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Positive polynomials and hyperdeterminants

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Abstract

Let F be a homogeneous real polynomial of even degree in any number of variables. We consider the problem of giving explicit conditions on the coefficients so that F is positive definite or positive semi-definite. In this note we produce a necessary condition for positivity, and a sufficient condition for non-negativity, in terms of positivity or semi-positivity of a one-variable *characteristic polynomial* of F. Also, we revisit the known sufficient condition in terms of Hankel matrices.

1. Introduction

Let $F = \sum_{|\lambda|=d} F_\lambda \, x^\lambda \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial in n variables, with real coefficients, homogeneous of even degree d. We are interested in giving explicit polynomial conditions on the coefficients F_λ so that F is positive definite or positive semi-definite.

Such conditions are known in the cases $d = 2$ (quadratic forms, positivity of principal minors) and $n = 2$ (binary forms, Sturm-Sylvester). We shall give conditions for any (n, d) via reduction to these two cases.

The tool used for the reduction is the *characteristic polynomial* of F, which is a fairly natural thing to consider once the notion of Determinant of a tensor is known. In the present case of symmetric tensors the Determinant is the discriminant. The notion of Determinant was developed (after Cayley and others) by Gelfand-Kapranov-Zelevinsky and we were motivated by some interesting questions posed in the Introduction of their work [7] on Hyperdeterminants.

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Inside the vector space of real homogeneous polynomials of degree d sit the convex cone of positive polynomials and the discriminant hypersurface. Part of our work will be to understand some of the geometry involved.

In this picture there are two further cones of classical relevance: the polynomials which are sums of squares and the sums of powers of linear forms; we refer to the monograph [13] by Reznick for more details on these cones and for references to the literature on positive polynomials. See also [10, 11] for the related Problem 17 of Hilbert.

In the last section we revisit the known sufficient condition for positivity in terms of positivity of the Hankel quadratic form (see [13]), making more explicit its relation to the Veronese embedding, to geometric plethysm (as in Fulton-Harris [5]) and to the coalgebra structure of the polynomial ring.

2. Positivity and the characteristic polynomial

Let F be a homogeneous polynomial of degree d in n variables $x_1, \ldots, x_n$ with coefficients in a field K. We will denote

$$K(n, d) \tag{2.1}$$

the K-vector space of all such polynomials. Its dimension is $N = \binom{n-1+d}{d}$.

For $K = \mathbb{R}$, the field of real numbers, we shall say that F is **positive** (resp. **non-negative**), written $F > 0$ (resp. $F \geq 0$), if $F(x) > 0$ for all $x \in \mathbb{R}^n - \{0\}$ (resp. $F(x) \geq 0$ for all $x \in \mathbb{R}^n$).

We are interested in obtaining conditions on the coefficients of F equivalent to $F > 0$ or to $F \geq 0$. We assume d is even, so that positive polynomials exist.

In case $d = 2$ such conditions are given by the well-known Sylvester's criterion:

If $F(x) = \sum_{1 \leq i,j \leq n} F_{ij} x_i x_j$ (with $F_{ij} = F_{ji} \in \mathbb{R}$) then

$$F > 0 \ \text{ if and only if } \ D_r(F) > 0, \ r = 1, \ldots, n \tag{2.2}$$

where $D_r(F) = \det(F_{ij})_{1 \leq i,j \leq r}$ is the $r \times r$ principal minor of the $n \times n$ symmetric matrix representing F.

Let us remark that the conditions $F \geq 0$ and $D_r(F) \geq 0$, $(r = 1, \ldots, n)$, are not equivalent. Actually, $F \geq 0$ is equivalent to

$$D_J(F) \geq 0, \ \forall J \subset \{1, \ldots, n\} \tag{2.3}$$

where $D_J(F) = \det(F_{ij})_{i,j \in J}$, see [6].

