

**PERIODIC PARABOLIC PROBLEMS WITH
NONLINEARITIES INDEFINITE IN SIGN**

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We give sufficient conditions (which are also necessary in many cases) on two nonnegative functions a, b that are possibly discontinuous and unbounded for the existence of nonnegative solutions for semilinear Dirichlet periodic parabolic problems of the form $Lu = \lambda a(x, t)u^p - b(x, t)u^q$ in $\Omega \times \mathbb{R}$, where $0 < p, q < 1$ and $\lambda > 0$. In some cases we also show the existence of solutions u_λ in the interior of the positive cone and that u_λ can be chosen such that $\lambda \rightarrow u_\lambda$ is differentiable and increasing. A uniqueness theorem is also given in the case $p \leq q$. All results remain valid for the corresponding elliptic problems.

1. Introduction

Let Ω be a $C^{2+\theta}$ bounded domain in \mathbb{R}^N , $N \geq 2$, $\theta \in (0, 1)$. For $T > 0$ and $1 \leq p \leq \infty$, let L_T^p be the Banach space of T -periodic functions f on $\Omega \times \mathbb{R}$ (i.e. satisfying $f(x, t) = f(x, t + T)$ a.e. $(x, t) \in \Omega \times \mathbb{R}$) such that $f|_{\Omega \times (0, T)} \in L^p(\Omega \times (0, T))$, equipped with the norm $\|f\|_{L_T^p} := \|f|_{\Omega \times (0, T)}\|_{L^p(\Omega \times (0, T))}$. Let C_T be the space of continuous and T -periodic functions on $\overline{\Omega} \times \mathbb{R}$ provided with the L^∞ norm, and let $C_T^{1+\theta, (1+\theta)/2}$ be the space of T -periodic functions belonging to $C^{1+\theta, (1+\theta)/2}(\overline{\Omega} \times \mathbb{R})$.

Let $\{a_{ij}\}, \{b_j\}$, $1 \leq i, j \leq N$, be two families of T -periodic functions satisfying $a_{ij} \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$, $a_{ij} = a_{ji}$ and $b_j \in L_T^\infty$, and assume that

$$\sum a_{ij}(x, t) \xi_i \xi_j \geq \alpha_0 |\xi|^2$$

for some $\alpha_0 > 0$ and all $(x, t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i, j entry is a_{ij} , let $\vec{b} = (b_1, \dots, b_N)$, let $0 \leq c_0 \in L_T^r$, $r > N + 2$,

2000 *Mathematics Subject Classification.* 35K20, 35P05, 35B10.

Key words. Periodic, parabolic, indefinite.

Partially supported by CONICET, SECYT-UNC, ANPCYT and Fundación Antorchas.

and let L be the parabolic operator given by

$$Lu = u_t - \operatorname{div}(A\nabla u) + \langle \bar{b}, \nabla u \rangle + c_0 u.$$

Let $W = \{u \in L^2((0, T), H_0^1(\Omega)) : u_t \in L^2((0, T), H^{-1}(\Omega))\}$. For a given $f \in L_T^2$, we will say that u is a (weak) solution of the periodic problem

$$(1.1) \quad \begin{cases} Lu = f & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \text{ } T\text{-periodic} \end{cases}$$

if u is T -periodic, $u|_{\Omega \times (0, T)} \in W$ and

$$\int_{\Omega \times (0, T)} \left[-u \frac{\partial h}{\partial t} + \langle A\nabla u, \nabla h \rangle + \langle \bar{b}, \nabla u \rangle h + c_0 u h \right] = \int_{\Omega \times (0, T)} f h$$

for all $h \in C_c^\infty(\Omega \times (0, T))$. For $u \in W$, the inequality $Lu \geq f$ (resp. \leq) in $\Omega \times \mathbb{R}$, $u = 0$ in $\partial\Omega \times \mathbb{R}$ will be understood in the analogous weak sense.

