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NORMALIZATION OF POINCARÉ SINGULARITIES *VIA* VARIATION OF CONSTANTS

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Abstract

We present a geometric proof of the Poincaré-Dulac Normalization Theorem for analytic vector fields with singularities of Poincaré type. Our approach allows us to relate the size of the convergence domain of the linearizing transformation to the geometry of the complex foliation associated to the vector field.
A similar construction is considered in the case of linearization of maps in a neighborhood of a hyperbolic fixed point.

1. Introduction

Let $X\colon U \subset \mathbb{C}^n \mapsto T\mathbb{C}^n$ be a vector field, holomorphic in the domain U; let $o \in U$ and $X(o) = 0$: o is a *Poincaré singular point* of X if the differential d_oX has eigenvalues $\lambda_1, \dots, \lambda_n$ satisfying:

$$0 \notin \text{convex hull of } \lambda_1, \dots, \lambda_n.$$

This is a geometric property of the complex foliation defined by X, namely any vector field $Y = gX$, g germ of unity at o, has at o a Poincaré singular point if X does. The geometric content of this condition is captured by the following remark by Arnold [**1**] (*Arnold's Transversality Condition*), which is crucial for our normalization method.

Let S_R be an Euclidean sphere in $\mathbb{C}^n$ of radius R; we say that X is *transversal* to S_R at $p \in S_R$ if $\langle X(p)\rangle^{\mathbb{R}} \oplus T_pS_R = (T_p\mathbb{C}^n)^{\mathbb{R}}$.

Theorem 1.1 (Arnold [**1**]). *Let o be a Poincaré singular point of X: then there exists $R_0 > 0$ such that for every $0 < R < R_0$, X is transversal to S_R.*

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We remark that R_0 depends on X only through its non-linear terms, being a linear vector field with Poincaré singularity transversal to S_R for all $R > 0$.

For a given choice of coordinates $z = (z_1, \dots, z_n)$, $z(o) = 0$, which we shall assume from now on, we can write:

$$X(z) = Az\,\partial_z + \cdots ,$$

where A is a $n \times n$ complex matrix and dots stand for nonlinear terms. We assume that in z coordinates: $A = S + \varepsilon N$, $\varepsilon > 0$, is the Jordan decomposition of A: N is the nilpotent part, made of blocks $N_k = \delta_{i,i+1}$, $i = 1, \dots, k-1$, and S the semisimple part, moreover $S = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We also introduce, for later use, the vector $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$.

Given an analytic diffeomorphisms f one can consider the *push forward* of the vector field X under f: $X_*(z) = df \cdot X(f^{-1}(z))$. Geometrically this represents the vector field X in the new coordinates system determined by f and we will say that X and X_* are analytically *conjugated*. For a given X a natural question is to determine the "simplest form" it can assume up to analytic conjugation, or given a vector field X_0 one can be interested in determining all the vector fields that are conjugated to it.

The most interesting case occurs when such a simplest form is the linear part of the vector field at the singular point. It is the *linearization* problem: it has been considered by Poincaré in his thesis [**7**], and solved by him in the case of Poincaré singularities. His results were later generalized by Dulac [**5**] to the *normalization problem*. Let us briefly recall what normalizing a vector field means. By a holomorphic change of coordinates:

$$w = z + h(z),$$

we try to reduce $X(z)$ to a simplest form, possibly to the linear vector field:

$$X_{\text{lin}} = Aw\,\partial_w.$$

Obstructions to realize this program are the *resonances*: there exist $\underline{m} = (m_1, \dots, m_n)$, $m_l \in \mathbb{N}$, $|\underline{m}| = m_1 + \cdots + m_n \geq 2$, and $j \in \{1, \dots, n\}$, such that:

$$\langle \underline{\lambda}, \underline{m} \rangle - \lambda_j = 0.$$

A *formal change of coordinates* leads to the following *formal normal form(s)* for the differential equation associated to the vector field, for all $j \in \{1, \ldots, n\}$:

$$\dot{w}_j = (Aw)_j + \sum_{\substack{\underline{m} \in \mathbb{N}^n : |\underline{m}| \geq 2 \\ \langle \underline{\lambda}, \underline{m} \rangle - \lambda_j = 0}} c_{\underline{m},j} w^{\underline{m}},$$

where we used the standard notation $w^{\underline{m}} = w_1^{m_1} \cdots w_n^{m_n}$ and $(c_{\underline{m},j})_{\underline{m},j} \subset \mathbb{C}$.

In the case of a Poincaré singular point there are at most finitely many resonant terms, and the non-resonant terms are bounded from below by some universal positive constant: $|\langle \underline{\lambda}, \underline{m} \rangle - \lambda_j| > c$, for all $\underline{m} \in \mathbb{N}^n$ and $j \in \{1, \ldots, n\}$, s.t. $\langle \underline{\lambda}, \underline{m} \rangle - \lambda_j \neq 0$. These remarks prevent the formal normalizing method from the *small divisor problem*; moreover, any normal form is in this case polynomial.