For the case $n = 2$ of binary forms we can use Sylvester's formulation of Sturm's Theorem [6, 2, 12]. To recall this, let $p \in \mathbb{R}[t]$ be a (monic) polynomial of degree d in one variable, over the real numbers. Consider the finite dimensional $\mathbb{R}$-algebra $A = \mathbb{R}[t]/(p)$ and denote its trace linear form $\mathrm{tr} : A \to \mathbb{R}$. For each $u \in A$ define a quadratic form on A by

$$Q_u(x) = \mathrm{tr}(ux^2) \tag{2.4}$$

Let $R(p) \subset \mathbb{C}$ be the set of complex roots of p. Then, Sylvester's theorem asserts that

$$\begin{aligned} \mathrm{rank}(Q_u) &= |\{r \in R(p)/\ u(r) \neq 0\}| \\ \mathrm{signature}(Q_u) &= |\{r \in R(p)/\ u(r) > 0\}| - |\{r \in R(p)/\ u(r) < 0\}|\,. \end{aligned} \tag{2.5}$$

Here $|\ \ |$ denotes the cardinality of a finite set. Suppose $p(0) \neq 0$ and denote P (resp. N) the number of positive (resp. negative) real roots of p. Choosing $u = 1$, we have that $\mathrm{sg}(Q_1)$ is the number $P+N$ of real roots of p. For $u = t$ we obtain $\mathrm{sg}(Q_t) = P-N$. Hence, $2P = \mathrm{sg}(Q_1) + \mathrm{sg}(Q_t)$. In particular, $P = 0$ (i. e. $p(t) > 0$ for $t > 0$) if and only if $\mathrm{sg}(Q_1) + \mathrm{sg}(Q_t) = 0$, a fact that will be useful later.

Let us denote $\nabla(n, d, \mathbb{C}) \subset \mathbb{C}(n, d)$ the set of singular polynomials of degree d in n variables, over the complex numbers. That is,

$$\nabla(n, d, \mathbb{C}) = \Big\{F \in \mathbb{C}(n, d)\ /\exists x \in \mathbb{C}^n - \{0\},\ \frac{\partial F}{\partial x_i}(x) = 0, \forall i\Big\}\,. \tag{2.6}$$

It is known (see e.g. [8]) that $\nabla(n, d, \mathbb{C})$ is an irreducible algebraic hypersurface of degree

$$D = n(d-1)^{n-1} \tag{2.7}$$

defined over the rational numbers. Therefore, there exists a polynomial (unique up to multiplicative constant)

$$\Delta = \Delta(n, d) \tag{2.8}$$

called the **discriminant**, such that

$$\nabla(n, d, \mathbb{C}) = \{F \in \mathbb{C}(n, d)\ /\ \Delta(F) = 0\}\,. \tag{2.9}$$

More precisely, writing a general polynomial

$$F = \sum_{|\lambda|=d} F_\lambda\, x^\lambda \quad \in \quad K(n, d) \tag{2.10}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, $|\lambda| = \sum_i \lambda_i$, $x^\lambda = \prod_i x_i^{\lambda_i}$ and $F_\lambda \in K$, so that $T_\lambda(F) = F_\lambda$ are coordinate functions on the vector space $K(n, d)$, we know Δ is a homogeneous polynomial in variables T_λ, of degree $D = n(d-1)^{n-1}$ and with rational coefficients. In other terms, Δ is an element of the D-th symmetric power of the rational vector space dual to $\mathbb{Q}(n, d)$.

We shall normalize Δ so that $\Delta(J) = 1$ where $J \in \mathbb{C}(n, d) - \nabla(n, d, \mathbb{C})$ is the polynomial

$$J(x) = \sum_{1 \leq j \leq n} x_j^d\,. \tag{2.11}$$

Restricting to the real numbers, we denote

$$\begin{aligned} \nabla = \nabla(n, d, \mathbb{R}) &= \nabla(n, d, \mathbb{C}) \cap \mathbb{R}(n, d) \\ &= \Big\{F \in \mathbb{R}(n, d)\ /\exists x \in \mathbb{C}^n - \{0\},\ \frac{\partial F}{\partial x_i}(x) = 0, \forall i\Big\} \\ &= \{F \in \mathbb{R}(n, d)\ /\ \Delta(F) = 0\} \end{aligned}$$

the set of real polynomials which have a singular point, real or complex.

Let us denote

$$\mathcal{P} = \mathcal{P}(n,d) = \{F \in \mathbb{R}(n,d)\ /\ F > 0\} \tag{2.12}$$

the set of all positive polynomials. It is easy to verify that $\mathcal{P}$ is an open convex cone in the vector space $\mathbb{R}(n,d)$. Here by "cone" we mean a set which is stable under multiplication by $\mathbb{R}_{>0}$. It is also easy to see that the closure of $\mathcal{P}$ (in the usual topology of $\mathbb{R}(n,d)$) is the closed convex cone

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}(n,d) = \{F \in \mathbb{R}(n,d)\ /\ F \geq 0\}\,. \tag{2.13}$$

With slight abuse of notation we write $\Delta : \mathbb{R}(n,d) \to \mathbb{R}$ for the polynomial function induced by the polynomial Δ.

