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HYPERBOLICITY IN A CLASS OF ONE–DIMENSIONAL MAPS

GREGORY J. DAVIS

Abstract

In this paper we provide a direct proof of hyperbolicity for a class of one-dimensional maps on the unit interval. The maps studied are degenerate forms of the standard quadratic map on the interval. These maps are important in understanding the Newhouse theory of infinitely many sinks due to homoclinic tangencies in two dimensions.

Introduction

In the theory of infinitely many sinks, two–dimensional invariant sets are formed when homoclinic tangencies between stable and unstable manifolds of a hyperbolic periodic point are formed. In order to show that infinitely many sinks occur in this situation, we must show that these invariant sets are hyperbolic ([**1**], [**4**], or [**6**]), which is a major undertaking.

When the homoclinic tangency is quadratic in nature, the two-dimensional problem has been thought of as a perturbation of the one–dimensional map ([**3**], [**5**]). In [**5**], a more complete and elegant proof was obtained by conjucating the hyperbolic invariant set for the quadratic map $f_b(x) = bx(1-x)$ to the two–dimensional invariant set in the two–dimensional infinitely many sinks problem. In the case where the homoclinic tangency is degenerate (i.e., of order r, $r = 4, 6, \dots$), the present proof is very long and involved [**1**]. If a conjucacy between the one and two–dimensional degenerate problems can be determined, then it may be possible to treat the higher order tangencies in two dimensions using the same type of ideas as presented in [**5**].

With the above motivation, we will examine the hyperbolicity of the following family of one–dimensional maps on the unit interval. The one-dimensional maps that we are concerned with are of the form

$$f_b(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right],\ b > 2^r,\ r = 4, 6, 8, \dots$$

where b is a real parameter and r is a fixed positive even integer. By studying these one–dimensional maps we will gain insight as to how the hyperbolicity of an invariant set near a degenerate homoclinic tangency in two–dimensions is justified. We will restrict our attention to the cases where $r > 2$, due to the fact that when $r = 2$ we obtain the well known and studied quadratic map $f_b(x) = bx(1-x)$; see [**2**], [**3**] or [**5**] for more information about this map.

Statements of results

It is easily seen that $f_b(x) = 0$ for $x = 0, 1$ and that $f_b(x)$ achieves its maximum value of $b/2^r$ when $x = 1/2$. In order for $I = [0,1]$ to be covered by itself under the map $f_b(x)$, b must be larger than 2^r. We are interested in the set Λ contained in I that is invariant for $f_b(x)$; i.e., $f_b(\Lambda) = \Lambda$. Explicitly, the invariant set which we are interested in for this map is given by $\Lambda = \bigcap_{i=0}^{\infty} f_b^{-i}(I)$. (The set Λ is the analogue of $\Lambda_n(t)$ in the two-dimensional problem [1]). It is our goal to show that Λ is a hyperbolic set for $f_b(x)$.

A *hyperbolic set* for f_b is a closed, bounded, invariant set Λ for which there exists an $K > 0$ such that for all $x \in \Lambda$ we have $|(f_b^k)'(x)| > 1$ for all $k > K$. If b is large enough, the hyperbolicity of Λ is relatively easy to establish.

Set $f_b^{-1}(I) = I_1 \cup I_2$, where $I_1 = [0, c]$, $c \in [0, 1/2]$, and $I_2 = [d, 1]$, $d \in [1/2, 1]$, so that $f_b(I_i) = I$, $i = 1, 2$. Then we have

Proposition 1.
Let $f_b(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right]$, $b > 2^r$, $r = 4, 6, 8, \ldots$ *then for* $b \geq 2^r\left(\frac{3r+2}{3r}\right)$, $|f_b'(x)| > 1$ *for all* $x \in f_b^{-1}(I)$.

The proof of proposition 1 and the remaining propositions in this section will be deferred to the next section.