Theorem 1.2 (Poincaré, Dulac)**.** *Let X be holomorphic in U and let $o \in U$ be a Poincaré singular point. Let X_0 be a polynomial normal form of X. Then there exists a neighborhood $V \subset U$ of o and an holomorphic diffeomorphisms H defined in V such that:*

$$H_* X = X_0.$$

Even if elementary, Poincaré's original proof of this result, as any other more recent proof (see *e.g.* [**4**]), is not explicit in determining the transformation H and its convergence domain. We shall give a geometric proof of this classical theorem, *via* a variation of constants approach. This will allow us to get a more explicit definition of the normalizing transformation, and will lead us to relate the size of the domain of the linearizing transformation to the transversality radius R_0 entering in Arnold's Transversality Condition. The key idea to obtain this result is classical: we use Hurwitz's Theorem to prove the existence of a local biholomorphisms; then, applying Cauchy's estimates, we can extend the domain of injectivity to the whole domain of definition.

The method used in the proof of our main result is an extension of the smooth normalization argument used by Sternberg [**8**]. Hence it is different from all perturbative-like, or KAM-like methods, where one try to push the non linear term of the vector field, to higher and higher order through an iterative algorithm (see Remark 2.4).

In Section 3 we will consider the case of discrete time dynamical systems, i.e. the *Siegel Center Problem*: linearization of a biholomorphic map in a neighborhood of a fixed point. We will present a geometric construction, similar to the one given for flows, which allows us to solve the Siegel Problem in the case of Poincaré fixed point, i.e. hyperbolic case, obtaining moreover an explicit bound on the size of the convergence domain of the linearizing map, related to geometric properties of the orbit space of the biholomorphism. We will also compare our result with other classical ones [**7**], [**6**].

To conclude this introduction, let us briefly mention the normalization problem for Siegel's singularities of analytic vector fields. In this case normalization is not always possible: one need some additional hypotheses on the growth rate of the small divisors and on the geometry of the foliation associated to the resonant normal form [**2**].

It would be interesting to deal with the Siegel case using ideas similar to those introduced here, to get Bruno's results. We could not succeed in developing this approach due to the fundamental role played in our geometric normalization of Poincaré singularities by Arnold's Transversality Condition, which is no longer true in the Siegel case.

2. Normalization *via* variation of constants

We start the description of our approach to normalization by a slight and straightforward generalization of the variation of constants formula. Let X be, in given z-coordinates, of the form:

$$X(z) = X_0(z) + X_1(z),$$

where X, X_0, X_1 are holomorphic in the common domain $U \in \mathbb{C}^n$. We denote by $\Phi_X^T(z)$ (respectively $\Phi_{X_0}^T(z)$) the complex flows of X (respectively X_0): they are both defined in a common domain in $\mathbb{C} \times \mathbb{C}^n$. We look for a T-depending holomorphic diffeomorphism $L_T(z)$ such that:

$$\Phi_X^T(z) = \Phi_{X_0}^T(L_T(z)), \tag{2.1}$$

hence, for sufficiently small positive Δ and R:

$$L_T(z) = \Phi_{X_0}^{-T} \circ \Phi_X^T(z) \quad \forall\ |T| < \Delta,\ \|z\| < R,$$

being $\|z\|$ the Euclidean norm in $\mathbb{C}^n$. This is the *Variation of Constant Transformation.*

Let us introduce an *integral representation* of this transformation, which turns to be well-suited for our use. Differentiating relation (2.1) w.r.t. time we get:

$$X_0(\Phi_X^T(z)) + X_1(\Phi_X^T(z)) = X_0(\Phi_X^T(z)) + d_{\Phi_X^T(z)}\Phi_{X_0}^T \dot{L}_T(z),$$

hence:

$$d_{\Phi_X^T(z)}\Phi_{X_0}^T \dot{L}_T(z) = X_1(\Phi_X^T(z)).$$

Therefore, integrating along any smooth path joining 0 and T lying inside the disk of radius Δ in $\mathbb{C}$, we obtain:

$$L_T(z) = z + \int_0^T d_{\Phi_X^s(z)} \Phi_{X_0}^{-s} X_1(\Phi_X^s(z))\, ds, \tag{2.2}$$

which is the *Variation of Constants Formula*. Let us explicitly observe that such a definition depends on X_0, *i.e.* on the chosen normal form in our case of use.