Theorem 2.1

$\mathcal{P} - \nabla$ is connected.

Proof. Let $F \in \mathcal{P} \cap \nabla$ be a positive singular polynomial. Let $x \in \mathbb{C}^n - \{0\}$ be a singular point of F. Since $F(x) = 0$ and $F > 0$ it follows that $x \in \mathbb{C}^n - \mathbb{R}^n$. The complex conjugate $\bar{x} \in \mathbb{C}^n$ is also a singular point of F, because F has real coefficients. It follows that F has two distinct singular points in $\mathbb{C}^n$.

The idea now is that polynomials with at least two singular points occur in codimension at least two and hence they do not disconnect $\mathcal{P}$. To prove this, let us denote

$$\nabla_2(n,d,\mathbb{C})$$

the Zariski closure of the set of complex polynomials with at least two distinct singular points. By a standard incidence correspondence argument, $\nabla_2(n,d,\mathbb{C}) \subset \mathbb{C}(n,d)$ is a complex algebraic variety of codimension two. Let us denote its real points

$$\nabla_2 = \nabla_2(n,d,\mathbb{R}) = \nabla_2(n,d,\mathbb{C}) \cap \mathbb{R}(n,d)\,. \tag{2.14}$$

We claim that $\mathcal{P} \cap \nabla = \mathcal{P} \cap \nabla_2$. The inclusion $\subset$ was observed just above, while the other one is clear since $\nabla_2 \subset \nabla$.

Then $\mathcal{P} \cap \nabla \subset \mathbb{R}(n,d)$ is an open subset of a real algebraic variety ∇_2 of real dimension $N-2$, where $N = \binom{n-1+d}{d} = \dim \mathbb{R}(n,d)$.

The Theorem now follows from Proposition 2.2 below. $\square$

Surely Proposition 2.2 is a well-known statement, but we shall give a proof due to lack of a suitable reference.

Proposition 2.2

Let $P \subset \mathbb{R}^N$ be a connected open set, $Y \subset \mathbb{R}^N$ a real algebraic variety of dimension d and denote $X = P \cap Y$. Then

a) For any family of supports h and every sheaf $\mathcal{L}$ of abelian groups

$$H^j_h(X,\mathcal{L}) = 0,\ \forall j > d \tag{2.15}$$

i.e. the h-cohomological dimension of X is $\leq d$.

In particular, $H^j_c(X,\mathbb{Z}) = 0$ for $j > d$, where H_c denotes cohomology with compact supports.

b) If $d \leq N-2$ then $P - X$ is connected.

Proof. We refer to [9] for general definitions.

To prove a), let us denote $S \subset Y$ the set of singular points, $T \subset Y$ the union of the irreducible components of Y of dimension $< d$ and $A = P \cap (S \cup T)$, which is a closed subset of X. Let us remark that

i) $S \cup T \subset \mathbb{R}^N$ is a real algebraic variety of dimension $< d$.

ii) $Y - (S \cup T)$ is a smooth manifold of dimension d. Then, its open subset $X - A$ is also a smooth manifold of dimension d.

Now we apply to (X, A) the theory of [9], (4.10). From the exact sequence of sheaves on X

$$0 \to \mathcal{L}_{X-A} \to \mathcal{L} \to \mathcal{L}_A \to 0 \tag{2.16}$$

one obtains a long exact sequence [9], (4.10.1) of cohomology with supports in h

$$\begin{aligned} \ldots \;\to\; & H^j_{h|X-A}(X - A, \mathcal{L}) \to H^j_h(X, \mathcal{L}) \to H^j_{h|A}(A, \mathcal{L}) \to \\ \to\; & H^{j+1}_{h|X-A}(X - A, \mathcal{L}) \to \ldots \end{aligned}$$

Let $j > d$. By induction on d and due to i) we know that $H^j_{h|A}(A, \mathcal{L}) = 0$. Since a smooth manifold of dimension d has h-cohomological dimension d ([9], (4.14.1) and (5.12)) we have $H^j_{h|X-A}(X - A, \mathcal{L}) = 0$. From the exact sequence it follows that $H^j_h(X, \mathcal{L}) = 0$, as claimed.