For $1 \leq r \leq \infty$ let $W_r^{2,1}(\Omega \times (t_0, t_1))$ be the Sobolev space of the functions $u \in L^r(\Omega \times (t_0, t_1))$, $u = u(x, t)$, $x = (x_1, \dots, x_N)$ such that u_t , u_{x_j} and $u_{x_i x_j}$ belong to $L^r(\Omega \times (t_0, t_1))$ for $1 \leq i, j \leq N$, and let $W_{r,T}^{2,1}$ be the space of T -periodic functions such that $u|_{\Omega \times (0, T)} \in W_r^{2,1}(\Omega \times (0, T))$. For $f \in L_T^r$, $r > 1$, we say that u is a strong solution of (1.1) if $u \in W_{r,T}^{2,1}(\Omega \times \mathbb{R})$ and the equation holds a.e. in the pointwise sense.

The existence of positive solutions for periodic parabolic problems of the form

$$(1.2) \quad \begin{cases} Lu = g(x, t, u) & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \text{ } T\text{-periodic} \end{cases}$$

has been widely studied (see e.g. [15] and the references therein). For applications we refer to [15], [5]. In [11] and [13], bifurcation of positive solutions for (1.2) was proved assuming that $\xi \rightarrow g(x, t, \xi)/\xi$ is nonincreasing in $(0, \infty)$ and that $g_\xi(x, t, 0)$ belongs to L_T^r for some $r > (N+2)/2$. On the other hand, in [12], existence results of positive solutions for (1.2) were given without monotonicity conditions on g and allowing $g_\xi(x, t, 0) = +\infty$, but assuming that $\inf_{0 < \sigma \leq \xi} (g(x, t, \sigma)/\sigma)$ belongs to L_T^r for some $r > (N+2)/2$. However, in many cases of interest

neither of the above conditions hold. A typical example of this situation is the problem

$$(1.3) \quad \begin{cases} Lu = \lambda a(x, t) u^p - b(x, t) u^q & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \text{ } T\text{-periodic} \end{cases}$$

where a, b are two nonnegative functions, $0 < p, q < 1$ and $\lambda > 0$.

Our aim in this paper is to study (1.3), where a, b belong to L_T^r with $r > N + 2$. Concerning this problem, we will show existence of nonnegative solutions for all $\lambda > 0$ under weak conditions on a and b (see Theorem 3.2 and also Remark 3.3), using iterative and fixed point methods combined with some facts about linear problems with weight. Also, under different assumptions on a and b , we will prove existence of solutions in the interior of the positive cone of $C_T^{1+\theta, (1+\theta)/2}$ for λ large enough and, for $p \leq q$, that u can be chosen such that $\lambda \rightarrow u_\lambda$ is differentiable and increasing (see Theorem 3.4). These last results will follow from a sub and supersolution approach together with the implicit function theorem. Finally, a uniqueness theorem for the solutions in the interior of the positive cone is given in Theorem 3.5 for the case $p \leq q$.

To avoid unnecessary complexity, we restrict ourselves to (1.3), but one can see that most of the results are still valid for increasing nonlinearities that behave like u^p near the origin and infinity. We mention also that as a consequence of our proofs all results remain true for the analogous elliptic problem.

In order to relate our results to others in the literature, let us mention that similar elliptic problems have been studied for example in [20] for $L = -\Delta$ and $a, b \in C(\overline{\Omega})$ using a variational approach, and recently for $a, b \in C^1(\Omega)$ and allowing these functions to have a singularity in the boundary in [14, Section 3] (see also the references therein). The particular case $Lu = mu^p$ in Ω , with $0 < p < 1$ and m changing sign, was treated in detail in [2] for $L = -\Delta$ and $m \in C^\theta(\overline{\Omega})$, $\theta \in (0, 1)$, using sub and supersolutions to construct nonnegative solutions, and there is also a result for the associated parabolic initial boundary value problem there. For the one dimensional elliptic problem (1.3), a precise description of the solution set when $p > q$ and $a = b \equiv 1$ was given in [8] for the operator $Lu = -(|u'|^{s-2} u')'$, $s > 1$. On the other hand, in the periodic parabolic case, a similar problem to (1.3) with $p = q$ was considered extensively in [16] for the heat operator with Neumann boundary condition, and it is also shown there that the nonnegative solutions are

not necessarily unique. Existence and multiplicity of changing sign solutions for similar periodic parabolic problems were also considered in [17]. We finally mention that related elliptic superlinear problems have been widely studied too, see e.g. [1] and references therein.

