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Tb theorems for Triebel-Lizorkin spaces over special spaces of homogeneous type and their applications*

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Abstract

The author first establishes the *Tb* theorems for Triebel-Lizorkin spaces by the discrete Calderón type reproducing formula and the Plancherel-Pôlya characterization for the Triebel-Lizorkin spaces. As an application of the *Tb* theorems, new characterizations of Triebel-Lizorkin spaces with minimum regularity and cancellation conditions are given over special spaces of homogeneous type.

1. Introduction

In the past years, there has been significant progress on the problem of proving the boundedness of generalized Calderón-Zygmund operators on various function spaces. The remarkable result is the famous $T1$ theorem of David and Journé in [3]. The $T1$ theorem, however, cannot be directly applied to the Cauchy integral operators on Lipschitz curves. Meyer observed that if 1 in the $T1$ theorem is allowed to be replaced by a bounded complex-valued function b which satisfies the condition $0 < \delta \leq \mathrm{Re} b(x)$ almost everywhere, then this result would imply the L^2 boundedness of the Cauchy integral operators on Lipschitz curves. McIntosh and Meyer proved such a theorem where the function 1 is replaced by accretive functions in [14]. David, Journé and Semmes gave more general conditions on L^∞ function b, which are so-called para-accretive functions. They proved that the function 1 in the $T1$ theorem can be replaced

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by para-accretive functions, namely the Tb theorem. Moreover, they showed that para-accretivity is also necessary in the sense that if the Tb theorem holds for a bounded function b, then b is para-accretive in [4]. The Tb theorem has been developed to other spaces, namely the Besov and Triebel-Lizorkin spaces. See [8, 9] and [11] for more details.

The main purpose of this paper is to establish the Tb theorems from $b\dot{F}_p^{\alpha,q}(X)$ to $b^{-1}\dot{F}_p^{\alpha,q}(X)$ for

$$\max\left\{\frac{d}{d+\alpha},\frac{d}{d+\alpha+\epsilon}\right\}<p<\infty,\quad \max\left\{\frac{d}{d+\alpha},\frac{d}{d+\alpha+\epsilon}\right\}<q\leq\infty,\ 0\neq|\alpha|<\epsilon$$

by the discrete Calderón type reproducing formula and the Plancherel-Pôlya characterization for Triebel-Lizorkin spaces, where d is the dimension of X. As an application of the Tb theorem, Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}(X)$ and $b^{-1}\dot{F}_p^{\alpha,q}(X)$ with

$$\max\left\{\frac{d}{d+\alpha},\frac{d}{d+\alpha+\epsilon}\right\}<p<\infty,\quad \max\left\{\frac{d}{d+\alpha},\frac{d}{d+\alpha+\epsilon}\right\}<q\leq\infty,\ 0<|\alpha|<\epsilon$$

can be characterized by more general operators whose kernels satisfy only half (depending on the sign of α) of the usual smoothness and cancellation conditions.

To state main results of this paper, we begin by recalling the definitions necessary for Triebel-Lizorkin spaces on spaces of homogeneous type and some basic facts about the Calderón-Zygmund operator theory. A *quasi-metric* ρ on a set X is a function $\rho : X\times X\to[0,\infty)$ satisfying:

(i) $\rho(x,y)=0$ if and only if $x=y$;
(ii) $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$;
(iii) There exists a constant $A\in[1,\infty)$ such that for all $x,y,z\in X$,

$$\rho(x,y)\leq A[\rho(x,z)+\rho(z,y)].$$

Any quasi-metric defines a topology, for which the balls $B(x,r)=\{y\in X:\rho(y,x)<r\}$ for all $x\in X$ and all $r>0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

DEFINITION 1.1 Let $d>0$ and $0<\theta\leq 1$. A *space of homogeneous type* $(X,\rho,\mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with supp $\mu=X$, and there exists a constant $0<C<\infty$ such that for all $0<r<\operatorname{diam}X$ and all $x,x',y\in X$,

$$|\rho(x,y)-\rho(x',y)|\leq C\rho(x,x')^{\theta}[\rho(x,y)+\rho(x',y)]^{1-\theta}. \tag{1.1}$$

$$\mu(B(x,r))\sim r^d, \tag{1.2}$$

Macías and Segovia have proved that one can replace the quasi-metric ρ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\overline{\rho}$ which yields the same topology on X as ρ such that $(X,\overline{\rho},\mu)$ is the space defined by Definition 1.1 for $d=1$ in [13]. From (1.2) it is easy to deduce $\mu(x)=0$ for all $x\in X$. This means that spaces of homogeneous type defined by Definition 1.1 are atomless measures spaces and for which each ball has the same Hausdorff dimension. Throughout this paper, we will always assume that $\mu(X) = \infty$.