Continuing with our discussion of the hyperbolicity of $f_b(x)$, if

$$2^r < b < 2^r\left(\frac{3r+2}{3r}\right),$$

then not all points in the pre-image of I, $f_b^{-1}(I)$, have $|f_b'(x)| > 1$. Fortunately, the absolute value of $f_b'(p)$ is greater than one where p is the fixed point of f_b which is an element of I_2. (See Figure 1).

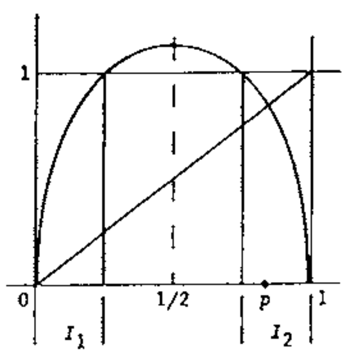

Figure 1

Proposition 2.

The fixed point p *of* $f_b(x)$ *which lies in the interval* I_2 *is a repellor, that is* $|f_b'(p)| > 1$.

Let $\hat{x}_i$ *be the points such that* $|f_b'(\hat{x}_i)| = 1$; $\hat{x}_i \in I_i$, $i = 1, 2$. *Define* $J_1 = [0, \hat{x}_1)$ *and* $J_2 = (\hat{x}_2, 1]$. *Clearly all* $x \in J_1 \cup J_2$ *have the property* $|f_b'(x)| > 1$.

Proposition 3.

If $x_0 \in \Lambda \cap [(I_1 \cup I_2) - (J_1 \cup J_2)]$, *then there is a positive integer* $k = k(x_0) > 1$ *such that* $|(f_b^k)'(x_0)| > 1$.

The idea behind Proposition 3 is that the values of x_0 that do not have $|(f_b)'(x_0)| > 1$ are mapped close to one and then near zero. When a point is mapped near zero, the point stays in the interval J_1, where the derivative is greater than one, for its next few iterates. The accumulative result is that the iterates for which the derivative is greater than one overcome the initial contraction of the point x_0.

We now have, by Propositions 1–3, the following result:

Main result.

Let $f_b(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right]$, $b > 2^r$, $r = 4, 6, 8, \ldots$ fixed. The invariant set $\Lambda = \bigcap_{i=0}^{\infty} f_b^{-i}(I)$ is a hyperbolic set for $f_b(x)$.

Proofs of results

Proof of Proposition 1:

Set $\alpha = \frac{3r+2}{3r}$, and $b_1 = 2^r\left(\frac{3r+2}{3r}\right) = 2^r\alpha$.

It is sufficient to show $|f_{b_1}'(x)| > 1$ since $|f_b'(x)| \geq |f_{b_1}'(x)|$ for $b > b_1$. Consider the function

$$f_{b_1}(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right] = \alpha[1 - (2x - 1)^r].$$

Differentiating $f_{b_1}(x)$ with respect to x, we obtain

$$f_{b_1}'(x) = -2r\alpha(2x - 1)^{r-1}.$$

Therefore, $|f_{b_1}'(x)| = |2r\alpha(2x - 1)^{r-1}$.

Let $x_i \in I$, where $i = 1, 2$, be the points such that $f_{b_1}(x_i) = 1$; i.e.,

$$\alpha[1 - (2x_i - 1)^r] = 1.$$

Solving for x_i we obtain,

$$1 - (2x_i - 1)^r = \frac{1}{\alpha}$$
$$(2x_i - 1)^r = 1 - \frac{1}{\alpha}$$

or

$$x_i = \frac{1}{2}\left[\pm\left(1-\frac{1}{\alpha}\right)^{\frac{1}{r}}+1\right].$$