2. Preliminaries and auxiliary results

We start collecting some known facts about the (weak) solution operator (denoted by L^{-1}) and strong solutions for problem (1.1).

Remark 2.1. (i) For $r > (N + 2)/2$ it holds that $L^{-1}(L_T^r) \subset C_T$ and $L^{-1}: L_T^r \rightarrow C_T$ is a compact and positive operator (see e.g. [4, Section 5], also [11, Remarks 2.1 and 2.2]).

(ii) Given $f \in L_T^r$, $r > N + 2$, there exists a unique solution $u \in W_{r,T}^{2,1}$ of (1.1) and the operator $L^{-1}: L_T^r \rightarrow W_{r,T}^{2,1}$ is continuous (see e.g. [19, Section 4]). Moreover, from the Sobolev imbedding theorems (e.g. [18, Lemma 3.3, p. 80]) it follows that $u \in C_T^{1+\theta, (1+\theta)/2}$; and if $f \geq 0$, the strong maximum principle (e.g. [5, Theorem 13.5]) gives that $u > 0$ in $\Omega \times \mathbb{R}$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega \times \mathbb{R}$, where ν denotes the outward unit normal to $\partial\Omega$.

We recall also some necessary facts about periodic parabolic problems with weight.

Remark 2.2. (i) Let $m \in L_T^r$, $r > (N + 2)/2$, and let

$$P(m) := \int_0^T \operatorname{esssup}_{x \in \Omega} m(x, t) dt.$$

Then $P(m) > 0$ is necessary and sufficient for the existence of a (unique) positive principal eigenvalue $\lambda_1(m)$ for the periodic problem $Lu = \lambda mu$ in $\Omega \times \mathbb{R}$, $u = 0$ on $\partial\Omega \times \mathbb{R}$ (cf. [10, Theorem 3.6]).

(ii) For $\lambda \in \mathbb{R}$, let $\mu_m(\lambda)$ be defined as the unique $\mu \in \mathbb{R}$ such that the Dirichlet periodic problem $Lu = \lambda mu + \mu_m(\lambda)u$ in $\Omega \times \mathbb{R}$ has a positive solution u . Then $\mu_m(\lambda)$ is well defined, $\mu_m(0) > 0$, μ_m is concave and continuous, and a given $\lambda \in \mathbb{R}$ is a principal eigenvalue associated to the weight m if and only if $\mu_m(\lambda) = 0$ (cf. [10, Lemma 3.2]). Also, if $\lambda_1(m)$ exists, then for $\lambda > 0$, $\mu_m(\lambda) > 0$ if and only if $\lambda < \lambda_1(m)$, and if $\lambda_1(m)$ does not exist, $\mu_m(\lambda) > 0$ for all $\lambda > 0$.

(iii) If $\mu_m(\lambda) > 0$ then for all $h \in L_T^r$ the T -periodic problem

$$(2.1) \quad \begin{cases} Lu = \lambda mu + h & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \text{ } T\text{-periodic} \end{cases}$$

has a unique solution $u \in W_{r,T}^{2,1}$, which is positive if $h > 0$, and the solution operator $h \rightarrow u$ for this problem is continuous from L_T^r into C_T (cf. [11, Lemma 2.9]). In particular, these conclusions apply for all $\lambda \geq 0$ if $\lambda_1(m)$ does not exist and for $0 \leq \lambda < \lambda_1(m)$ when $\lambda_1(m)$ exists.

Conversely, if (2.1) has a positive solution h and $\lambda_1(m)$ exists, then $\lambda < \lambda_1(m)$ (cf. [13, Remark 2.1 (e)]).

(iv) The following comparison principle holds: if $m_1, m_2 \in L_T^r$, $P(m_1) > 0$ and $m_1 \leq m_2$ in $\Omega \times \mathbb{R}$, then $\lambda_1(m_1) \geq \lambda_1(m_2)$ and, if in addition $m_1 < m_2$ in a set of positive measure, then $\lambda_1(m_1) > \lambda_1(m_2)$ (cf. [10, Remark 3.7]).

The following lemma states a maximum principle for noncylindrical domains.