It is well known that the spaces of homogeneous type introduced by Coifman and Weiss in [2] include $\mathbb{R}^n$, the boundaries of bounded Lipschitz domains in $\mathbb{R}^n$, the d-torus in $\mathbb{R}^n$, C^∞−compact Riemannian manifolds and many other models. In particular, the Lipschitz manifolds introduced recently by Triebel in [15] and isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [16] that the d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set, the generalized Sierpinski carpet and so forth.

DEFINITION 1.2 A complex-valued bounded function b defined on X is said to be a *para-accretive function* if there exists a constant $C > 0$ and $\kappa \in (0, 1]$ such that for all balls $B \subset X$, there is a ball $B' \subset B$ with $\kappa\mu(B) \leq \mu(B')$ satisfying

$$\frac{1}{\mu(B)}\Big|\int_{B'} b(x)d\mu(x)\Big| \geq C > 0.$$

For $\eta \in (0, \theta]$, we let $C_0^\eta(X)$ be the set of all functions having compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space. Moreover, if b is a para-accretive function, M_b denote the corresponding multiplication operator and $bC_0^\eta(X)$ denote the image of $C_0^\eta(X)$ under M_b with the natural topology, which means that $f \in bC_0^\eta(X)$ if and only if $f = bg$ for some $g \in C_0^\eta(X)$ and we define $\|f\|_{bC_0^\eta(X)} = \|g\|_{C_0^\eta(X)}$. Let b_1 and b_2 be two para-accretive functions. Suppose that T is a continuous linear mapping from $b_1C_0^\eta(X)$ to $(b_2C_0^\eta(X))'$, associated to a kernel $K(x, y)$ in the sense that

$$< Tf, g >= \int_X \int_X g(x)b_2(x)K(x, y)b_1(y)f(y)d\mu(x)d\mu(y)$$

for all f and g in C_0^η with disjoint supports.

Assume that $K(x, y)$ satisfies the pointwise conditions

$$|K(x, y)| \leq C\rho(x, y)^{-d}, \tag{1.3}$$

$$|K(x, y) - K(x', y)| \leq C\rho(x, x')^\epsilon\rho(x, y)^{-d-\epsilon} \quad \text{for } \rho(x, x') \leq \rho(x, y)/(2A), \tag{1.4}$$

$$|K(x, y) - K(x, y')| \leq C\rho(y, y')^\epsilon\rho(x, y)^{-d-\epsilon} \quad \text{for } \rho(y, y') \leq \rho(x, y)/(2A). \tag{1.5}$$

Assume also that T satisfies the Weak Boundedness Property, denote this by $T \in WBP(X)$,

$$| < Tf, g > | \leq Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}$$

for all f and g in $C_0^\eta(X)$ with diameters of supports not greater than r.

We now recall the definitions of the following approximations to the identity and the test function of type (β, γ) which were introduced in [4, 8, 9].

DEFINITION 1.3 A sequence $\{S_k\}_{k\in\mathbb{Z}}^b$ of operators is said to be an *approximation to the identity associated to a para-accretive function* b if $S_k(x,y)$, the kernel of S_k, are functions from $X\times X$ into $\mathbb{C}$ such that for all $k\in\mathbb{Z}$ and all x,x',y and y' in X, and some $0<\epsilon\le\theta$ and $C>0$,

$$|S_k(x,y)|\le C\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}};\tag{1.6}$$

$$|S_k(x,y)-S_k(x',y)|\le C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\tag{1.7}$$

for $\rho(x,x')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$|S_k(x,y)-S_k(x,y')|\le C\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\tag{1.8}$$

for $\rho(y,y')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\begin{aligned}&\big|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]\big|\\&\le C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}\end{aligned}\tag{1.9}$$

for $\rho(x,x')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(y,y')\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int S_k(x,y)b(x)d\mu(x)=1\tag{1.10}$$

for all $k\in\mathbb{Z}$;

$$\int S_k(x,y)b(y)d\mu(y)=1\tag{1.11}$$

for all $k\in\mathbb{Z}$.