For all $x \in f_{b_1}^{-1}(I)$,

$$|f'_{b_1}(x)| \geq |f'_{b_1}(x_i)|,$$

and

$$\begin{aligned}
|f'_{b_1}(x_i)| &= \left|2r\alpha\left(1-\frac{1}{\alpha}\right)^{\frac{r-1}{r}}\right| \\
&\geq \left|2r\left(\frac{\alpha-1}{\alpha}\right)^{\frac{r-1}{r}}\right| \\
&= 2r\left[\frac{\left(\frac{3r+2}{3r}\right)-1}{\frac{3r+2}{3r}}\right]^{\frac{r-1}{r}} \\
&= 2r\left[\frac{\frac{2}{3r}}{\frac{3r+2}{3r}}\right]^{\frac{r-1}{r}} \\
&= 2r\left(\frac{2}{3r+2}\right)^{\frac{r-1}{r}} \\
&= \frac{4r}{3r+2}\left(\frac{2}{3r+2}\right)^{\frac{-1}{r}}.
\end{aligned}$$

Therefore $|f'_{b_1}(x)| \geq \frac{4r}{3r+2}\left(\frac{3r+2}{2}\right)^{\frac{1}{r}} > 1$.

Thus $|f'_{b_1}(x)| > 1$ for all $x \in f_{b_1}^{-1}(I)$, and Proposition 1 is proven. ■

Proof of Proposition 2:

The idea of this proof is to show that $\hat{x}_2 < p < 1$ which implies $|f'_b(p)| > 1$. Consider $g_b(x) = f_b(x) - x$, and note that $g_b(p) = 0$.

Set $z = (x - 1/2)$, then

$$\begin{aligned}
g_b(x) &= f_b\left(z+\frac{1}{2}\right) - \left(z+\frac{1}{2}\right) \\
&= b\left(\frac{1}{2^r} - z^r\right) - \left(z+\frac{1}{2}\right) \\
&= -b\left(z^r + \frac{z}{b} + \frac{2^{r-1}-b}{2^r b}\right).
\end{aligned}$$

Define the function $h_b(z)$ to be

$$h_b(z) = z^r + \frac{z}{b} + \frac{2^{r-1}-b}{2^r b}.$$

Evaluating $h_b(z)$ at $z = (p - 1/2)$ and $z = 0$ yields the following:

$$h_b(p - 1/2) = 0,$$

and

$$h_b(0) = \frac{2^{r-1} - b}{2^r b} < 0$$

since $b > 2^r$. Because $h_b(y)$ has only one variation in sign, Descartes's rule of signs implies that $h_b(y)$ has at most one positive real root; however, we already know that $z = (p - 1/2)$ is a positive root of $h_b(z)$. Therefore, to prove that $\hat{x}_2 < p$, that is $p \in J_2$, it is sufficient to show that $h_b(\hat{z}) < 0$ where $\hat{z} = (\hat{x}_1 - 1/2)$. First it is necessary to calculate $\hat{z}$. Recall that

$$f_b(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right],$$

and

$$f_b'(x) = -rb\left(x - \frac{1}{2}\right)^{r-1}.$$

The point $\hat{x}_2$ satisfies the relation $f_b'(\hat{x}_2) = -1$; that is

$$f_b'(\hat{x}_2) = -rb\left(\hat{x}_2 - \frac{1}{2}\right)^{r-1} = -1.$$

Replacing $(\hat{x}_2 - 1/2)$ with $\hat{z}$ we obtain

$$-rb\hat{z}^{r-1} = -1, \text{ or } \hat{z} = (rb)^{\frac{-1}{r-1}}.$$

We now proceed to show that $h_b(\hat{z}) < 0$.