Lemma 2.3. *Let $D \subset \Omega \times \mathbb{R}$ be a domain, let $0 \leq g \in L^r(D)$, $r > N+2$, and let $u \in W_{r,T}^{2,1}(\Omega \times \mathbb{R})$ such that*

$$\begin{cases} Lu = g & \text{in } D \\ u \geq 0 & \text{on } \partial D. \end{cases}$$

Then $u \geq 0$ in D .

Proof: Since $u \in C(\overline{D})$, there exists $c := \min_{\overline{D}} u$. Suppose $c < 0$, and let $(x_0, t_0) \in \overline{D}$ such that $u(x_0, t_0) = c$. Then $(x_0, t_0) \in D$. Let $Q \subset D$ be a smooth bounded cylinder of the form $Q = \Omega_0 \times (t_0 - \delta, t_0 + \delta)$, $\delta > 0$, such that $x_0 \in \partial\Omega_0$. Taking into account that $\nabla u(x_0, t_0) = 0$, the maximum principle in Proposition 13.2 in [5] says that $u \equiv c$ in Q . Now, choosing adequately such Q 's (for example, taking $\partial\Omega_0$ "planar" around x_0) we can cover a neighborhood $B_\rho(x_0) \times (t_0 - \delta, t_0 + \delta)$ with a finite number of them, and hence it follows that the set $C := \{(x, t) \in D : u(x, t) = c\}$ is open. Since it is also closed we have $C = D$. But then $u < 0$ on ∂D . Contradiction. \square

Remark 2.4. In Lemma 2.3, the regularity assumptions on the coefficients of L can be weakened. In fact, it is clear from the proof that it suffices that they hold in each compact subset of D .

3. The main results

Lemma 3.1. *Let $b, f \in L_T^r$, $r > N + 2$, $b \geq 0$. Then, there exists a unique $u \in W_{r,T}^{2,1}$ solution of*

$$(3.1) \quad \begin{cases} Lu + b|u|^{q-1}u = f & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \text{ } T\text{-periodic.} \end{cases}$$

Moreover, the solution operator $S: C_T \rightarrow C_T$ is compact and increasing. In particular, $f > 0$ implies $Sf > 0$.

Proof: Let $v \in C_T$. From Remark 2.1 (ii) we have that there exists a unique solution $u \in W_{r,T}^{2,1}$ of the Dirichlet periodic problem

$$(3.2) \quad Lu + b|v|^{q-1}v = f \quad \text{in } \Omega \times \mathbb{R}.$$

We note that the solution operator $S_f: C_T \rightarrow C_T$ is compact. Indeed, Lebesgue's dominated convergence theorem gives that the map $v \rightarrow |v|^{q-1}v$ is continuous from C_T into L_T^s for all $s \geq 1$, and therefore $v \rightarrow -b|v|^{q-1}v + f$ is continuous from C_T into L_T^l for $l > N + 2$. Thus, the compactness follows from Remark 2.1 i).

For $R > 0$, let $\overline{B}_R^{C_T}$ be the closed ball in C_T with center at 0 and radius R . We claim that $S_f: \overline{B}_R^{C_T} \rightarrow \overline{B}_R^{C_T}$ for R large enough. Indeed, from (3.2) we have $|Lu| \leq |f| + R^q b$ and thus

$$-(L^{-1}|f| + R^q L^{-1}b) \leq u \leq L^{-1}|f| + R^q L^{-1}b.$$

Hence, $|u| \leq c_1 + c_2 R^q$ where c_1, c_2 do not depend on u and the claim follows. So, the Schauder fixed point theorem (e.g. [9, Corollary 11.2]) gives a solution for (3.1).

Suppose now there exist two solutions $u, w \in W_{r,T}^{2,1}$ for (3.1). Then

$$L(u - w) = b(|w|^{q-1}w - |u|^{q-1}u) \quad \text{in } \Omega \times \mathbb{R}.$$

Since $(|w|^{q-1}w - |u|^{q-1}u) / (w - u)$ is always positive when $u \neq w$, the maximum principle in Lemma 2.3 implies $u = w$.