Another way to characterize a singular point o of Poincaré type is the following one, see also [**2**, §II, p. 165]. There exists a line l_{ω_0} in $\mathbb{C}$: $l_{\omega_0} = \{t\omega_0 + \eta : \omega_0 \in S^1, \eta \in \mathbb{C}^*, t \in \mathbb{R}\}$, such that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of d_oX are contained in the halfspace $\mathcal{H}_{\underline{\lambda}} \subset \mathbb{C}$ defined by l_{ω_0} and such that $0 \notin \mathcal{H}_{\underline{\lambda}}$. Let $l_{\omega_0}^{\perp} = \{it\omega_0 : t \in \mathbb{R}\}$ and let $({l_{\omega_0}}^{\perp})^+$ be the "positive" halfline not contained in $\mathcal{H}_{\underline{\lambda}}$. We remark that we can change ω_0 into ω, with $|\arg(\omega-\omega_0)| < \theta$, $\theta > 0$ sufficiently small, keeping the geometric characterization of the Poincaré singularity. For a fixed choice of ω_0, let μ_j be the *distance* of λ_j from l_{ω_0}, $j \in \{1, \ldots, n\}$, and let $\alpha = \min\{\mu_1, \ldots, \mu_n\}$, $\beta = \max\{\mu_1, \ldots, \mu_n\}$.

We are now able to state our main result, which is a version of Theorem 1.2 containing a more explicit definition and a geometric estimate of the domain of convergence of the normalizing transformation.

Theorem 2.1. *Let $X\colon U \in \mathbb{C}^n \mapsto T\mathbb{C}^n$ be a holomorphic vector field and let o be a Poincaré singular point of X. Let R_0 be the transversality radius appearing in the Arnold's Transversality Condition: hence $B_{R_0} \subset U$. Let:*

$$X(z) = X_0(z) + X_1(z),$$

where $X_0(z)$ is a polynomial normal form of X, whose linear part is in Jordan canonical form: $S + \varepsilon N$, $\varepsilon > 0$, and X_1 is m-flat, $m > \max\left\{\deg X_0, \frac{q(\beta+\varepsilon)}{\alpha+\varepsilon}\right\}$, being $q > 1$. Then:

$$L(z) = \lim_{t\to+\infty} \left\{ z + \int_0^t d_{\Phi_X^{is\omega}(z)} \Phi_{X_0}^{-is\omega} X_1(\Phi_X^{is\omega}(z))\, ds \right\},$$

where integration is along any halfline not contained in $\mathcal{H}_{\underline{\lambda}}$, is a normalizing biholomorphism in a neighborhood of 0:

$$L_* X = X_0.$$

Moreover in the linearizing case, i.e. when $X_0 = Az\,\partial_z$ and it is not resonant or X_1 is m-flat, $L(z)$ is a linearizing biholomorphism defined in a domain containing the Euclidean ball B_{R_0} where Arnold's Transversality Condition holds.

Remark 2.2. The statement of the above theorem applies when X is in a "prepared" normal form $X = X_0 + X_1$: while this condition can always be satisfied after a m-degree polynomial change of coordinates, this should be taken into account on applying the bounds on the convergence domain of the linearization. On the other hand, such "prepared" normal form transformation has no influence on the estimate on m.

Remark 2.3. Let $X = Az\,\partial_z + X_1$ then if X_1 is sufficiently flat, we can *linearize* X even if A is resonant. We recall that the maximal modulus of resonance in the Poincaré case is bounded by β/α, hence giving a simple interpretation of the order m of flatness appearing in the statement of the theorem.

On the other hand if the order of X_1 is too small, performing the polynomial change of variables to put X into the "prepared normal form", one cannot avoid the "introduction" of resonant monomials of small degree, s.t. in the "prepared normal form", X_0 will be no longer linear and our result guarantees only normalizability.

As a final remark, we observe that some explicit bound on the size of the convergence domain of the normalizing transformation can be obtained not only in the linearization case, but in the general case of normalization, too. This bounds, if needed *e.g.* in applied bifurcation problem, can be easily deduced from the following proof, but we will omit them as they have not such a synthetic and geometric interpretation as in the case of linearization.

The rest of this section is devoted to the proof of Theorem 2.1: first we will deal with the general normalization problem, and then we will show how to modify the arguments in the simpler case of linearization in order to get the estimate on the size of the linearization domain.

Without loss of generality we suppose that $\mathcal{H}_{\underline{\lambda}} = \{z \in \mathbb{C} : \Re z < 0\}$: all the arguments in the proof transfer literally to the general case just considering as integration path $(l_{\omega_0}^{\perp})^+$ instead of the real positive semi-axis. From a geometric point of view this choice corresponds to a time reparametrization of the complex foliation associated to X, by a complex non-zero factor. Under this assumptions we have:

$$\beta = \Re\lambda_n \leq \Re\lambda_{n-1} \leq \cdots \leq \Re\lambda_1 = \alpha < 0,$$

where we changed the previous definitions of α, β by switching sign: this has no effects on the statement of the theorem and will simplify notations.

A fundamental remark is that under these hypotheses the differential equation with real independent variable t and complex phase space U

given by:

$$\frac{dz}{dt} = X(z), \tag{2.3}$$

defines an analytic flow $\Phi^t_X(z)$ for $z \in B_{R_0}$ and $t > 0$, moreover:

$$\lim_{t\to+\infty} \Phi^t_X(z) = 0.$$

Of course, this is nothing but Arnold's Transversality Condition.