Remark 1.1 By Coifman's construction in [3, 4], if b is a given para-accretive function, one can construct an approximation to the identity of order θ such that $S_k(x,y)$ has a compact support when one variable is fixed, namely, there is a constant $C>0$ such that for all $k\in\mathbb{Z}, S_k(x,y)=0$ if $\rho(x,y)\ge C2^{-k}$.

Remark 1.2 We also remark that in the sequel, if the approximation to the identity as in Definition 1.2 exists, then all the results still hold when b and b^{-1} are bounded. It seems that we do not need to assume that b is a para-accretive function. However, in [4], it was proved that the existence of the approximation to the identity as in Definition 1.2 is equivalent to the para-accretivity of b.

DEFINITION 1.4 Fix two exponents $0<\beta\le\theta$ and $\gamma>0$. Suppose that b is a para-accretive function. A function f defined on X is said to be a test function of type (β,γ) centered at $x_0\in X$ with width $t>0$ if f satisfies the following conditions:

$$|f(x)|\le C\frac{t^\gamma}{(t+\rho(x,x_0))^{d+\gamma}};\tag{1.12}$$

$$|f(x)-f(x')| \le C\left(\frac{\rho(x,x')}{t+\rho(x,x_0)}\right)^{\beta}\frac{t^{\gamma}}{(d+\rho(x,x_0))^{d+\gamma}} \tag{1.13}$$

for $\rho(x,x') \le \frac{1}{2A}(t+\rho(x,x_0))$;

$$\int_X f(x)b(x)d\mu(x) = 0. \tag{1.14}$$

If f is a test function of type (β,γ) centered at x_0 with width $d>0$, we write $f \in \mathcal{M}_b(x_0,d,\beta,\gamma)$, and the norm of f in $\mathcal{M}_b(x_0,d,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{M}_b(x_0,d,\beta,\gamma)} = \inf\{C\ge 0: \text{ (1.12) and (1.13) hold}\}.$$

We denote by $\mathcal{M}_b(\beta,\gamma)$ the class of all $f\in\mathcal{M}_b(x_0,1,\beta,\gamma)$ for some fixed $x_0\in X$. It is easy to see that $\mathcal{M}_b(x_1,d,\beta,\gamma)=\mathcal{M}_b(\beta,\gamma)$ with the equivalent norms for all $x_1\in X$ and $d>0$. Furthermore, it is also easy to check that $\mathcal{M}_b(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}_b(\beta,\gamma)$. We denote by $(\mathcal{M}_b(\beta,\gamma))'$ the dual space of $\mathcal{M}_b(\beta,\gamma)$ consisting of all linear functionals $\mathcal{L}$ from $\mathcal{M}_b(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that for all $f\in\mathcal{M}_b(\beta,\gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{M}_b(\beta,\gamma)}.$$

We denote by $\langle h,f\rangle$ the natural pairing of elements $h\in(\mathcal{M}_b(\beta,\gamma))'$ and $f\in\mathcal{M}_b(\beta,\gamma)$, Since $\mathcal{M}_b(x_1,d,\beta,\gamma)=\mathcal{M}_b(\beta,\gamma)$ with the equivalent norms for all $x_1\in X$ and $d>0$. Thus, for all $h\in(\mathcal{M}_b(\beta,\gamma))'$, $\langle h,f\rangle$ is well defined for all $f\in\mathcal{M}_b(x_0,d,\beta,\gamma)$ with $x_0\in X$ and $d>0$. In what follows, for a given $0<\epsilon\le\theta$, we let $\widetilde{\mathcal{M}}_b(\beta,\gamma)$ be the completion of the space $\mathcal{M}_b(\epsilon,\epsilon)$ in $\mathcal{M}_b(\beta,\gamma)$ when $0<\beta,\gamma<\epsilon$.

Let b be a para-accretive function. As usual, we write

$$b\mathcal{M}_b(\beta,\gamma)=\{f: f=bg \text{ for some } g\in\mathcal{M}_b(\beta,\gamma)\}.$$

If $f\in b\mathcal{M}_b(\beta,\gamma)$ and $f=bg$ for some $g\in\mathcal{M}_b(\beta,\gamma)$, then the norm of f is defined by

$$\|f\|_{b\mathcal{M}_b(\beta,\gamma)}=\|g\|_{\mathcal{M}_b(\beta,\gamma)}.$$

By this definition, it is easy to see that

$$f\in(b\widetilde{\mathcal{M}}_b(\beta,\gamma))' \text{ if and only if } bf\in(\widetilde{\mathcal{M}}_b(\beta,\gamma))',$$

where we define $bf\in(\widetilde{\mathcal{M}}_b(\beta,\gamma))'$ by

$$\langle bf,g\rangle=\langle f,bg\rangle$$

for all $g\in\widetilde{\mathcal{M}}_b(\beta,\gamma)$.