To end the lemma, we note that the compactness of S follows reasoning as in the beginning of the proof, and the fact that S is increasing follows by Lemma 2.3 in a similar way as above. \square

Theorem 3.2. *Let $0 < p, q < 1$, and let $0 \leq a, b \in L_T^r$, $r > N + 2$, such that the following condition holds*

(H1) *There exists an open set $B := \Omega_0 \times (t_0, t_1) \subset \Omega \times \mathbb{R}$ such that $a(x, t) > 0$ in a subset of B of positive measure and $b \equiv 0$ in B .*

Then, for all $\lambda > 0$ there exists a (nontrivial) nonnegative solution $u \in W_{r,T}^{2,1}$ of (1.3).

Proof: Let $0 \leq v \in C_T$ and $\lambda > 0$. By the above lemma there exists a unique positive solution $u \in W_{r,T}^{2,1}$ of (3.1) with $f := \lambda av^p$. Let S_f be the operator defined by $S_f v = S(\lambda av^p)$ where S is the solution operator of Lemma 3.1. Then S_f is compact and nondecreasing. Note that if $v \equiv c$ for some constant c large enough, then $S_f v \leq v$. Indeed, let $u = S_f v$. From (3.1) it follows that $Lu \leq \lambda av^p$ and so $\|u\|_\infty \leq Kc^p$ for some K independent of c .

Take $v \equiv c \gg 0$, consider the nonincreasing sequence $\{S_f^j v\}_{j=0}^\infty$ and let $u_\infty \geq 0$ be its limit. We have

$$(3.3) \quad \begin{cases} Lu_{j+1} = \lambda au_j^p - bu_{j+1}^q & \text{in } \Omega \times \mathbb{R} \\ u_{j+1} = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u_{j+1} \text{ } T\text{-periodic,} \end{cases}$$

where $u_j = S_f^j v$. Now, by Lebesgue's theorem the right side of (3.3) converges in L_T^r , thus $\{u_{j+1}\}_{j=1}^\infty$ converges in $W_{r,T}^{2,1}$ (and so also in C_T). Then $u_\infty \in W_{r,T}^{2,1}$ and $u_\infty = 0$ on $\partial\Omega \times \mathbb{R}$. Going to the limit in (3.3) we find that $Lu_\infty = \lambda au_\infty^p - bu_\infty^q$ in $\Omega \times \mathbb{R}$. It remains to see that $u_\infty \neq 0$. We proceed by contradiction. Suppose that $u_\infty = 0$. Since u_j converges to u_∞ in C_T , there exists j_0 such that $u_j \leq 1$ for $j \geq j_0$. Observe also that

$$(3.4) \quad u_k(x, t) > 0 \text{ for } (x, t) \in B, \quad k \in \mathbb{N} \cup \{0\}.$$

Indeed, (3.4) clearly holds for $k = 0$ and if it holds for $k = j$, since

$$\begin{cases} Lu_{j+1} = \lambda au_j^p > 0 & \text{in } B \\ u_{j+1} \geq 0 & \text{on } \partial_P B \\ u_{j+1} \text{ } T\text{-periodic,} \end{cases}$$

where

$$(3.5) \quad \partial_P B := (\partial\Omega_0 \times (t_0, t_1)) \cup (\Omega_0 \times \{t_0\}),$$

the maximum principle gives (3.4) for $k = j + 1$. Taking into account (3.4) and that $u_{j+1} \leq u_j$, a computation gives, for $p < s < 1$ and $j \geq j_0$,

$$\begin{aligned}
L(u_{j+1}^{1-s}) &= (1-s)u_{j+1}^{-s}L(u_{j+1}) \\
&\quad + s(1-s)u_{j+1}^{-s-1}\langle A\nabla u_{j+1}, \nabla u_{j+1} \rangle + sc_0u_{j+1}^{1-s} \\
(3.6) \quad &\geq (1-s)u_{j+1}^{-s}\lambda au_j^p \\
&\geq (1-s)\lambda au_j^{p-s} \\
&\geq (1-s)\lambda a \qquad \qquad \qquad \text{in } B.
\end{aligned}$$