$$\begin{aligned}
h_b(\hat{z}) &= \hat{z}^r + \hat{z}/b + (2b)^{-1} - 2^{-r} \\
&= (rb)^{\frac{-1}{r-1}} + b^{-1}(rb)^{\frac{1}{r-1}} + (2b)^{-1} - 2^{-r} \\
&< \left(\frac{1}{r2^r}\right)^{\frac{r}{r-1}} + \frac{1}{2^r}\left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}} + \frac{1}{2\cdot 2^r} - 2^{-r}; \text{ since } b > 2^r \\
&= \left(\frac{1}{r2^r}\right)^{\frac{r}{r-1}} + \frac{1}{2^r}\left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}} - \frac{1}{2^{r+1}} \\
&= \frac{1}{2^r}\left[\left(\frac{1}{r^r 2^{2r}}\right)^{\frac{1}{r-1}} + \left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}} - \frac{1}{2}\right] \\
&= \frac{1 + r - 2^{\frac{1}{r-1}} r^{\frac{r}{r-1}}}{2^r(2r)^{\frac{r}{r-1}}}.
\end{aligned}$$

Set $A = \frac{1}{2^r(2r)^{\frac{r}{r-1}}}$, then

$$\begin{aligned} h_b(\hat{z}) &< A\left(1 + r - r(2r)^{\frac{1}{r-1}}\right) \\ &< A\left(1 + r - r\left[1 + \frac{1}{r-1}\ln(2r)\right]\right) \qquad (*) \\ &= A\left(1 - \frac{r}{r-1}\ln(2r)\right) \\ &= A\left(1 - \left(1 + \frac{1}{r-1}\right)\ln(2r)\right) \\ &< A\left(-\frac{\ln(2r)}{r-1}\right), \text{ since } \ln(2r) > 1 \text{ for } r \geq 2 \\ &< 0. \end{aligned}$$

Thus, we will have shown that $h_b(\hat{z}) < 0$ as soon as equation $(*)$ is verified; however, equation $(*)$ is true due to the fact that

$$\begin{aligned} (2r)^{\frac{1}{r-1}} &= e^{\frac{1}{r-1}\ln(2r)} \\ &= 1 + \frac{1}{r-1}\ln(2r) + \text{ (higher order terms)} \\ &< 1 + \frac{1}{r-1}\ln(2r) \end{aligned}$$

where (higher order terms) > 0 since $\frac{1}{r-1}\ln(2r) > 0$ for $r \geq 2$.

Therefore, $h_b(\hat{z}) < 0$, which implies $p \in J_2$ or $|f_b'(p)| > 1$, and the proof of Proposition 2 is complete. ■

Proof of Proposition 3:

Let $1 + g$ be the maximum value obtained by $f_b(x)$ where $x \in I$; i.e., let $1 + g = f_b(1/2)$. Choose any $x_0 \in \Lambda \cap [I_1 \cup I_2) - (J_1 \cup J_2)]$. Define $\delta = \delta(x_0) \geq 0$ so that the distance between $f_b(x_0)$ and $1 + g$ is $b\delta^r$, and define $\gamma = \gamma(x_0) \geq 0$ to be the distance between $f_b(x_0)$ and 1. Then $f_b'(x_0) = rb\delta^{r-1}$, and $\gamma \leq b\delta^r$.

(See Figure 2).

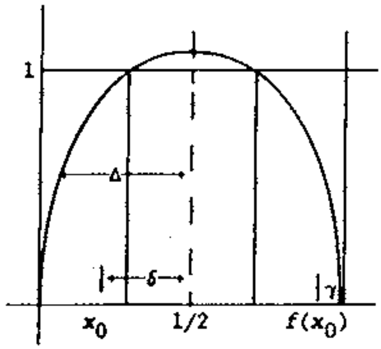

Figure 2

Let $\lambda_{\max}$ be the maximum value of $f_b'(x)$ where $x \in I$; that is, let

$$\lambda_{\max} = f_b'(0) = rb(1/2)^{r-1}.$$

Define $k = k(x_0) \geq 2$ to be the integer such that

$$\lambda_{\max}^{k-1}\gamma \geq \frac{1}{2},$$

but

$$\lambda_{\max}^{j}\gamma < \frac{1}{2}$$

for $j < k-1$. We now have that

$$\lambda_{\max}^{k-1} \geq \frac{1}{2\gamma} \geq \frac{1}{2b\delta^r}.$$

Define Δ, where $0 \leq \Delta \leq 1/2$, to be the value of δ for which

$$b\Delta^r = \frac{1}{2} + g$$

Then

$$b\Delta^r = \frac{1}{2} + g = f_b(1/2) = \frac{b}{2^r} - \frac{1}{2}.$$

Solving for Δ^r we find that

$$\Delta^r = \frac{1}{b}\left(\frac{b}{2^r} - \frac{1}{2}\right) = \frac{1}{b2^r}(b - 2^{r-1}).$$