In particular, when $\mathcal{L} = \mathbb{Z}_X$ is the constant sheaf $\mathbb{Z}$ on X and h is the family of compact subsets of X, we obtain the statement about cohomology with compact supports.

To prove b) we apply the Poincare-Lefschetz duality theorem ([3], (7.13)) for the N-dimensional manifold P and the closed set $X \subset P$. This Theorem implies that

$$H^{N-1}_c(X) = H_1(P, P - X)\,. \tag{2.17}$$

Combining with a) we obtain $H_1(P, P - X) = 0$ and from the homology sequence of the pair $(P, P - X)$ we deduce $H_0(P - X) = H_0(P) = \mathbb{Z}$. □

Now we deduce a Corollary of importance for our present purpose.

Corollary 2.3

Let $F \in \mathbb{R}(n, d)$. If $F \geq 0$ then $\Delta(F) \geq 0$.

Proof. We want to show that Δ is non-negative on $\bar{\mathcal{P}}$. It is clear that

$$\mathcal{P} - \nabla \subset (\Delta > 0) \cup (\Delta < 0) \tag{2.18}$$

where we denote $(\Delta > 0) = \{F/\ \Delta(F) > 0\}$. By Theorem 2.1, $\mathcal{P} - \nabla$ is connected and the polynomial J defined in (2.11) belongs to the intersection $(\mathcal{P} - \nabla) \cap (\Delta > 0)$, so we obtain $\mathcal{P} - \nabla \subset (\Delta > 0)$, that is, $\mathcal{P} \subset (\Delta \geq 0)$, and by continuity $\bar{\mathcal{P}} \subset (\Delta \geq 0)$, as we wanted to prove. □

Example 2.4 Les us consider the case $(n, d) = (2, 4)$ of binary quartics, written $F = \sum_{0 \leq i \leq 4} F_i\ x_1^i x_2^{4-i}$. Over the complex numbers we have $\nabla(2, 4, \mathbb{C}) \subset \mathbb{C}(2, 4)$

of respective complex dimensions 4 and 5. Their real parts $\nabla \subset \mathbb{R}(2,4)$ have real dimensions 4 and 5, but ∇ contains the open set $\mathcal{P} \cap \nabla$ which has dimension 3 and is a "component" of ∇ in the sense that it is not contained in the closure of $\nabla - \mathcal{P} \cap \nabla$, with respect to the usual (not Zariski) topology of $\mathbb{R}(2,4)$.

More explicitly, $\mathcal{P} \cap \nabla$ consists of the quartics with two double complex roots, i. e. of the form $F = (ax_1^2 + bx_1x_2 + cx_2^2)^2$ with $b^2 - 4ac < 0$ (so we see again it has dimension 3) and it is clear that these are not limit of real quartics with only one double root.

Let $V \subset \mathbb{R}(2,4)$ denote the 3-dimensional subspace of those F's with $F_4 = 1$ and $F_3 = 0$. We refer to [8], page 381, for a drawing of the two-dimensional real algebraic variety $\nabla \cap V$, but let's point out that $(\mathcal{P} \cap \nabla) \cap V$ (curve of quartics of the form $(x_1^2 + cx_2^2)^2$ with $c > 0$) is lacking in that picture and should be added as a curve pointing out and going through the point $F = x_1^4$ marked "quadruple root".

DEFINITION 2.5 For $F \in K(n,d)$ we define the **characteristic polynomial** of F (with respect to J)

$$\chi(F;J)(t) = \Delta(F + tJ) \ \in \ K[t] \tag{2.19}$$

where $J(x) = \sum_{1 \le j \le n} x_j^d$, as in (2.11).

Remark 2.6 a) The definition depends on the choice of J, but for simplicity we may write $\chi(F)$ instead of $\chi(F;J)$. In fact, our choice of a positive J is rather arbitrary.

b) Since J and Δ have rational coefficients it follows that $\chi(F)$ has coefficients in K if F has coefficients in K.

c) $\chi(F)$ is a polynomial in t of degree D as in (2.7), and we may write

$$\chi(F)(t) = \sum_{0 \le j \le D} \Delta_j(F)\ t^j \tag{2.20}$$

where $\Delta_0 = \Delta$ and Δ_j is a homogeneous polynomial in the coefficients of F, of degree $D - j$ for $j = 0, \ldots, D$. Also, by our normalization of Δ in (2.11) it follows that $\chi(F)$ is monic.

d) The roots of $\chi(F)$ are the values of t such that $F + tJ$ is singular, that is, they parametrize the intersections of the discriminant hypersurface ∇ with the pencil spanned by F and J. These roots may have the right to be called "eigenvalues of F" (with respect to J).