Also,

$$(3.7) \quad u_{j+1} \geq 0 \text{ on } \partial_P B.$$

From (3.6) and (3.7), the maximum principle implies $u_{j+1}^{1-s} \geq z$ on B for $j \geq j_0$ where $z \in W_r^{2,1}(B)$ is the (nonnegative and nontrivial) solution of

$$\begin{aligned}
Lz &= (1-s)\lambda a \quad \text{in } B, \\
z &= 0 \qquad \qquad \text{on } \partial_P B.
\end{aligned}$$

Thus $u_\infty^{1-s} \geq z$ and so $u_\infty = 0$ is impossible. \square

Remark 3.3. Let us make some comments on the condition (H1). Let $m \in L_T^r$, $r > N+2$, $m = m^+ - m^-$, where as usual $m^+ = \max(m, 0)$ and $m^- = \max(-m, 0)$. By the above theorem there exists a nonnegative solution of the Dirichlet periodic problem

$$Lu = \lambda m^+ u^p - m^- u^q \quad \text{in } \Omega \times \mathbb{R}$$

for all $\lambda > 0$ provided that (H1) holds. Moreover, if $m \in C_T$, (H1) is also necessary by the maximum principle. In particular, taking $\lambda = 1$ and $p = q$ it follows that $Lu = mu^p$ in $\Omega \times \mathbb{R}$ has a nonnegative solution if and only if $m > 0$ at some point. This generalizes (and gives a different proof of) the existence results known in the elliptic case (e.g. [2]) as well as in the periodic parabolic case.

We focus now on the existence of strictly positive solutions for (1.3). We note that, for general $p, q \in (0, 1)$, the only information available from the proof of Theorem 3.2 about the solution u provided there is that $u(x, t) > 0$ for each B satisfying (H1) which, in general, does not imply that $u(x, t) > 0$ for all $(x, t) \in \Omega \times \mathbb{R}$.

Let

$$\Omega_\varepsilon := \{x \in \Omega : \delta(x) > \varepsilon\},$$

$$A_\varepsilon := \Omega - \Omega_\varepsilon.$$

$$\delta(x) := \text{dist}(x, \partial\Omega)$$

$$P^\circ := \text{interior of the positive cone of } C_T^{1+\theta, (1+\theta)/2}.$$

Theorem 3.4. *Let $0 < p, q < 1$, and let $0 \leq a, b \in L_T^r$, $r > N + 2$.*

Assume

(H2) *Either there exist $\varepsilon, K > 0$ and $\gamma > 1 - q - 1/(N + 2)$ such that $b \leq K\delta^\gamma$ a.e. $(x, t) \in A_\varepsilon \times \mathbb{R}$, or $q > 1 + 1/r - 1/(N + 2)$.*

Then, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ problem (1.3) has a solution $u \in W_{r,T}^{2,1} \cap P^\circ$.

Assume in addition that

(H3) *$p \leq q$, and either there exist $\varepsilon, K > 0$ and $\gamma > 1 - p - 1/(N + 2)$ such that $a \leq K\delta^\gamma$ a.e. $(x, t) \in A_\varepsilon \times \mathbb{R}$, or $p > 1 + 1/r - 1/(N + 2)$.*

Then, there exists $\lambda_0 > 0$ such that a solution $u = u_\lambda$ of (1.3) can be chosen such that $\lambda \rightarrow u_\lambda$ is a C^1 increasing map from (λ_0, ∞) into $W_{r,T}^{2,1} \cap P^\circ$.

Proof: Assume in (H1) that e , K and γ exist and let $v \in C_T$ such that $v \geq c\delta$ for some $c > 0$. Since $1 - q < \gamma + 1/(N + 2)$ we can pick α and σ with $\sigma > N + 2$ such that $1 - q < \alpha < \gamma + 1/\sigma$. Now,

$$\|b/v^\alpha\|_{L^\sigma(A_\varepsilon \times (0,T))}^\sigma = \int_{A_\varepsilon \times (0,T)} b^\sigma/v^{\alpha\sigma} \leq c^{-\alpha\sigma} K^\sigma \int_{A_\varepsilon \times (0,T)} \delta^{\sigma(\gamma-\alpha)} < \infty$$

the last inequality because $\sigma(\gamma - \alpha) > -1$. So, $b/v^\alpha \in L_T^s$ for $s := \min(\sigma, r) > N + 2$. If in (H1) it holds that $q > 1 + 1/r - 1/(N + 2)$ then $1 - q < 1/s - 1/r$ for some $s > N + 2$ and so we can pick α such that $1 - q < \alpha < 1/s - 1/r$. Now, by Hölder's inequality,