Let $\lambda_{\min}$ be the minimum value of $f_b'(x)$ for $x \in [0, 1/2 - \Delta]$,

$$\lambda_{\min} = f_b'(1/2 - \Delta) = rb\Delta^{r-1}.$$

Using our formula for Δ^r we see that Δ^{r-1} is given by

$$\Delta^{r-1} = (\Delta^r)^{\frac{r-1}{r}} = \left(\frac{1}{b2^r}(b - 2^{r-1})\right)^{\frac{r-1}{r}}$$

Therefore,

$$\lambda_{\min} = rb\left(\frac{1}{b2^r}(b - 2^{r-1})\right)^{\frac{r-1}{r}}.$$

Combining the formulas for $\lambda_{\max}$ and $\lambda_{\min}$ we obtain the following ratio:

$$\frac{\lambda_{\min}}{\lambda_{\max}^{\frac{r-1}{r}}} = (b - 2^{r-1})^{\frac{r-1}{r}} r^{\frac{1}{r}} b^{\frac{2-r}{r}} 2^{\frac{1-r}{r}} > 1.$$

From this relation we have that

$$\lambda_{\min}^{k-1} \geq \left(\lambda_{\max}^{\frac{r-1}{r}}\right)^{k-1} (b - 2^{r-1})^{\frac{r-1}{r}} r^{\frac{1}{r}} b^{\frac{2-r}{r}} 2^{\frac{1-r}{r}}.$$

We will now show that $(f_b^k)'(x_0) > 1$

$$(f_b^k)'(x_0) \geq rb\delta^{r-1}\lambda_{\min}^{k-1}$$

Therefore,

$$\begin{aligned}
(f_b^k)'(x_0) &\geq (\lambda_{\max}^{k-1})^{\frac{r-1}{r}} (b - 2^{r-1})^{\frac{r-1}{r}} r^{\frac{1}{r}} b^{\frac{2-r}{r}} 2^{\frac{1-r}{r}} \\
&\geq rb\delta^{r-1} \frac{1}{2^{\frac{r-1}{r}} b^{\frac{r-1}{r}} \delta^{r-1}} (b - 2^{r-1})^{\frac{r-1}{r}} r^{\frac{1}{r}} b^{\frac{2-r}{r}} 2^{\frac{1-r}{r}} \\
&\geq 2^{\frac{2-2r}{r}} r^{\frac{1+r}{r}} b^{\frac{3-r}{r}} (b - 2^{r-1})^{\frac{r-1}{r}} \\
&= 2^{\frac{2-2r}{r}} r^{\frac{r+1}{r}} \left(\frac{(b - 2^{r-1})^{r-1}}{b^{r-3}}\right)^{\frac{1}{r}} \\
&= \frac{r^{\frac{r+1}{r}}}{4^{\frac{r-1}{r}}} \left(\frac{(b - 2^{r-1})^{r-1}}{b^{r-3}}\right)^{\frac{1}{r}} \\
&> 1.
\end{aligned}$$

It should be noted that the above argument is valid only for the values of x_0 for which $\gamma = \gamma(x_0)$ has the property

$$\lambda_{\max}^{k-1}\gamma \geq \frac{1}{2}, \text{ but } \lambda_{\max}^{j}\gamma < \frac{1}{2}$$

where $j < k-1$ and $k = k(x_0) \geq 2$.