Now, we can introduce the new Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}(X), b^{-1}\dot{F}_p^{\alpha,q}(X)$ via approximations to the identity in [5].

Definition 1.5 Suppose that $-\epsilon<\alpha<\epsilon$, $\{S_k\}_{k\in\mathbb{Z}}^b$ be an approximation to the identity associated to a para-accretive function b and let $D_k=S_k-S_{k-1}$ for $k\in\mathbb{Z}$. Suppose β and γ satisfying

$$\max\{\alpha,0,-\alpha+\max\{0,d(1/p-1)\}\}<\beta<\epsilon \tag{1.15}$$

and

$$\max\{\alpha - d/p, \max\{0, d(1/p-1)\}, -\alpha + d(1/p-1)\} < \gamma < \epsilon.$$

The *Triebel-Lizorkin space* $b\dot{F}_p^{\alpha,q}(X)$ for

$$\max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < p < \infty \quad \text{and} \quad \max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < q \le \infty$$

is the collection of $f \in (\widetilde{\mathcal{M}}_b(\beta,\gamma))'$ such that

$$\|f\|_{b\dot{F}_p^{\alpha,q}(X)} = \left\| \left\{ \sum_{k=-\infty}^{\infty} [2^{k\alpha}|D_k(f)|]^q \right\}^{1/q} \right\|_{L^p(X)} < \infty\,;$$

The *Triebel-Lizorkin space* $b^{-1}\dot{F}_p^{\alpha,q}(X)$ for

$$\max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < p < \infty \quad \text{and} \quad \max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < q \le \infty$$

is the collection of $f \in (b\widetilde{\mathcal{M}}_b(\beta,\gamma))'$ such that

$$\|f\|_{b^{-1}\dot{F}_p^{\alpha,q}(X)} = \left\| \left\{ \sum_{k=-\infty}^{\infty} [2^{k\alpha}|D_k(bf)|]^q \right\}^{1/q} \right\|_{L^p(X)} < \infty\,.$$

The Tb theorems on these new Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}(X), b^{-1}\dot{F}_p^{\alpha,q}(X)$ with $|\alpha| < \epsilon$ were proved in [5] and [9]. More precisely, they showed that T is bounded from $b\dot{F}_p^{\alpha,q}(X)$ to $b^{-1}\dot{F}_p^{\alpha,q}(X)$ with $|\alpha| < \epsilon$,

$$\max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < p < \infty \quad \text{and} \quad \max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < q \le \infty$$

if its kernel satisfies (1.3), (1.4) and (1.5), $T(b) = T^*(b) = 0$ and $M_bTM_b \in WBP(X)$. In this paper, we first prove the Tb theorems on Triebel-Lizorkin spaces for the kernel of operator T satisfying only half smoothness and moment conditions. These results can be stated as follows.

Theorem 1

Let $0 < \epsilon \le \theta$, $-\epsilon < \alpha < 0$. Suppose that $T^*(b) = 0, M_bTM_b \in WBP(X)$, and $K(x,y)$, the kernel of T, satisfies (1.3) and (1.5), then T is bounded from $b\dot{F}_p^{\alpha,q}(X)$ to $b^{-1}\dot{F}_p^{\alpha,q}(X)$ for $\frac{d}{d+\alpha+\epsilon} < p < \infty, \frac{d}{d+\alpha+\epsilon} < q \le \infty$.

Theorem 2

Let $0 < \epsilon \le \theta$, $0 < \alpha < \epsilon$. Suppose that $T(b) = 0, M_bTM_b \in WBP(X)$, and $K(x,y)$, the kernel of T, satisfies (1.3) and (1.4), then T is bounded from $b\dot{F}_p^{\alpha,q}(X)$ to $b^{-1}\dot{F}_p^{\alpha,q}(X)$ for $\frac{d}{d+\alpha} < p < \infty, \frac{d}{d+\alpha} < q \le \infty$.

Before state the second main result, we recall the following construction given by Christ in [1], which plays a key role for the development of the theory of functions on spaces of homogeneous type.