We can extend this remark to obtain a kind of asymptotic stability of the origin as a singular point of a differential equation defined by X and with independent variable T varying in a sectorial neighborhood, centered on the real positive semiaxis, of infinity in the Riemann sphere. In fact, for any $0 < R^0 < R_0$ we can find $\theta > 0$ such that the equation:

$$\frac{dz}{dt} = i\omega X(z), \tag{2.4}$$

where $t \in \mathbb{R}$ and $|\arg(\omega)| < \theta$, defines real flows which, by the same arguments we used for the equation (2.3), have the origin as an asymptotically stable stationary point. Therefore such real flows imbed into the complex flow $\Phi^T_X(z)$ of $X(z)$ which turns to be defined for $(T, z) \in \mathcal{S} \times \{z : \|z\| < R^0\}$, where $\mathcal{S} = \{T : |T| < \Delta\} \cup \{T = t\omega, |\arg(\omega)| < \theta, t > 0\} = \mathbb{D}_\Delta \cup \mathcal{C}_{0,\theta}$.

We shall prove now that, for small enough $R > 0$:

$$L(T, z) =: L_T(z) \colon \mathcal{S} \times \mathbb{D}_R \mapsto \mathbb{C}^n.$$

The first step is to obtain an estimate for the growth rate of $\|\Phi^t_X(z)\|$. For all $j \in \{1, \dots, n\}$ we have:

(2.5)
$$\begin{aligned}
\frac{d}{dt}\left|\left(\Phi^t_X(z)\right)_j\right|^2 &= \left[\frac{d}{dt}\left(\Phi^t_X(z)\right)_j\right]\overline{\left(\Phi^t_X(z)\right)_j} + \left(\Phi^t_X(z)\right)_j\left[\frac{d}{dt}\overline{\left(\Phi^t_X(z)\right)_j}\right]\\
&= X_j(\Phi^t_X(z))\overline{\left(\Phi^t_X(z)\right)_j} + \left(\Phi^t_X(z)\right)_j\overline{X_j(\Phi^t_X(z))}\\
&= 2\Re\lambda_j|\left(\Phi^t_X(z)\right)_j|^2\\
&\quad + 2\varepsilon\Re\left((N\Phi^t_X(z))_j\overline{\left(\Phi^t_X(z)\right)_j}\right) + \mathcal{O}\left(|\left(\Phi^t_X(z)\right)_j|^3\right).
\end{aligned}$$

Hence for all $\delta > 0$ we get, for all $\|z\| < R$, $R > 0$ small enough:

$$\frac{d}{dt}\|\Phi^t_X(z)\|^2 \le 2(\alpha + \varepsilon + \delta)\|\Phi^t_X(z)\|^2,$$

and therefore:

$$\|\Phi_X^t(z)\|^2 \leq e^{2(\alpha+\varepsilon+\delta)t} R^2. \tag{2.6}$$

In order to get an estimate for the integral from of $L_T(z)$ given by (2.2) we need to prove firstly that $d_{\Phi_X^s(z)}\Phi_{X_0}^{-s}$ is defined for $\|z\| < R$, R sufficiently small, and for all $T \in \mathcal{S}$, then we must find a suitable asymptotic estimate of it.

For sufficiently small $|T|$ and $\|z\|$, the couple $(\Phi_X^T(z), d_{\Phi_X^T(z)}\Phi_{X_0}^{-T})$ is the solution of the following Cauchy problem for a system at variation type:

$$\begin{cases} \dot{w} = X(w) \\ \dot{W} = -\dfrac{\partial X_0}{\partial w} W \\ w(0) = z \\ W(0) = E, \end{cases}$$

where E is the identity matrix in $\mathbb{C}^n$, $w \in \mathbb{C}^n$ and $W \in \mathbb{C}^n \times \mathbb{C}^n$. Therefore the existence of $t \mapsto (w(T), W(T)) = (\Phi_X^T(z), d_{\Phi_X^T(z)}\Phi_{X_0}^{-T})$ for every real $t > 0$ and $\|z\| < R^0$ follows from the asymptotic stability of the origin as a singular point of (2.3) and from basic theory of linear ordinary differential equations. To get the desired asymptotic estimate for $d_{\Phi_X^t(z)}\Phi_{X_0}^{-t}$ we consider the above system with fixed z and writing $X_0(w) = Aw + g(w)$ we obtain the equation at variation:

$$\dot{W} = -AW - \frac{\partial g}{\partial w}(\Phi_X^t) W.$$

From this equation is readily obtained the following inequality for the norm of linear operators:

$$\|d_{\Phi_X^t(z)}\Phi_{X_0}^{-t}\| \leq \|e^{-tA}\| + \mathcal{O}(\|\Phi_X^t\|),$$

and then for every $0 < R < R^0$ there exits $\delta > 0$ such that for every $t > 0$:

$$\|d_{\Phi_X^t(z)}\Phi_{X_0}^{-t})\| \leq e^{-(\beta+\varepsilon-\delta)t} R. \tag{2.7}$$