In addition, the above argument is only legitimate in the case when $\delta < |1/2 - f_b^{-1}(1-\gamma_{\max})|$, where $\gamma_{\max}$ is the value of γ for which $\lambda_{\max}\gamma_{\max} = 1/2$. In Lemma 4 we will prove the existence of certain parameter values b for which there exist values of x_0 that do not satisfy the above condition. These values of b are shown to also have the property of $|(f_b^k)'(x_0)| > 1$, where $k = 2$, but an argument different from the one above is required to show this fact. This argument will be presented in Lemma 5. However, before we prove Lemma 4 and Lemma 5, it will be necessary to provide additional notation.

Set

$$\delta_{\max} = |1/2 - f_b^{-1}(1-\gamma_{\max})|$$

and define the intervals L_1 and L_2 to be

$$L_1 = I_1 \cap [1/2 - \delta_{\max},\, 1/2];\ L_2 = I_2 \cap [1/2,\, 1/2 + \delta_{\max}].$$

Lemma 4.

There exist intervals of parameter values $M(r) \subset (2^r, \alpha 2^r]$ such that $L_i \cup J_i \neq I_i$, $i = 1, 2$, where $\alpha = \left(\frac{3r+2}{3r}\right)$.

Proof:

Define $\widetilde{x}_i$, where $i = 1,2$, to be the values of x for which $f_b(\widetilde{x}_i) = 1 - \gamma_{\max}$, and define $\hat{x}_i$, where $i = 1,2$, to be the values of x for which $|f_b'(\hat{x}_i)| = 1$.

To prove Lemma 4 it is sufficient to show that there is an interval $M(r)$ of parameter values b for which $|1/2 - \widetilde{x}_i| < |1/2 - \hat{x}_i|$.

Solving $\lambda_{\max}\gamma_{\max} = 1/2$ for $\gamma_{\max}$ we obtain

$$\gamma_{\max} = \frac{1}{2\lambda_{\max}}.$$

Recall that

$$\lambda_{\max} = \frac{rb}{2^{r-1}}.$$

Therefore, $\gamma_{\max} = \frac{2^r}{4rb}$, and hence, $1 - \gamma_{\max} = 1 - \frac{2^r}{4rb}$.

The above equation implies that the value $\widetilde{x}_i$ satisfies

$$f_b(\widetilde{x}_i) = 1 - \frac{2^r}{4rb};$$

that is, $\widetilde{x}_i$ satisfies

$$\left[\frac{1}{2^r} - \left(\widetilde{x}_i - \frac{1}{2}\right)^r\right] = 1 - \frac{2^r}{4rb}.$$

Solving this equation for $\widetilde{x}_i$ we find that

$$\widetilde{x}_i = \mp\left(\frac{2^r}{4b^2r} - \frac{1}{b} + \frac{1}{2^r}\right)^{\frac{1}{r}} + \frac{1}{2}$$

which implies that

$$\left|\frac{1}{2} - \widetilde{x}_i\right| = \left(\frac{2^r}{4b^2r} - \frac{1}{b} + \frac{1}{2^r}\right)^{\frac{1}{r}}.$$

Due to the fact that the values $\hat{x}_i$ of x satisfy $|f_b'(\hat{x}_i)| = 1$, we have

$$|-rb(\hat{x}_i - 1/2)^{r-1}| = 1.$$

Upon solving this equation for $\hat{x}_i$, we see that

$$\hat{x}_i = \mp\left(\frac{1}{rb}\right)^{\frac{1}{r-1}} + \frac{1}{2}.$$

Therefore,

$$\left|\frac{1}{2} - \hat{x}_i\right| = \left(\frac{1}{rb}\right)^{\frac{1}{r-1}}.$$