Lemma 1.6

Let X be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some (possible finite) index set, and constants $\delta \in (0,1)$ and $C_1, C_2 > 0$ such that

(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ *for each fixed* k *and* $Q_\alpha^k \cap Q_\beta^k = \emptyset$ *if* $\alpha \neq \beta$;

(ii) *for any* α, β, k, l *with* $l \geq k$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \emptyset$;

(iii) *for each* (k, α) *and each* $l < k$ *there is a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$;

(iv) $diam(Q_\alpha^k) \leq C_1 \delta^k$;

(v) *each* Q_α^k *contains some ball* $B(z_\alpha^k, C_2 \delta^k)$, *where* $z_\alpha^k \in X$.

In fact, we can think of Q_α^k as being a dyadic cube with a diameter roughly δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$. See [10] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}, \tau \in I_k$, we will denote by $Q_\tau^{k,\nu}, \nu = 1, \cdots, N(k, \tau, M)$, the set of all cubes $Q_{\tau'}^{k+M} \subset Q_\tau^k$, where M is a fixed large positive integer. Let $m_{Q_\tau^{k,\nu}}(E_k(f))$ be averages of $E_k(f)$ over $Q_\tau^{k,\nu}$.

As an application of the Tb theorem, new characterizations of Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}(X)$ with $0 \neq |\alpha| < \epsilon$,

$$\max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < p < \infty \quad \text{and} \quad \max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+\epsilon+\alpha}\right\} < q \leq \infty$$

with minimum regularity and cancellation conditions are given over spaces of homogeneous type.

Theorem 3

A sequence $\{S_k(x,y)\}_{k\in\mathbb{Z}}^b$ of functions from $X \times X$ into $\mathbb{C}$ satisfies (1.6), (1.8) and (1.10) of Definition 1.3 above and $E_k = S_k - S_{k-1}$. For $f \in \widetilde{\mathcal{M}}_b(\beta, \gamma)$ with β, γ satisfying (1.15). If $-\epsilon < \alpha < 0$, $\frac{d}{d+\alpha+\epsilon} < p < \infty$, $\frac{d}{d+\alpha+\epsilon} < q \leq \infty$, then

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(\tau,k,M)} [2^{k\alpha} m_{Q_\tau^{k,\nu}}(|E_k(f)|) \chi_{Q_\tau^{k,\nu}}]^q \right\}^{1/q} \right\|_{L^p(X)} \sim \|f\|_{b\dot{F}_p^{\alpha,q}(X)}; \tag{1.16}$$

For $f \in b\widetilde{\mathcal{M}}_b(\beta, \gamma)$ with β, γ satisfying (1.15). If $-\epsilon < \alpha < 0$, $\frac{d}{d+\alpha+\epsilon} < p < \infty$, $\frac{d}{d+\alpha+\epsilon} < q \leq \infty$, then

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(\tau,k,M)} [2^{k\alpha} m_{Q_\tau^{k,\nu}}(|E_k(bf)|) \chi_{Q_\tau^{k,\nu}}]^q \right\}^{1/q} \right\|_{L^p(X)} \sim \|f\|_{b^{-1}\dot{F}_p^{\alpha,q}(X)} \tag{1.17}$$

Theorem 4

A sequence $\{S_k(x,y)\}_{k\in\mathbb{Z}}^b$ of functions from $X \times X$ into $\mathbb{C}$ satisfies (1.6), (1.7) and (1.11) of Definition 1.3 above and $E_k = S_k - S_{k-1}$. For $f \in \widetilde{\mathcal{M}}_b(\beta, \gamma)$ with β, γ satisfying (1.15). If $0 < \alpha < \epsilon$, $\frac{d}{d+\alpha} < p < \infty$, $\frac{d}{d+\alpha} < q \leq \infty$, then

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} [2^{k\alpha} |E_k(f)|]^q \right\}^{1/q} \right\|_{L^p(X)} \sim \|f\|_{b\dot{F}_p^{\alpha,q}(X)}; \tag{1.18}$$

For $f \in b\widetilde{\mathcal{M}}_b(\beta,\gamma)$ *with* β,γ *satisfying (1.15). If* $0<\alpha<\epsilon$, $\frac{d}{d+\alpha}<p<\infty$, $\frac{d}{d+\alpha}<q\le\infty$, *then*