We can give now a uniform bound on $\|L_t(z)\|$ for $t > 0$ and $\|z\| < R$. Recalling that $\|X_1(w)\| \leq C\|w\|^m$ and $m > \frac{q(\beta+\varepsilon)}{\alpha+\varepsilon}$, so that $(\beta + \varepsilon - \delta) - m(\alpha + \varepsilon + \delta) > 0$ for sufficiently small δ, from the estimates (2.6) and (2.7) we get:

$$
\begin{aligned}
\|L_t(z) - z\| &= \left\| \int_0^t d_{\Phi_X^s(z)} \Phi_{X_0}^{-s} X_1(\Phi_X^s(z))\, ds \right\| \\
&\le C\|z\|^m \int_0^t e^{[-(\beta+\varepsilon-\delta)+m(\alpha+\varepsilon+\delta)]s}\, ds \\
&\le \frac{C}{(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)} \\
&\quad \times R^m(1-e^{[-(\beta+\varepsilon-\delta)+m(\alpha+\varepsilon+\delta)]t}),
\end{aligned} \tag{2.8}
$$

for $t > 0$ and $\|z\| < R$. Hence every L_t, $t > 0$, maps the Euclidean ball B_R into $B_{R+R'}$ where $R' = \frac{C}{(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)} R^m$.

An analogous estimate leads to the proof of the convergence of $t \mapsto L_t$. In fact, let us suppose for the moment that τ is real and with sufficiently small modulus, then:

$$
\begin{aligned}
\|L_{t+\tau}(z) - L_t(z)\| &= \left\| \int_t^{t+\tau} d_{\Phi_X^s(z)} \Phi_{X_0}^{-s} X_1(\Phi_X^s(z))\, ds \right\| \\
&\le C\|z\|^m \frac{e^{-[(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)]t}}{[(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)]} \\
&\quad \times (1-e^{-[(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)]\tau}).
\end{aligned} \tag{2.9}
$$

Therefore from the Cauchy condition and the hypothesis on m, it follows that:

$$\lim_{t\to+\infty} L_t(z) = L(z),$$

uniformly when $\|z\| < R$. By the same argument with obvious modifications we get:

$$\lim_{\substack{T\to\infty \\ T\in\mathcal{S}}} L_T(z) = L(z),$$

uniformly when $\|z\| < R$.

To end the proof of existence of a locally defined normalizing transformation we need to show that $L(z)$ conjugates the vector field to X_0 i.e.: $L_*X = X_0$ and it is a biholomorphism in a neighborhood of the origin.

To prove the first claim is enough to show that L conjugates the corresponding flows, namely:

$$\Phi_{X_0}^{-\tau} \circ L \circ \Phi_X^{\tau} = L \quad \forall\, \tau.$$

This is obvious writing:

$$L = \lim_{\substack{T \to \infty \\ T \in \mathcal{S}}} \Phi_{X_0}^{-t} \circ \Phi_X^t, \tag{2.10}$$

as in this case:

$$\Phi_{X_0}^{-\tau} \circ L \circ \Phi_X^{\tau} = \lim_{\substack{T \to \infty \\ T \in \mathcal{S}}} \Phi_{X_0}^{-(t+\tau)} \circ \Phi_X^{t+\tau} = L(z).$$

So the first claim is proved if (2.10) holds; let us prove it. Because:

$$L_T(z) = \Phi_{X_0}^{-T} \circ \Phi_X^T(z) = z + \int_0^T d_{\Phi_X^s(z)} \Phi_{X_0}^{-s} X_1(\Phi_X^s(z))\, ds,$$

for sufficiently small $|T|$ and $\|z\| < R$, let us define $t_0 = \sup\{t > 0 :$ for every $\|z\| < R$ and $\tau \in [0, t) : \|\Phi_{X_0}^{-\tau} \circ \Phi_X^{\tau}(z)\| < +\infty\}$, and let us suppose by contradiction that $t_0 < \infty$. Then there exists a sequence $(t_m, z^{(m)})$ such that $t_m \to t_0$, $\|z^{(m)}\| < R$ and:

$$\lim_{m \to \infty} \left\| z^{(m)} + \int_0^{t_m} d_{\Phi_X^s(z)} \Phi_{X_0}^{-s} X_1(\Phi_X^s(z))\, ds \right\| = +\infty.$$

This contradicts the bound $L_t(B_R) \subset B_{R+R'}$, $t > 0$, from which the claim follows. The proof that L is a biholomorphism, locally invertible in a neighborhood of the origin, follows from Weierstrass' Theorem applied to the family of analytic maps $\{L_t\}$ and from the Inverse Function Theorem together with the remark that $d_0 L =$ identity.

This ends the proof of the existence of the normalizing transformation.

Let us come now to the case $X_0 = A$ and non-resonant (or X_1 is sufficiently flat), *i.e.* the case when the linearizing map is:

$$L_T(z) = z + \int_0^T e^{-sA} X_1(\Phi_X^s(z))\, ds. \tag{2.11}$$

We must prove that $L(z)$ is a linearizing biholomorphism defined in B_{R_0}: this will follows from the proof of the analogous claim for B_{R^0}.