$$\left\| \frac{b}{v^\alpha} \right\|_{L_T^s}^s \leq \|b\|_{L_T^r}^s \|v^{-\alpha}\|_{L_T^{\left(\frac{r}{s}\right)'}}^s \leq c^{-\alpha s} \|b\|_{L_T^r}^s \left(\int_{\Omega \times (0,T)} \delta^{-\alpha \frac{rs}{r-s}} \right)^{\frac{r-s}{r}} < \infty,$$

the last inequality because $\alpha \frac{rs}{r-s} < 1$. Thus in any case we have $b/v^\alpha \in L_T^s$ for some $\alpha > 1 - q$ and $s > N + 2$. Then Theorem 3.4 in [12] gives a positive solution $u_\lambda \in L_T^\infty$ for the problem

$$(3.8) \quad \begin{cases} (L + b/v^\alpha) u_\lambda = \lambda a u_\lambda^p & \text{in } \Omega \times \mathbb{R} \\ u_\lambda = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u_\lambda \text{ } T\text{-periodic,} \end{cases}$$

for all $\lambda > 0$. Recalling Remark 2.1 (ii) we have that $u_\lambda \in P^\circ$, and, due to the homogeneity, it holds that $u_\lambda = \lambda^{1/(1-p)}U$ for some $U \in P^\circ$ solution of $(L + b/v^\alpha)U = aU^p$ in $\Omega \times \mathbb{R}$, $U = 0$ on $\partial\Omega \times \mathbb{R}$. Hence, since $\alpha \geq 1 - q$, there exists $\lambda_0 > 0$ such that $u_\lambda \geq v^{\alpha/(1-q)}$ for all $\lambda > \lambda_0$. Therefore, from (3.8) we get $Lu_\lambda \leq \lambda au_\lambda^p - bu_\lambda^q$ in $\Omega \times \mathbb{R}$.

On the other hand, since we can construct supersolutions w_λ of (1.3) such that $w_\lambda \geq c$ in $\Omega \times \mathbb{R}$ for all $c > 0$ and all $\lambda > 0$ (see [12, Lemma 2.3]), Theorem 1 in [7] applies and thus the first assertion of the theorem is proved.

Let $\lambda > \lambda_0$, let u_λ be the solution of (1.3) found above, and let $m_\lambda := \lambda p a u_\lambda^{p-1} - q b u_\lambda^{q-1}$, $\tilde{m}_\lambda := \lambda a u_\lambda^{p-1} - b u_\lambda^{q-1}$. Note that (H2) and (H3) imply that m_λ and \tilde{m}_λ belong to L_T^s for some $s > N + 2$. We claim that the implicit function theorem can be applied in the point (λ, u_λ) . Indeed, to see this it suffices to show that for a given $g \in L_T^s$ there is a unique solution $h \in W_{r,T}^{2,1}$ of the problem

$$(3.9) \quad \begin{cases} (L - m_\lambda)h = g & \text{in } \Omega \times \mathbb{R} \\ h = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ h \text{ } T\text{-periodic,} \end{cases}$$

and that the solution operator S_λ for this problem is continuous. Consider first the case when $\lambda_1(m_\lambda)$ exists. From (1.3) it follows that $\lambda_1(\tilde{m}_\lambda) = 1$. Since $\tilde{m}_\lambda \geq m_\lambda/q$, recalling the comparison principle in Remark 2.2 (iv) we obtain

$$(3.10) \quad 1 \leq q\lambda_1(m_\lambda) < \lambda_1(m_\lambda)$$

and so $\mu_{m_\lambda}(1) > 0$. If $\lambda_1(m_\lambda)$ does not exist the same conclusion holds. Hence, by the results in Remark 2.2 (iii) and Remark 2.1 (ii) in both cases S_λ is well defined and continuous on $W_{r,T}^{2,1}$.