Evaluating our formulas for $|1/2 - \widetilde{x}_i|$ and $|1/2 - \hat{x}_i|$ when the parameter b is equal to 2^r yields

$$\left|\frac{1}{2} - \widetilde{x}_i\right| = \left(\frac{2^r}{4(2^r)^2r} - \frac{1}{2^r} + \frac{1}{2^r}\right)^{\frac{1}{r}} = \left(\frac{1}{4r2^r}\right)^{\frac{1}{r}} = \frac{1}{2}\left(\frac{1}{4r}\right)^{\frac{1}{r}}$$

and

$$\left|\frac{1}{2} - \hat{x}_i\right| = \left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}} = \frac{1}{2}\left(\frac{1}{2r}\right)^{\frac{1}{r-1}}.$$

Hence when $b = 2^r$, we have $|1/2 - \widetilde{x}_i| < |1/2 - \hat{x}_i|$ which is due to the fact that

$$\left(\frac{1}{4r}\right)^{r-1} < \left(\frac{1}{2r}\right)^{r} \text{ for } r > 4,$$

or equivalently,

$$\left(\frac{1}{4r}\right)^{\frac{1}{r}} < \left(\frac{1}{2r}\right)^{\frac{1}{r-1}}.$$

Therefore, when the parameter b is equal to 2^r and $r > 4$, the intervals L_i are not void. As the value of the parameter b is increased, the distance between the points $\hat{x}_i$ and $1/2$ decreases and the distance between the points $\widetilde{x}_i$ and $1/2$ increases. When the parameter $b \geq \alpha 2^r$, we have by Proposition 1 $|f_b'(x)| > 1$ for all values of $x \in f_b^{-1}(I)$. Due to the continuity of $f_b(x)$ with respect to the parameter b, there is a parameter value b^* where $b^* \in (2^r, \alpha 2^r]$ for which $L_i \cup J_i = I_i$ whenever $b \geq b^*$. Define $M(r)$ to be the interval $(2^r, b^*) \subset (2^r, \alpha 2^r]$. Hence, if $b \in M(r)$ then $L_i \cup J_i \neq I_i$, and the proof of Lemma 4 is complete. ■

Corollary.
If $r = 4$, then $M(r) = \phi$.

Proof:
Evaluating

$$\left|\frac{1}{2} - \widetilde{x}_i\right| = \left(\frac{2^r}{4b^2 r} - \frac{1}{b} + \frac{1}{2^r}\right)^{\frac{1}{r}}, \text{ and } \left|\frac{1}{2} - \hat{x}_i\right| = \left(\frac{1}{rb}\right)^{\frac{1}{r-1}}$$

when $r = 4$ and $b = 2^4$ yields

$$\left|\frac{1}{2} - \widetilde{x}_i\right| = \frac{1}{4} = \left|\frac{1}{2} - \hat{x}_i\right|.$$

Therefore, when $r = 4$, $L_i \cup J_i = I_i$, and $M(r) = \phi$. ■

Define $K_i = I_i - (L_i \cup J_i)$, where $i = 1, 2$, for $r \geq 6$ and $b \in M(r)$.

Lemma 5.
If $b \in M(r)$ and $x_o \in K_i$, then $|f_b'(x_0) \cdot f_b'(f_b(x_0))| > 1$.

Proof:
It is enough to show that $|f_b'(\widetilde{x}_2) \cdot f_b'(f_b(\hat{x}_2))| > 1$ as $f_b'(x_0) \geq f_b'(\widetilde{x}_2)$ and $f_b'(f_b(x_0)) \geq f_b'(f_b(\widetilde{x}_2))$ for all $x_0 \in K_i$.