The next Proposition gives a necessary condition for non-negativity.

Proposition 2.7

Let $F \in \mathbb{R}(n,d)$. If $F \ge 0$ then $\chi(F)(t) \ge 0$ for all $t \in \mathbb{R}_{\ge 0}$.

Proof. Since $J \ge 0$, for $F \ge 0$ and $t \ge 0$ we have $F + tJ \ge 0$. By Corollary 2.3, $\chi(F)(t) = \Delta(F + tJ) \ge 0$, as wanted. □

Remark 2.8 a) The Proposition gives a necessary condition for non-negativity in principle explicitly computable via applying the Sylvester criterion (2.5) to the characteristic polynomial $\chi(F)$.

b) As kindly pointed out to us by Jiawang Nie (Berkeley), the converse to Proposition 2.7 fails for $d = 2$.

c) Regarding the choice of J mentioned in (2.6) a), let us fix a finite number of positive polynomials $J_1, \dots, J_m \in \mathbb{R}(n,d)$. As in Proposition 2.7, if $F \geq 0$ then $\chi(F; J_i)(t) \geq 0$ for all $t \in \mathbb{R}_{\geq 0}$ and all $i = 1, \dots, m$. It would be interesting to know if this necessary condition for non-negativity is also sufficient, for some choice of m and $J_1, \dots, J_m$.

d) Similarly, for $F \in \mathbb{R}(n,d)$ define a generalized characteristic polynomial

$$\chi(F; J_1, \dots, J_m)(t_1, \dots, t_m) = \Delta(F + \sum_{1=1}^{m} t_i J_i) \ \in \mathbb{R}[t_1, \dots, t_m]\,. \tag{2.21}$$

The same proof of Proposition 2.7 gives that a necessary condition for $F \geq 0$ is

$$\chi(F; J_1, \dots, J_m)(t_1, \dots, t_m) \geq 0, \ \forall t_i \geq 0\,. \tag{2.22}$$

As kindly pointed out to us by Michel Coste, the converse to Proposition 2.7 holds replacing $\geq$ by $>$. More precisely,

Proposition 2.9

Let $F \in \mathbb{R}(n,d)$. If $\chi(F)(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$ then $F > 0$.

Proof. If $\chi(F)(t) = \Delta(F + tJ) > 0$ for all $t \in \mathbb{R}_{\geq 0}$ the polynomials $F + tJ$ with $t \in [0, +\infty]$ are all non-singular and define a smooth family of smooth varieties $X_t = (F + tJ = 0)$. By Thom's Lemma such a family is locally trivial and hence the fibers X_t are all diffeomorphic. Since the set of zeros of J is equal to $\{0\}$, the same is true for $F + tJ$ for all $t \geq 0$. In particular, the set of zeros of F is $\{0\}$, so that $F > 0$ or $F < 0$. If $F < 0$ then there exists $t \geq 0$ such that $F + tJ$ has a non-trivial zero, which is a contradiction. Therefore $F > 0$, as wanted.

Here is an alternative proof, due to the Referee: Because of homogeneity, we consider homogeneous polynomials to be real functions on the unit sphere $S \subset \mathbb{R}^n$. The function M which maps a continuous real valued function to its minimum value $M(G) = \min\{G(x) | x \in S\}$ is continuous with respect to the supremum norm on the space $\mathcal{C}(S, \mathbb{R})$ of continuous functions. Now suppose F is not strictly positive on S, i.e. $M(F) < 0$. Since J is positive and S is compact, we have that $M(F + bJ) > 0$ for some positive real number b. Since M is continuous, there is some $0 < a < b$ such that $M(F + aJ) = 0$. But then $F + aJ$ has a critical point outside of the origin and hence $\chi(F)(a) = 0$. □

Now we obtain further conditions for positivity by combining the previous constructions with the operation of restriction to a linear subspace.