Let (α, β) be a maximal interval in which $\lambda \rightarrow u_\lambda$ is a C^1 map into $W_{r,T}^{2,1} \cap P^\circ$. Observe that $\lambda \rightarrow u_\lambda$ is increasing. Indeed, differentiating (1.3) with respect to λ gives

$$(L - m_\lambda) \frac{\partial u}{\partial \lambda} = b u_\lambda^p \geq 0 \quad \text{in } \Omega \times \mathbb{R}.$$

If $\lambda_1(m_\lambda)$ exists, Remark 2.2 (iii) and (3.10) imply $\frac{\partial u}{\partial \lambda} > 0$. If $\lambda_1(m_\lambda)$ does not exist, then μ_{m_λ} is positive everywhere, thus $\mu_{m_\lambda}(1) > 0$ and so, again by Remark 2.2 (iii) we get $\frac{\partial u}{\partial \lambda} > 0$ also in this case.

Now, suppose $\beta < \infty$, and let $u_j \in P^\circ$ be the solutions of (1.3) corresponding to some sequence $\lambda_j \rightarrow \beta^-$. Recalling Remark 2.1 (i), a standard compactness argument gives some u_β solution of (1.3) for $\lambda = \beta$.

Moreover, since $\lambda \rightarrow u_\lambda$ is increasing we have $u_\beta \in P^\circ$. But then reasoning as above we can apply the implicit function theorem in the point (β, u_β) , contradicting the maximality of (α, β) . \square

Concerning the matter of uniqueness or multiplicity of positive solutions, there are examples of different nonnegative solutions for similar problems to (1.3) with $p = q$ both in the elliptic and the periodic parabolic case (cf. [2], [16]), so we cannot expect uniqueness here. Moreover, when $p > q$, $a = b \equiv 1$, $L = -\Delta_p$ and $N = 1$, it is shown in [8] that for some λ there are exactly two solutions in the interior of the positive cone for (1.3), and therefore we cannot expect a general uniqueness theorem for the case $p > q$ neither. However, when $p \leq q$ the solution in the interior of the positive cone for (1.3) (if such a solution exists) turns out to be unique as the following theorem shows. Its proof is inspired in the proof of Proposition 2.2 in [6] and uses a change of variable introduced (for the elliptic case and $b = 0$) by L. Nirenberg (see [3, Appendix II, Method IV]).

Theorem 3.5. *Let $0 < p \leq q < 1$, $\lambda > 0$, and let $0 \leq a, b \in L_T^r$, $r > N + 2$. Then there exists at most one solution $u \in W_{r,T}^{2,1} \cap P^\circ$ of (1.3).*

Proof: Suppose that there exist $u, v \in W_{r,T}^{2,1} \cap P^\circ$ solutions of (1.3), and let $C := \{(x, t) \in \Omega \times \mathbb{R} : u(x, t) > v(x, t)\}$. Let s be such that $p \leq s \leq q$. A simple computation gives that $h(\xi) := (\lambda a(x, t)\xi^p - b(x, t)\xi^q)/\xi^s$ is nonincreasing in $(0, \infty)$ a.e. $(x, t) \in \Omega \times \mathbb{R}$. For $(x, t) \in C$, define $w(x, t) := \int_{v(x,t)}^{u(x,t)} \xi^{-s} d\xi$. Then clearly $w = 0$ on ∂C and taking into account that $u, v \in P^\circ$ it also follows that w is continuous in \overline{C} . Now, since $h(\xi)$ and ξ^{s-1} are nonincreasing, a computation shows that, in C ,

$$\begin{aligned} Lw - c_0 w &= u^{-s} Lu - v^{-s} Lv + su^{-s-1} \langle A \nabla u, \nabla u \rangle \\ &\quad - sv^{-s-1} \langle A \nabla v, \nabla v \rangle - c_0 (u^{1-s} - v^{1-s}) \\ &\leq h(u) - h(v) + su^{-s-1} \langle A \nabla u, \nabla u \rangle - sv^{-s-1} \langle A \nabla v, \nabla v \rangle \\ &\leq su^{s-1} \langle \nabla u / u^s + \nabla v / v^s, A \nabla w \rangle. \end{aligned}$$

Thus, recalling the maximum principle in Lemma 2.3 and Remark 2.4, the theorem follows. \square

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Rebut el 13 de març de 2006.