$$

From Assumption 1, 2 we obtain $\|Tx\| < \infty$. Thus T maps Φ_1 into itself. The operator T is a contraction. In fact, let x_1, $x_2 \in \Phi_1$,

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\leq \sum_{r=0}^{n-1} |Z_1(n,r)|\,|F(r,x_1(r)) - F(r,x_2(r))| \\ &+ \sum_{r=n}^{\infty} |Z_2(n,r)|\,|F(r,x_1(r)) - F(r,x_2(r))| \\ &\leq \|x_1 - x_2\| \left\{ \sum_{r=0}^{n-1} |Z_1(n,r)|g(r) + \sum_{r=n}^{\infty} |Z_2(n,r)g(r)| \right\}. \end{aligned}$$

This implies that T is contraction in Φ_1. ■

Remark. If all solutions of (2.2') tends to zero as $n \to \infty$ and the assumptions of Lemma 3.3 hold, then all solutions of (2.2) tend to zero as $n \to \infty$.

Remark. Let assumptions of theorem 3.4 hold and initial condition x_0 of the solution $x(n)$ of (2.2) implies

$$|x_0| < (1+q_1)^{-1}[F_2 - q_1(f_0 Y^{-1} + F_1)]$$

where

$$F_2 - q_1(f_0 Y^{-1} + F_1) > 0.$$

Then for each solution of (2.2) with $x_0 \neq 0$ corresponds solution $y(n)$ of (2.2') with $y_0 \neq 0$ such that (2.7) holds and conversely.

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