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty}[2^{k\alpha}|E_k(bf)|]^q\Big\}^{1/q}\Big\|_{L^p(X)}\sim\|f\|_{b^{-1}\dot{F}_p^{\alpha,q}(X)}. \tag{1.19}$$

Theorem 3 and 4 were proved in [10] for the Triebel-Lizorkin space when $0<|\alpha|<\epsilon, 1<p,q<\infty, b=1$. The second purpose of this paper is to give a uniform treatment. To be precise, to deal with the case where $0<\alpha<\epsilon$ and $p,q>1$, the main tools used in [10] were the continuous Calderón reproducing formula and the $T1$ theorem. The proof of the case where $-\epsilon<\alpha<0$, and $p,q>1$ then follows from the duality argument. However, the continuous Calderón reproducing formula and duality argument do not work for the cases where either p or q, or both p and q are less than or equal to 1. The key feature of Theorem 3 and 4 is to use the discrete Calderón reproducing formula and the Plancherel-Pôlya characterization of Triebel-Lizorkin spaces $b\dot{F}_p^{\alpha,q}(X)$ and $b^{-1}\dot{F}_p^{\alpha,q}(X)$ with $|\alpha|<\epsilon$,

$$\max\Big\{\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+\alpha}\Big\}<p<\infty \quad\text{and}\quad \max\Big\{\frac{d}{d+\epsilon},\frac{d}{d+\epsilon+\alpha}\Big\}<q\le\infty$$

developed in [5, 9], and the Tb theorems are established above. Moreover, we will provide variants of the discrete Calderón-type reproducing formula. See $\dot{T}^b_{N,M}$ below. It is an interesting question that if the left hand sides of expressions in (1.16) and (1.17) can be replaced by the left hand side of expressions in (1.18) and (1.19).

A brief description of the contents of this paper as follows. The proof of Theorem 1 and Theorem 2 will be finished in Section 2. In Section 3 we give an application of the Tb theorem. The proof of Theorem 3 is given in Section 4, Theorem 4 can be proved in similar way.

2. Tb Theorem

We recall the discrete Calderón reproducing formulae in [9], which will play a crucial role, it can be stated as follows.

Lemma 2.1

Let $\{S_k\}^b_{k\in\mathbb{Z}}$ *be an approximation to the identity associated to a para-accretive function* b *as in Definition 1.3 and let* $D_k=S_k-S_{k-1}$ *for* $k\in\mathbb{Z}$. *Then there exists a family of functions* $\{\widetilde{D}_k(x,y)\}_{k\in\mathbb{Z}}$ *such that for any fixed* $y_\tau^{k,\nu}\in Q_\tau^{k,\nu}, k\in\mathbb{Z},\tau\in I_k$ *and* $\nu\in\{1,\dots,N(k,\tau,M)\}$ *and all* $f\in\mathcal{M}_b(\beta,\gamma)$ *with* $0<\beta,\gamma<\theta$,

$$f(x)=\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau,M)}D_k(f)(y_\tau^{k,\nu})\int_{Q_\tau^{k,\nu}}b(x)\widetilde{D}_k(x,y)b(y)d\mu(y)$$

where $diam(Q_\tau^{k,\nu})\sim2^{k+M}$ *for* $k\in\mathbb{Z},\tau\in I_k,\nu\in\{1,\dots,N(k,\tau,M)\}$ *and a fixed large* $M\in\mathbb{N}$, *the series converges in the sense that for* $g\in(\mathcal{M}_b(\beta',\gamma'))'$ *with* $\beta'<\beta$ *and* $\gamma'<\gamma$,

$$\lim_{M,N\to\infty}\left\langle \sum_{|k|\le M}\sum_{\tau\in I_k,\rho(x_0,z_\tau^k)\le N}\sum_{\nu=1}^{N(k,\tau,M)} D_k(f)(y_\tau^{k,\nu})\int_{Q_\tau^{k,\nu}} b(\cdot)\widetilde{D}_k(\cdot,y)b(y)d\mu(y), g\right\rangle = \langle f,g\rangle.$$

Moreover, $\widetilde{D}_k(x,y), k\in\mathbb{Z}$, satisfy the following estimates: for $\epsilon', 0<\epsilon'<\epsilon$, where ϵ is the regularity exponent of S_k, there exists a constant $C>0$ such that

$$|\widetilde{D}_k(x,y)|\le C\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{d+\epsilon'}}; \tag{2.1}$$

$$|\widetilde{D}_k(x,y)-\widetilde{D}_k(x,y')|\le C\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon'}\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{d+\epsilon'}} \tag{2.2}$$

for $\rho(y,y')|\le\frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int_X \widetilde{D}_k(x,y)b(y)d\mu(y)=\int \widetilde{D}_k(x,y)b(x)d\mu(x)=0 \tag{2.3}$$

for all $k\in\mathbb{Z}$.

