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Q-PERFECT GROUPS AND UNIVERSAL Q-CENTRAL EXTENSIONS

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Abstract

Using results of Ellis-Rodríguez Fernández, an explicit description by generators and relations is given of the mod q Schur multiplier, and this is shown to be the kernel of a universal q-central extension in the case of a q-perfect group, i.e. one which is generated by commutators and q-th powers. These results generalise earlier work K. Dennis and Brown-Loday.

A group G will be called *q-perfect*, where q is a non-negative integer, if G is generated by its commutator subgroup $[G,G]$ and the elements of the form g^q for all $g \in G$. An extension of groups

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1 \tag{1}$$

will be called *q-central* if it is a central extension and every element of A has order dividing q. The q-central extension (1) will be called *universal* if for any other q-central extension

$$1 \longrightarrow A' \longrightarrow E' \longrightarrow G \longrightarrow 1$$

there is a unique morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 1 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

The existence of universal q-central extensions of q-perfect groups in the classical case $q = 0$ is well known, and a similar argument yields the general case. That is, the universal coefficient theorem yields an exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_1(G), H_2(G,\mathbb{Z}_q)) \longrightarrow H^2(G, H_2(G,\mathbb{Z}_q)) \longrightarrow \mathrm{Hom}(H_2(G), H_2(G,\mathbb{Z}_q)) \longrightarrow 0.$$

The q-perfect condition implies that the Ext term is zero, and so there is a unique element e in $H^2(G, H_2(G, \mathbb{Z}_q))$ which maps to the canonical morphism $H_2(G) \to H_2(G, \mathbb{Z}_q)$ induced by the coefficient morphism $\mathbb{Z} \to \mathbb{Z}_q$. This cohomology element e determines the universal q-central extension.

However in the case $q = 0$ Miller in [M] gives for this extension group an explicit construction by generators and relations (see [K] for a recent account, and [B-L] for an account deduced from the non-abelian tensor product). The aim of this paper is to show that results of [E-R] yield a similar construction, in terms of generators and relations, for the group E of the universal q-central extension

$$1 \longrightarrow H_2(G, \mathbb{Z}_q) \longrightarrow E \longrightarrow G \longrightarrow 1$$

for any q-perfect group G.

The group E of this universal q-central extension is the special case $G\Delta^q G$ of a construction $N\Delta^q G$ defined in [E-R] for any group G and normal subgroup N. We need to derive some extra properties of that construction. While doing so it seems worth putting the construction in slightly greater generality to allow possibly wider uses, especially as this causes no extra difficulty in the proofs.

Let $\nu : N \to G$ be a crossed module. The reader not familiar with this term can think of ν as the inclusion of a normal subgroup. We write ${}^g n$ for the operation of an element $g \in G$ on an element $n \in N$, and let G operate on itself by conjugation: ${}^x y = xyx^{-1}$. We write $[x, y] = xyx^{-1}y^{-1}$. We suppose N operates on G via ν and the conjugation action of G on itself.

Let q be any non-negative integer . The group $N \otimes^q G$ is to have generators $n \otimes g$ and $\{n\}$ for all $n \in N$ and $g \in G$, subject to the relations

$$n \otimes gh = (n \otimes g)({}^g n \otimes {}^g h), \tag{2}$$

$$nm \otimes g = ({}^n m \otimes {}^n g)(n \otimes g), \tag{3}$$

$$\{n\}(m \otimes g)\{n\}^{-1} = {}^{n^q} m \otimes {}^{n^q} g, \tag{4}$$

$$[\{n\}, \{m\}] = n^q \otimes m^q, \tag{5}$$

$$\{nm\} = \{n\} \left[\Pi_{i=1}^{q-1} (n^{-1} \otimes ({}^{n^{1-q+i}} m)^i)\right] \{m\}, \tag{6}$$

$$\{[n, g]\} = (n \otimes g)^q, \tag{7}$$

for all g, $h \in G$ and m, $n \in N$.

In order to understand the point of these relations, note that a morphism $\xi : N \otimes^q G \to G$ may be defined on the generators by

$$\xi(n \otimes g) = [n, g], \ \xi\{n\} = n^q.$$

It is straightforward to prove that ξ is well defined, since the relations (2)-(7) are externalisations of rules for commutator and power operations.