Recall that

$$\widetilde{x}_2 = \left(\frac{2^r}{4b^2 r} - \frac{1}{b} + \frac{1}{2^r}\right)^{\frac{1}{r}} + \frac{1}{2};\ \hat{x}_2 = \left(\frac{1}{rb}\right)^{\frac{1}{r-1}} + \frac{1}{2}.$$

Since

$$f_b(x) = b\left[\frac{1}{2^r} - \left(x - \frac{1}{2}\right)^r\right]$$

and

$$f_b'(x) = -rb\left(x - \frac{1}{2}\right)^{r-1},$$

we have

$$f_b'(\widetilde{x}_2) = -rb\left(\frac{2^r}{4rb^2} + \frac{1}{2^r} - \frac{1}{b}\right)^{\frac{r-1}{r}}$$

$$f_b(\hat{x}_2) = b\left[\frac{1}{2^r} - \left(\frac{1}{rb}\right)^{\frac{r}{r-1}}\right]$$

$$f_b'(f_b(\hat{x}_2)) = -rb\left(b\left[\frac{1}{2^r} - \left(\frac{1}{rb}\right)^{\frac{r}{r-1}}\right] - \frac{1}{2}\right)^{r-1}.$$

By substituting the above expressions for $f_b(\hat{x}_2)$ and $f_b'(f_b(\hat{x}_2))$ into $|f_b'(\widetilde{x}_2) \cdot f_b'(f_b(\hat{x}_2))|$ we obtain the following:

$$
\begin{aligned}
|f_b'(\widetilde{x}_2) \cdot f_b'(f_b(\hat{x}_2))| &= r^2 b^2 \left(\frac{2^r}{4rb^2} + \frac{1}{2^r} - \frac{1}{b}\right)^{\frac{r-1}{r}} \left(b\left[\frac{1}{2^r} - \left(\frac{1}{rb}\right)^{\frac{r}{r-1}}\right] - \frac{1}{2}\right)^{r-1} \\
&> r^2 2^{2r} \left(\frac{2^r}{4rb^2}\right)^{\frac{r-1}{r}} \left(2^r\left[\frac{1}{2^r} - \left(\frac{1}{rb}\right)^{\frac{r}{r-1}}\right] - \frac{1}{2}\right)^{r-1} \\
&= r^2 2^{2r} \frac{2^r}{4rb^2} \left(\frac{4rb^2}{2^r}\right)^{\frac{1}{r}} \left(\left[1 - 2^r\left(\frac{1}{rb}\right)^{\frac{r}{r-1}}\right] - \frac{1}{2}\right)^{r-1} \\
&> \frac{r2^{3r}}{4 \cdot 2^{2r}\left(1 + \frac{2}{3r}\right)^2} (4r2^r)^{\frac{1}{r}} \left[\frac{1}{2} - \frac{2^r}{r2^r}\left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}}\right]^{r-1} \\
&= \frac{r2^{r-1} r^{\frac{1}{r}} 4^{\frac{1}{r}}}{\left(1 + \frac{2}{3r}\right)^2} \left[\frac{1}{2} - \frac{1}{r}\left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}}\right]^{r-1} \\
&> \frac{r2^{r-1}}{\left(1 + \frac{2}{3r}\right)^2} \left[\frac{1}{2} - \frac{1}{r}\left(\frac{1}{r2^r}\right)^{\frac{1}{r-1}}\right]^{r-1} \\
&= \frac{r}{\left(1 + \frac{2}{3r}\right)^2} \left[1 - \frac{1}{r}\left(\frac{1}{2r}\right)^{\frac{1}{r-1}}\right]^{r-1} \\
&> \frac{r}{\left(1 + \frac{2}{3r}\right)^2} \left[1 - \frac{1}{r}\right]^{r-1} \\
&> \frac{r}{\left(1 + \frac{1}{9}\right)^2} \cdot \frac{1}{e} \\
&= \frac{.81r}{e} > \frac{r}{4} > 1.
\end{aligned}
$$

Therefore, we have shown that whenever the parameter b is an element of $M(r)$ and x_0 is an element of K_i, then $|f_b'(x_0) \cdot f_b'(f_b(x_0))| > 1$. ■

As we have now completed the proofs of Lemma 4 and Lemma 5, we have also completed the proof of Proposition 3. ■

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Department of Mathematics
University of Wisconsin–Green Bay
Green Bay, WI 54311
U.S.A.

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