Firstly we observe that as e^{-sA} is globally defined and $\Phi_X^s(z)$ is defined for $\|z\| < R^0$ and $T \in \mathcal{S}$, $L_T(z)$ is well-defined in $B_{R^0} \times \mathcal{S}$. Moreover in the same domain we get the estimate:

$$\begin{aligned} \|L_t(z) - z\| &= \left\| \int_0^T e^{-sA} X_1(\Phi_X^s(z))\, ds \right\| \\ &< C' + \frac{C}{(\beta + \varepsilon - \delta) - m(\alpha + \varepsilon + \delta)} R^m, \end{aligned} \tag{2.12}$$

where $\|X_1(w)\| < C\|w\|^m$ for $\|w\| < R$ and:

$$C' = \left\| \int_0^{s_0} e^{-sA} X_1(\Phi_X^s(z))\, ds \right\|,$$

for some s_0. Therefore each L_T, $(z,T) \in B_{R^0} \times \mathcal{S}$, maps B_{R^0} into $B_{R^0+C'+R'}$, where $R' = \frac{C}{(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)} R^m$. With a similar argument we get that, if $\Re T > s_0$:

$$\begin{aligned} \|L_{T+\tau}(z) - L_T(z)\| &< \left\| \int_T^{T+\tau} e^{-sA} X_1(\Phi_X^s(z))\, ds \right\| \\ &< CR^m \frac{e^{-s_0[(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)]}}{(\beta+\varepsilon-\delta)-m(\alpha+\varepsilon+\delta)}, \end{aligned} \tag{2.13}$$

therefore:

$$\lim_{t\to+\infty} L_t(z) = L(z).$$

Hence we have a family $\{L_t\}_{t>0}$ of biholomorphisms from B_{R^0} to $B_{R^0+C'+R'}$, converging in B_{R^0} to L: we will adapt a classical argument [**3**] concerning sequences of automorphisms to prove that L is a biholomorphism on B_{R^0}, too. Let us denote J_{L_t} and J_L respectively the jacobians of L_t and L, of course:

$$\lim_{t\to\infty} J_{L_t} = J_L,$$

uniformly on compact subsets of B_{R^0}. It is a straightforward consequence of Hurwitz's Theorem and of the equality $J_L(0)=1$ that $J_L(z)\neq 0$ for every $z \in B_{R^0}$. Therefore the proof that L is a biholomorphism from B_{R^0} onto its image will end if we prove that L is injective in B_{R^0}.

Let us suppose, by contradiction, that there exist $z^{(1)}, z^{(2)} \in B_{R^0}$ such that $L(z^{(j)}) = w$, $j = 1,2$. Let $B_r(z^{(j)})$, $j = 1,2$ be two Euclidean balls centered at $z^{(j)}$ and having the same radius r, such that:

$$B_r(z^{(j)}) \subset B_{R^0} \quad \text{and} \quad B_r(z^{(1)}) \cap B_r(z^{(2)}) = \emptyset.$$

We claim that there exists $R > 0$, depending on $|J_L(z^{(j)})|$, R^0 and R', and $t_0 > 0$ such that for every $t > t_0$, $j = 1,2$, we have:

$$B_R(w) \subset L_t\left(B_r(z^{(j)})\right). \tag{2.14}$$

This leads to a contradiction: in fact from (2.14) it follows that for sufficiently large $t > t_0$ there exist two points $w^{(j)} \in B_r(z^{(j)})$, $j \in \{1,2\}$, such that $L_t(w^{(j)}) = w$, for $j \in \{1,2\}$, which is impossible because L_t is a biholomorphism on B_{R^0}. Let us prove (2.14). For $j = 1,2$, there exists

$t_0 > 0$ such that if $t > t_0$ then $|J_{L_t}(w^{(j)})| \geq \frac{1}{2}|J_L(w^{(j)})| > 0$. From Cauchy inequalities we get, for $t > t_0$, $l, k = 1, \dots, n$:

$$\left|\frac{\partial (L_t)_l}{\partial z_k}\right| < \frac{R'}{r},$$

and therefore $\|d_w(L_t)^{-1}\| > \sigma\|w\|$, where $\sigma > 0$ depends on R', R^0, r, n, $|J_l(\omega^{(j)})|$ but is independent of t, for $t > t_0$. Hence

$$\|d_{z^{(j)}}(L_t)\| < \sigma\|w\|, \tag{2.15}$$

for every $w \in \mathbf{C}^n$. Another application of Cauchy inequalities leads to the following estimates of the error made substituting the linear approximation to the complete Taylor series, holding true when $\|z - z^{(j)}\| < \frac{r}{n}$:

$$\begin{aligned}\left|\sum_{k\in\mathbf{N}^n:|k|\geq 2}\frac{1}{k!}D^k(L_t)_l(z-z^{(j)})^k\right| &\leq R'\left\{\frac{1}{r^2}n^2\|z-z^{(j)}\|^2\right.\\ &\qquad\left.+\frac{1}{r^3}\|z-z^{(j)}\|^3+\cdots\right\}\\ &\leq \frac{rR'n^2\|z-z^{(j)}\|^2}{r^2(r-n\|z-z^{(j)}\|)}.\end{aligned} \tag{2.16}$$