The construction $N\Delta^q G$ introduced in [E-R] assumes N is a normal subgroup of G, and is the quotient of $N \otimes^q G$ by the relations

$$n \otimes \nu n = 1, \tag{8}$$

for all $n \in N$. The image of $n \otimes g$ in $N\Delta^q G$ is written $n \wedge g$.

The new information we add on these constructions is first that there is an operation of G on $N \otimes^q G$ which on generators is given by

$$(9) \qquad {}^k(n \otimes g) = {}^k n \otimes {}^k g, \; {}^k\{n\} = (n^q \otimes k)^{-1}\{n\},$$

for all g, $k \in G$, $n \in N$. In order to prove that (9) yields a well defined operation on $N \otimes^q G$, we have to prove that this operation preserves the relations in the sense that the operation of $k \in G$ on the left-hand side of a relation gives the right-hand side, modulo the relations (2)-(7).

In the case of the relations (2) and (3), this has already been verified for the non-abelian tensor product $N \otimes G$ in [B-L], which has generators $n \otimes g$ and relations corresponding to (2) and (3), and it follows that the same holds here. Further we can assume for $n \otimes g$ in $N \otimes^q G$ properties analogous to those for $n \otimes g$ in $N \otimes G$ which are proved in Proposition 2.3 of [B-L].

For the relation (4), we find the operation of $k \in G$ on the left-hand side gives

$$\begin{aligned}(n^q \otimes k)^{-1}\{n\}({}^k m \otimes {}^k g) &= (n^q \otimes k)^{-1}({}^{n^q k} m \otimes {}^{n^q k} g)\{n\} \text{ by (4)} \\ &=^{[k,n^q]} ({}^{n^q k} m \otimes {}^{n^q k} g)(n^q \otimes k)^{-1}\{n\} \text{ by [B-L] Prop. 2.3(c)} \\ &= ({}^{kn^q} m \otimes {}^{kn^q} g)\,{}^k\{n\}, \text{ as required.}\end{aligned}$$

The relation (5) is more tricky. We state it in the form

$$\{n\}\{m\}\{n\}^{-1} = (n^q \otimes m^q)\{m\}.$$

On acting with k on the left hand side of this we get

$$\begin{aligned}&{}^k(\{n\}\{m\}\{n\}^{-1}) \\ &\qquad = (n^q \otimes k)^{-1}\{n\}(m^q \otimes k)^{-1}\{m\}\{n\}^{-1}(n^q \otimes k) \\ &\quad = (n^q \otimes k)^{-1}\{n\}(m^q \otimes k)^{-1}\{n\}^{-1}(n^q \otimes m^q)\{m\}(n^q \otimes k) \text{ by (5)} \\ &\quad = (n^q \otimes k)^{-1}\{n\}(m^q \otimes k)^{-1}\{n\}^{-1}(n^q \otimes m^q)^{\,m^q}(n^q \otimes k)\{m\} \\ &\qquad = (n^q \otimes k)^{-1\;n^q}(m^q \otimes k)^{-1}(n^q \otimes m^q)^{\,m^q}(n^q \otimes k)\{m\} \text{ by (4)}\end{aligned}$$

On acting with k on the right hand side we get

$${}^k(n^q \otimes m^q)(m^q \otimes k)^{-1}\{m\}.$$

Thus if we write $a = n^q$, $b = m^q$, the equality of these is an identity

$$(10) \qquad (k \otimes a)^{\,a}(k \otimes b)(a \otimes b)^{\,b}(a \otimes k) = {}^k(a \otimes b)(k \otimes b).$$

To prove this, it is easiest to use the relation between crossed squares and cat^2-groups established in [**L**]. Consider the crossed square constructed in [**B-L**, Proposition 2.15]:

$$\begin{array}{ccc} N \otimes G & \xrightarrow{\kappa} & G \\ {\scriptstyle\kappa}\downarrow & & \downarrow \\ N & \longrightarrow & G \end{array}$$

where κ is the commutator morphism $n \otimes g \mapsto [n, g]$. The group $N \otimes G$ embeds in the big group of the cat^2-group corresponding to this crossed square, and in this embedding, the element $n \otimes g$ can be represented as a commutator in the big group. Thus to check the identity in $N \otimes G$ corresponding to (10) it is sufficient to check the corresponding commutator identity. But this identity is well known, and may be checked directly.