Let V be a finite dimensional real vector space and denote $S^d(V^*)$ the d-th symmetric power of the dual of V, thought of as the space of homogeneous polynomials of degree d in V. As in (2.6) and (2.9) denote $\nabla_V \subset S^d(V^*)$ the set of singular polynomials and $\Delta_V \in S^D((S^d(V^*))^*) = S^D(S^d(V))$ the discriminant.

Let $V \subset \mathbb{R}^n$ be a linear subspace. For $F \in \mathbb{R}(n,d) = S^d((\mathbb{R}^n)^*)$, denote $F_V \in S^d(V^*)$ the restriction of F to V. We may now formulate a stronger necessary condition for non-negativity:

Proposition 2.10

Let $F \in \mathbb{R}(n,d)$. If $F \geq 0$ in $\mathbb{R}^n$ then $\Delta_V(F_V) \geq 0$ for every linear subspace $V \subset \mathbb{R}^n$. Also, the characteristic polynomial $\chi_V(F)(t) = \Delta_V(F_V + tJ_V)$ is ≥ 0 for $t \geq 0$

Proof. It is clear that if $F \geq 0$ in $\mathbb{R}^n$ then its restriction F_V is also ≥ 0 in V. Applying Corollary 2.3 in V we obtain $\Delta_V(F_V) \geq 0$, as claimed. The claim about $\chi_V(F)$ is immediate as in Proposition 2.7. $\square$

EXAMPLE 2.11 For $J \subset \{1,\ldots,n\}$ let $V_J = \{x \in \mathbb{R}^n / x_i = 0, \forall i \notin J\}$, the J-th coordinate plane. It follows that if $F \geq 0$ in $\mathbb{R}^n$ then

$$\Delta_{V_J}(F) \geq 0,\ \forall J \subset \{1,\ldots,n\}\,. \tag{2.23}$$

Notice that in case $d=2$, these Δ_{V_J} coincide with the diagonal minors D_J of (2.3).

Remark 2.12 In case $d=2$, the conditions of Example 2.11 are equivalent to $F \geq 0$ (see (2.3)). For $d > 2$ this is no longer true, as it is easily seen in the case of binary forms.

3. Review of positivity via the Hankel quadratic form

In this section we review a well-known sufficient condition for non-negativity of $F \in \mathbb{R}(n,d)$ in terms of a quadratic form $h(F)$ associated to F, see for example [13]. See also [1] for some related recent results.

Let V be a vector space of dimension n over a field K. The multiplication of the symmetric algebra $S(V^*)$ induces maps

$$\mu : S^m(S^d(V^*)) \to S^{md}(V^*)\,. \tag{3.1}$$

These are homomorphisms between linear representations of $\mathrm{GL}(V)$. They may be interpreted geometrically as the pull-back of homogeneous polynomials of degree m under the d-th Veronese map

$$\mathbb{P}V \to \mathbb{P}S^d(V) \tag{3.2}$$

sending $v \in V$ to $v^d \in S^d(V)$ (see [5]).

Let $K = \mathbb{R}$ and fix m, d. Consider the vector space $U = S^d(V)$ and suppose $G \in S^m(U^*)$ is non-negative (resp. positive). Then the restriction of G to the d-th Veronese variety in $\mathbb{P}U$ is clearly non-negative (resp. positive) and hence $\mu(G) \in S^{md}(V^*)$ is non-negative (resp. positive).

For $m = 2$ in particular, we have

$$\mu : S^2(S^d(V^*)) \to S^{2d}(V^*) \tag{3.3}$$

and if the quadratic form $G \in S^2(U^*)$ is ≥ 0 (resp. > 0) then the homogeneous polynomial $\mu(G) \in S^{2d}(V^*)$ is ≥ 0 (resp. > 0).

On the other hand, suppose we have a map

$$h : S^{2d}(V^*) \to S^2(S^d(V^*)) \tag{3.4}$$

such that $\mu \circ h =$ identity. It follows that if the quadratic form $h(F)$ on $U = S^d(V)$ is ≥ 0 (resp. > 0) then $F = \mu(h(F)) \in S^{2d}(V^*)$ is ≥ 0 (resp. > 0). This is the sufficient condition mentioned above.