Therefore there exists $\delta > 0$, depending on n, R', r, $|J_L(z^{(j)})|$, but independent of $t > t_0$, such that if $\|z - z^{(j)}\| < \delta$ then:

$$\|L_t(z) - L_t(z^{(j)}) - d_{z^{(j)}}L_t(z - z^{(j)})\| < \frac{1}{2}\sigma\|z - z^{(j)}\|.$$

Combining this inequality and (2.15) we obtain:

$$\|L_t(z) - L_t(z^{(j)}\| > \frac{1}{2}\sigma\|z - z^{(j)}\|,$$

hence for every $t > t_0$, $j = 1, 2$:

$$B_{\frac{1}{3}\sigma\delta}(z^{(j)}) \subset L_t(B_r(z^{(j)}),$$

and (2.14) follows as a consequence of the convergence of the L_t's to L. The proof is concluded by remarking that the conjugacy functional equation locally satisfied by L extends to B_{R^0} by analytic continuation.

Remark 2.4. Let us show that for any finite t, the transformed vector field $(L_t)_*X$ has non-linearities of the same order of X. This is completely different from classical methods of perturbative theory where one looks for diffeomorphisms ϕ, such that ϕ_*X has non linear terms of order higher than X.

To simplify assume A to be non-resonant and let us write $X_1 = \sum_{|\underline{k}|=m} X_{1,\underline{k}} z^{\underline{k}} \partial_z + \mathcal{O}(|z|^p)$, $p > m$. Fix $t > 0$ and look for $h_t(z)$, s.t.: $L_t(z) = z + h_t(z) + \cdots$. From (2.11) we get:

$$(h_t(z))_j = \sum_{|\underline{k}|=m} \frac{e^{t(\langle \underline{\lambda}, \underline{k} \rangle - \lambda_j)} - 1}{\langle \underline{\lambda}, \underline{k} \rangle - \lambda_j} X_{1,\underline{k},j} z^{\underline{k}},$$

where $X_{1,\underline{k},j}$ is the j-th component of $X_{1,\underline{k}}$. Hence we get:

$$(L_t)_*(Az\partial_z + X_1(z)) = Az\partial_z + \sum_{j \in \{1,\dots,n\},\, |\underline{k}|=m} e^{t(\langle \underline{\lambda}, \underline{k} \rangle - \lambda_j)} X_{1,\underline{k},j} z^{\underline{k}} \partial_{z_j} + \cdots .$$

3. Linearization of biholomorphic maps

In this section we study the case of discrete time systems: iteration of biholomorphisms, with particular interest in the problem of their linearization. We will present a modified version of the construction previously given for flows, which allows us to solve the linearization problem in the case of hyperbolic fixed point.

Let us consider an analytic diffeomorphisms of n complex variables fixing the origin: $F \in \mathrm{Diff}^{\omega}(\mathbb{C}^n, 0)$, $F(0) = 0$ and assume that in the chosen coordinates z, it has the form: $F(z) = Az + F_1(z)$, where $A = S + \varepsilon N$, $\varepsilon > 0$, is the Jordan Canonical form of dF_0, whereas $\|F_1(z)\| = \mathcal{O}(\|z\|^2)$.

Let $\mu_1, \dots, \mu_n$ be its eigenvalues, we will assume that the origin is a fixed point of *Poincaré type* [1], namely:

$$\sup_j |\mu_j| < 1, \tag{3.1}$$

and let us also introduce the vector $\underline{\mu} = (\mu_1, \dots, \mu_n)$.

Once again we are interested in the possibility of "reducing" the given system to a "simplest form", $F_0(z)$, through an analytical change of variables $H(z)$ which locally conjugates F and F_0:

$$H \circ F = F_0 \circ H. \tag{3.2}$$

The "simplest form" could be the linear map $F_{\mathrm{lin}}(z) = Az$, but *resonances* [2] can be an obstruction. In presence of resonances the "simplest

[1] If $\inf_j |\mu_j| > 1$, we will consider the map $\tilde{F} = F^{-1}$.

[2] In the discrete time case resonances are couples $j \in \{1, \dots, n\}$, $\underline{m} \in \mathbb{N}^n$, s.t. $|\underline{m}| \geq 2$ and $\underline{\mu}^{\underline{m}} - \mu_j = 0$.

form" is given by the following *normal form*, for all $j \in \{1, \dots, n\}$:

$$(F_0(z))_j = (Az)_j + \sum_{\substack{\underline{m} \in \mathbb{N}^n : |\underline{m}| \geq 2 \\ \underline{\mu}^{\underline{m}} - \mu_j = 0}} b_{\underline{m},j} w^{\underline{m}}, \tag{3.3}$$

where $(b_{\underline{m},j})_{\underline{m},j} \subset \mathbb{C}$.