The relation (6) presents similar problems. After some fidding, one is required to verify an identity of the form

$$\begin{aligned} &{}^{n^{-q}}((nm)^q \otimes k)^{-1} \Pi_{i=1}^{q-1}(n^{-1} \otimes ({}^{n^{1-q+i}}m)^i) \\ &\qquad = {}^{n^{-q}}(n^q \otimes k)^{-1} \Pi_{i=1}^{q-1}({}^{k}n^{-1} \otimes ({}^{kn^{1-q+i}}m)^i)(m^q \otimes k)^{-1}. \end{aligned}$$

Again, in the big group corresponding to the crossed square for $N \otimes G$, as above, this translates to a valid commutator identity. Therefore it holds in $N \otimes^q G$.

The preservation of the relation (7) is easily verified, using the crossed square rule Proposition 2.3(d) of [**B-L**]. So the operation by k preserves the relations and so defines an operation of G on $N \otimes^q G$.

We now assert that, with this operation, the morphism $\xi : N \otimes^q G \to G$ defined above is a crossed module. The first rule for a crossed module, that $\xi({}^g x) = g(\xi x)g^{-1}$, is clear. The second rule, that $yxy^{-1} = {}^{\xi y}x$, involves three checks which have not already been done in the proof in [**B-L**] that $\kappa : N \otimes G \to G$ is a crossed module. Two of these follow immediately from (4) and (5), while the third, that

$$(m \otimes g)\{n\}(m \otimes g)^{-1} = {}^{[m,g]}\{n\},$$

reduces to the rule that $(m \otimes g) = ([m, g] \otimes b)({}^b m \otimes {}^b g)$, which is a consequence of a crossed square rule for $N \otimes G$.

As a consequence of the crossed module rules, Ker ξ is central in $N \otimes^q G$.

It is now easy to verify that, with this action and the h-map $N \times G \to N \otimes^q G$, $(n, g) \mapsto n \otimes g$, the square

$$\begin{array}{ccc} N \otimes^q G & \xrightarrow{\xi} & G \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle 1} \\ N & \xrightarrow{i} & G \end{array}$$

becomes a crossed square [**L**]. A useful consequence is that the action of G on Ker ξ is trivial (see [**B-L**, p.113]).

We now give a relation between $N \otimes^q G$ and $N \otimes G$.

Proposition 11. *There is an exact sequence*

$$N \otimes G \xrightarrow{\alpha} N \otimes^q G \longrightarrow N^{ab} \longrightarrow 1$$

where N^{ab} denotes the group N made Abelian.

Proof: There is a morphism $\alpha : N \otimes G \to N \otimes^q G$ which satisfies $\alpha(n \otimes g) = n \otimes g$ for all $n \in N$, $g \in G$. By (4), Im α is normal in $N \otimes^q G$. The quotient is, by the relations (5) and (6), generated by elements $\{n\}$ for $n \in N$ with the relations $\{nm\} = \{n\}\{m\}$ and $[\{n\}, \{m\}] = 1$ for all $n, m \in N$. The proposition follows. ■

Corollary 12. *If N and G are finite, then so also is $N \otimes^q G$.*

Proof: This follows from the main result of [**E**], that $N \otimes G$ is finite if N and G are finite. ■

We will show in Remark 19 that the morphism α is in general not injective.

The relation of this new construction to that given in [**E-R**] is now easy to establish, and is analogous to the relation between $N \otimes G$ and $N \wedge G$ established in [**B-L**]. That is, there is an exact sequence

$$\Gamma(N/[N,G]) \xrightarrow{\Psi} N \otimes^q G \longrightarrow N\Delta^q G \longrightarrow 1 \tag{13}$$

where Ψ is defined by $\gamma([n]) \mapsto n \otimes n$. Here Γ and γ are as in [**B-L**] (Γ is Whitehead's universal quadratic functor). Hence we obtain:

(14) *If N and G are finite, then so also is $N\Delta^q G$.*

Since the image of the morphism Ψ in (13) is G-invariant, the crossed module properties of $\xi : N \otimes^q G \to G$ are inherited by $\partial : N\Delta^q G \to G$.