What we shall do next is to explicitly construct such an h. It will be the well-known Hankel quadratic form $h(F)$ associated to a homogeneous polynomial $F \in S^{2d}(V^*)$ of even degree $2d$ (see [13]). The definition will be based on the co-algebra structure of the symmetric algebra. It will follow in particular that h is linear and $\mathrm{GL}(V)$-equivariant. In [13] these constructions are based on an inner product on $S^{2d}(V^*)$. Let us remark that since $S^{2d}(V^*)$ is an irreducible representation of $\mathrm{GL}(V)$, such an equivariant h is unique up to multiplicative constant. Thus, the construction below may be considered as another example of plethysm as in [5].

Let V be a vector space of dimension n over a field K. For each $d \in \mathbb{N}$ we have a natural map

$$V^{\otimes d} \otimes V^{*\otimes d} \to K \tag{3.5}$$

given on elementary tensors by

$$(v_1 \otimes \cdots \otimes v_d) \otimes (\varphi_1 \otimes \cdots \otimes \varphi_d) \mapsto \frac{1}{d!} \sum_{\sigma \in \mathbb{S}_d} \prod_{i=1}^{d} < \varphi_i, v_{\sigma(i)} > \, . \tag{3.6}$$

This map factors through the quotient and gives a map of symmetric powers

$$(\,,\,) : S^d(V) \otimes S^d(V^*) \to K \tag{3.7}$$

with similar formula for elementary tensors (monomials). This induces a linear map (called polarization, see e.g. [4])

$$\wp : S^d(V) \to (S^d(V^*))^* \, . \tag{3.8}$$

Let us remark that $(\,,\,)$ and $\wp$ are equivariant for the natural actions of $\mathrm{GL}(V)$.

Let $\{e_1, \ldots, e_n\}$ be an ordered basis of V and denote $\{x_1, \ldots, x_n\}$ the dual basis of V^*, so that $< x_i, e_j >= \delta_{ij}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_i \alpha_i = d$, denote as usual $e^\alpha = e_1^{\alpha_1} e_2^{\alpha_2} \ldots e_n^{\alpha_n} \in S^d(V)$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \in S^d(V^*)$. Then $\{e^\alpha\}_{|\alpha|=d}$ (resp. $\{x^\alpha\}_{|\alpha|=d}$) is a basis of $S^d(V)$ (resp. of $S^d(V^*)$). Also, it is easy to check from the explicit formulas above that

$$(x^\beta, e^\alpha) = \frac{\alpha!}{d!}\, \delta_{\alpha\beta} \tag{3.9}$$

where $\alpha! = \prod_i \alpha_i!$. It follows that

$$\wp(e^\alpha)\Big(\frac{d!}{\beta!}\, x^\beta\Big) = \delta_{\alpha\beta} \, . \tag{3.10}$$

Writing $D^\alpha = \wp(e^\alpha) \in (S^d(V^*))^*$ we have

$$D^\alpha\Big(\frac{d!}{\beta!}\, x^\beta\Big) = \delta_{\alpha\beta} \tag{3.11}$$

and the isomorphism $\wp$ may be written as

$$\wp\Big(\sum_{|\alpha|=d} a_\alpha e^\alpha\Big) = \sum_{|\alpha|=d} a_\alpha D^\alpha \, . \tag{3.12}$$

Now we look at the structure of co-algebra in the symmetric algebra

$$S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)\,. \tag{3.13}$$

Consider the multiplication map of the symmetric algebra $S(V^*)$

$$\mu : S^d(V^*) \otimes S^e(V^*) \to S^{d+e}(V^*) \tag{3.14}$$

and the diagram

$$\begin{array}{ccc} (S^{d+e}(V^*))^* & \xrightarrow{\mu^*} & (S^d(V^*)\otimes S^e(V^*))^* \cong (S^d(V^*))^* \otimes (S^e(V^*))^* \\ \wp\big\uparrow & & \big\uparrow\wp\otimes\wp \\ S^{d+e}(V) & \xrightarrow{h} & S^d(V)\otimes S^e(V) \end{array} \tag{3.15}$$

where the vertical arrows are isomorphisms and h is defined so that the diagram commutes. It easily follows from the definitions that the effect of h on basis elements is, with sums extended to the pairs (α,β) such that $|\alpha| = d,\ |\beta| = e,\ \alpha+\beta=\gamma$:

$$h(e^\gamma) = \sum c_{\alpha\beta}\; e^\alpha \otimes e^\beta \tag{3.16}$$

where $c_{\alpha\beta} = \frac{d!}{\alpha!}\frac{e!}{\beta!}\frac{(\alpha+\beta)!}{(d+e)!}$. In terms of the basis elements $E^\alpha = \frac{d!}{\alpha!}\; e^\alpha \in S^d(V)$,

$$h(E^\gamma) = \sum\; E^\alpha \otimes E^\beta\,. \tag{3.17}$$

Applying this to V^* we obtain $\mathrm{GL}(V)$-equivariant maps

$$h : S^{d+e}(V^*) \to S^d(V^*) \otimes S^e(V^*) \tag{3.18}$$

such that for $X^\alpha = \frac{d!}{\alpha!}\; x^\alpha \in S^d(V^*)$

$$h(X^\gamma) = \sum\; X^\alpha \otimes X^\beta \tag{3.19}$$

Proposition 3.1

With the notation above and any d, e, the composition

$$S^{d+e}(V^*) \xrightarrow{\ h\ } S^d(V^*)\otimes S^e(V^*) \xrightarrow{\ \mu\ } S^{d+e}(V^*) \tag{3.20}$$

is the identity.

Proof. To carry out this elementary calculation, let us compute on basis elements x^γ as above

$$\mu(h(x^\gamma)) = \mu\Big(\sum c_{\alpha\beta}\; x^\alpha\otimes x^\beta\Big) = \sum c_{\alpha\beta}\; x^{\alpha+\beta} = \Big(\sum c_{\alpha\beta}\Big)\; x^\gamma = x^\gamma\,.$$

The last equality amounts to

$$\sum \frac{d!}{\alpha!}\frac{e!}{\beta!} = \frac{(d+e)!}{(\alpha+\beta)!} \tag{3.21}$$

and this formula is easily checked by multiplying

$$\Big(\sum_{i=1}^{n} x_i\Big)^d = \sum_{|\alpha|=d} \frac{d!}{\alpha!}\, x^\alpha \quad \text{and} \quad \Big(\sum_{i=1}^{n} x_i\Big)^e = \sum_{|\beta|=e} \frac{e!}{\beta!}\, x^\beta \tag{3.22}$$

and equating coefficients of x^γ. $\square$

Now we specialize (3.1) to the case $d = e$. Since the multiplication of the symmetric algebra is commutative, by restriction we obtain $\mathrm{GL}(V)$-equivariant maps, still denoted μ and h

$$S^{2d}(V^*) \xrightarrow{\ h\ } S^2(S^d(V^*)) \xrightarrow{\ \mu\ } S^{2d}(V^*) \tag{3.23}$$

defined by the formulas above and satisfying $\mu \circ h =$ identity.

This is the desired explicit definition of h as in (3.4). Hence, we obtain the sufficient condition: if the quadratic form $h(F)$ is positive (resp. non-negative) then $F \in S^{2d}(V^*)$ is positive (resp. non-negative).

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References

1. G. Blekherman, Convexity properties of the cone of nonnegative polynomials, *Discrete Comput. Geom.* **32** (2004), 345–371.
2. M. Demazure, Les notes informelles de calcul formel, Hermite déjà... http://www.stix.polytechnique.fr/publications/1984-1994.html
3. A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin-New York, 1980.
4. I. Dolgachev, *Lectures on Invariant Theory*, London Mathematical Society Lectures Notes Series, **296**, Cambridge University Press, 2003.
5. W. Fulton and J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991.
6. F.R. Gantmacher, *The Theory of Matrices*, AMS Chelsea Publishing, Providence, RI, 1998.
7. I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Hyperdeterminants, *Adv. Math.* **96** (1992), 226–263.
8. I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser Boston, Inc., Boston, MA, 1994.
9. R. Godement, *Topologie Algebrique et Theorie des Faisceaux*, Hermann, Paris, 1958.
10. I. Kaplansky, *Hilbert's Problems*, Lecture Notes-University of Chicago, 1977.
11. A. Pfister, *Hilbert's Seventeenth Problem and Related Problems on Definite Forms*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math. **28**, Northern Illinois 1974), Amer. Math. Soc., Providence, RI, 1976.
12. C. Procesi, Positive symmetric functions, *Adv. in Math.* **29** (1978), 219–225.
13. B. Reznick, Sums of even powers of real linear forms, *Mem. Amer. Math. Soc.* **96** (1992), 8–155.