The main result of this section is

Theorem 3.1. *Let $F \colon U \subset \mathbb{C}^n \mapsto \mathbb{C}^n$ be a biholomorphisms fixing the origin, assume moreover the origin to be a fixed point of non-resonant Poincaré type. Let the chosen coordinates such that $dF_0 = S + \varepsilon N$, $\varepsilon > 0$, is in Jordan canonical form, hence $F(z) = (S + \varepsilon N)z + F_1(z)$ and F_1 is m-flat, $m > q\frac{|\log \min |\mu_j||}{|\log(\max |\mu_j| + \varepsilon)|}$, $q > 1$. Then:*

$$L(z) = z + \sum_{l=0}^{+\infty} \Delta_l(z),$$

where $\Delta_l(z) = (S + \varepsilon N)^{-l} f_1 \circ F^l(z)$ and $f_1 = (S + \varepsilon N)^{-1} F_1$, is a linearizing biholomorphism in a neighborhood of 0*:*

$$AL = L \circ F.$$

The linearizing map has a convergence domain containing an euclidean ball of radius R_δ explicitly estimated in the proof by (3.5).

The rest of the section is devoted to prove this result. Let us assume A to be non-resonant, hence the normal form reduces to the linear part of F. We are looking for a family of maps $L_l(z)$, defined for $l \in \mathbb{N}$ and $\|z\|$ sufficiently small, such that: $F^{\circ l}(z) = A^l L_l(z)$, $A = (S + \varepsilon N)$. Let us introduce the "one-step" map: $\Delta_l(z) = L_{l+1}(z) - L_l(z)$. Under our assumptions we obtain:

$$\Delta_l(z) = A^{-l} f_1 \circ F^{\circ l}(z), \tag{3.4}$$

where $f_1 = A^{-1} F_1$.

For all positive integer l, we trivially have the following properties:

(1) $L_l(0) = 0$;
(2) $(dL_l)_0 =$ identity;
(3) $L_{l+1} = z + \sum_{k=0}^{l} \Delta_k(z)$.

From the existence of $L(z) = \lim_{l \to +\infty} L_l(z)$, we also get:

$$AL = L \circ F,$$

namely L linearizes F.

Let us now prove the existence of the previous limit. Because there are no resonances we can perform a polynomial change of coordinates such that F is in some "prepared form" where $F_1(z)$ has order m, with m arbitrary large. We remark that $\Delta_l(z)$ and $F_1(z)$ have the same order.

Let us call $\rho^* = \max_j |\mu_j|$ and $\rho_* = \min_j |\mu_j|$, then for any $\delta > 0$ we can find $R_\delta > 0$ such that:

$$\|F(z)\| \leq (\rho^* + \varepsilon + \delta)\|z\| \quad \forall\, \|z\| < R_\delta. \tag{3.5}$$

This is in some sense the analogous of the Arnold Transversality Condition: it ensures that orbits of F intersect transversally (in fact enter into) the euclidean ball of radius R_δ.

By assumption $\rho^* < 1$ (Poincaré case), hence we can choose $\delta > 0$ such that: $\rho^* + \varepsilon + \delta < 1$. Using the m-flatness of f_1 we claim that there exists a positive constant C s.t.:

$$\|A^{-l} f_1 \circ F^{\circ l}(z)\| \leq C\rho_*^{-l}(\rho^* + \varepsilon + \delta)^{lm}\|z\|^m. \tag{3.6}$$

Because $m \geq q|\log \rho_*|/|\log(\rho^* + \varepsilon)|$, we have: $(\rho^* + \varepsilon + \delta)^m/\rho_* = \vartheta < 1$ for δ small enough. Then:

$$\|\Delta_l(z)\| \leq C\vartheta^l\|z\|^m, \tag{3.7}$$

for all $\|z\| < R_\delta$.

The existence of the limit for $L_l(z)$ follows by Cauchy criterium and the estimate:

$$\|L_{l+k}(z) - L_l(z)\| \leq \left\| \sum_{p=l+1}^{l+k} \Delta_p(z) \right\| \leq C\|z\|^m \frac{\vartheta^{l+1}(1 - \vartheta^k)}{1 - \vartheta}. \tag{3.8}$$

The proof that L is a biholomorphism from R_δ onto its image follows the same lines as the analogous result in the case of vector fields. And this concludes the proof.

Remark 3.2. This linearization procedure is new and it is different from the classical ones of Poincaré [**7**] or Koenigs [**6**], in fact for any finite l, $L_l(z)$ doesn't "push" the non-linearities of the given biholomorphisms to higher and higher orders. This construction is also different from the Césaro mean, thanks to the presence of the term f_1, which also increases the speed of the convergence.

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