We now establish a weak universal property for $\xi : N \otimes^q G \to G$.

Proposition 15. *Let $\pi : E \to G$ be an epimorphism with kernel which is central and has every element of order dividing q. Then there is a morphism $\eta : N \otimes^q G \to E$ such that $\pi h = \xi$.*

Proof: Here η is defined on the generators by

$$\eta(n \otimes g) = [n', g'], \; \eta\{n\} = (n')^q,$$

where $n', g' \in E$ are elements such that $\pi n' = \nu n$, $\pi g' = g$. The fact that $[n', g']$ is independent of the choice of n', g' follows from centrality, and the

independence of $(n')^q$ follows from the fact that the kernel of π consists of elements of order dividing q. That η is well defined on $N \otimes^q G$ is now easy to verify. ■

We now draw some consequences from the exact homology sequence

$$(16)\quad H_3(G,\mathbb{Z}_q) \longrightarrow H_3(G/N,\mathbb{Z}_q) \longrightarrow \operatorname{Ker}(N\Delta^q G \longrightarrow G) \longrightarrow H_2(G,\mathbb{Z}_q)$$
$$\longrightarrow H_2(G/N,\mathbb{Z}_q) \longrightarrow N/N\#_q G \longrightarrow H_1(G,\mathbb{Z}_q) \longrightarrow H_1(G/N,\mathbb{Z}_q) \longrightarrow 0$$

established in [**E-R**]. Here $\mathbb{Z}_q$ is the integers mod q, and

$$N\#_q G = \xi(N \otimes^q G) = \partial(N\Delta^q G)$$

is the subgroup of G generated by commutators in G and qth powers of elements of N. It follows immediately from (16) that
(17) Ker $(N\Delta^q G \to g)$ *consists of elements whose order divides* q^2 *and* Ker $(G\Delta^q G \to G)$ *consists of elements whose order divides* q.

We can now prove the main result of this paper, which uses the notion of q-perfect group and q-central extension given at the start.

Proposition 18. *If G is a q-perfect group, then universal q-central extensions of G are isomorphic to the sequence*

$$1 \longrightarrow H_2(G,\mathbb{Z}_q) \longrightarrow G\Delta^q G \xrightarrow{\partial} G \longrightarrow 1.$$

Proof: Let D denote the kernel of ∂. We have already pointed out that $1 \to D \to G\Delta^q G \to G\#_q G \to 1$ is a q-central extension. (We emphasise that the centrality was a consequence of the crossed module rules for ∂.) It is a q-central extension of G if and only if G is q-perfect. Further, it is a consequence of (17) that $D = H_2(G,\mathbb{Z}_q)$, as pointed out in [**E-R**].

Suppose that $1 \to A' \xrightarrow{i} E' \xrightarrow{\pi} G \to 1$ is a q-central extension of G. As already pointed out, there is a morphism $\eta : G\Delta^q G \to E$ such that $\pi\eta = \partial$. Suppose now that G is q-perfect, and that ζ is another such morphism. Then $\eta\zeta^{-1}$ is a function $G\Delta^q G \to E$ with values in A, and so is a morphism since the extension is central. If G is q-perfect, then so also is $G\Delta^q G$, by rules already proved and the rule

$$[g,h] \wedge [g',h'] = [g \wedge g', h \wedge h']$$

proved in [**B-L**, Proposition 2.3]. Hence $\eta\zeta^{-1}$ is constant, since A is abelian with every element of order dividing q. ■

Remark 19. We can now show that in general the morphism $\alpha : G \otimes G \to G \otimes^q G$ is not injective. If G is perfect, then these groups contain as subgroups H_2G and $H_2(G,\mathbb{Z}_q)$ respectively. There are perfect groups G for which the natural morphism $H_2G \to H_2(G,\mathbb{Z}_q)$ induced by α is not injective.

For example, if G is the simple group $B_3(3)$, then H_2G is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$ ([K, p. 283]).

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