We need the following Lemmas, whose proofs are similar to [9, 10].

Lemma 2.2

With the same notation as in Theorem 1. For $k,k'\in\mathbb{Z}$, $x,y\in X$, then

$$|D_kM_bTM_b\widetilde{D}_{k'}(x,y)|\le C[1+|k-k'|](2^{(k-k')\epsilon'}\wedge 1)\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}}.$$

Lemma 2.3

With the same notation as in Theorem 2. For $k,k'\in\mathbb{Z}$, $x,y\in X$, then

$$|D_kM_bTM_b\widetilde{D}_{k'}(x,y)|\le C[1+|k-k'|](2^{(k'-k)\epsilon'}\wedge 1)\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}}.$$

Proof of Theorem 1. By Lemma 2.1 and Theorem 2.2 in [5], for $f\in\widetilde{\mathcal{M}}_b(\beta,\gamma)$ we have

$$\begin{aligned}&\|T(f)\|_{b^{-1}\dot{F}_p^{\alpha,q}(X)}\\ &\le C\Big\|\Big\{\sum_{l=1}^{\infty}\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\Big[\inf_{z\in Q_\tau^{l,\nu}}\Big|\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_\tau^{l,\nu})^{-\alpha/d}\\ &\quad\times\int_{Q_{\tau'}^{k',\nu'}} D_lM_bTM_b\widetilde{D}_{k'}(z,x)d\mu(x)M_bD_{k'}(f)(y_{\tau'}^{k',\nu'})\Big|\chi_{Q_\tau^{l,\nu}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}\end{aligned}$$

where $y_{\tau'}^{k',\nu'}$ are any point in $Q_{\tau'}^{k',\nu'}$.

From Lemma A.2 in [7] and the Fefferman-Stein vector-valued inequality in [6], it follows that

$$\begin{aligned}
&\|T(f)\|_{b^{-1}\dot{F}_p^{\alpha,q}(X)}\\
&\le C\Big\|\Big\{\sum_{l=1}^{\infty}\Big[\sum_{k'=1}^{\infty}2^{(l-k')\alpha}2^{-k'd}[1+|k'-l|]\big(2^{(l-k')\epsilon'}\wedge 1\big)2^{[k'-(l\wedge k')]d/r}2^{(l\wedge k')d}\\
&\quad\times\Big[\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|M_bD_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r\Big]^{1/r}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\le C\Big\|\Big\{\sum_{l=1}^{\infty}\sum_{k'=1}^{\infty}\Big[2^{(l-k')\alpha}2^{-k'd}[1+|k'-l|]\big(2^{(l-k')\epsilon'}\wedge 1\big)2^{[k'-(l\wedge k')]d/r}2^{(l\wedge k')d}\Big]^{q\wedge 1}\\
&\quad\times\Big[\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|M_bD_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r\Big]^{q/r}\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\le C\Big\|\Big\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\Big[\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\le C\|f\|_{b\dot{F}_p^{\alpha,q}(X)},
\end{aligned}$$

where the second and the third inequalities follow from the facts

$$\sup_l\sum_{k'=1}^{\infty}2^{(l-k')\alpha}2^{-k'd}[1+|k'-l|]\big(2^{(l-k')\epsilon'}\wedge 1\big)2^{[k'-(l\wedge k')]d/r}2^{(l\wedge k')d}<\infty$$

and

$$\sup_{k'}\sum_{l=1}^{\infty}\Big[2^{(l-k')\alpha}2^{-k'd}[1+|k'-l|]\big(2^{(l-k')\epsilon'}\wedge 1\big)2^{[k'-(l\wedge k')]d/r}2^{(l\wedge k')d}\Big]^{q\wedge 1}\le C$$

for $-\epsilon'<\alpha<0$, $\min(1,p,q)>r>\frac{d}{d+\epsilon'+\alpha}$.

This shows Theorem 1.

The main difference of proof between Theorem 2 and Theorem 1 is that Lemma 2.2 should be replaced by Lemma 2.3. We omit these details.

3. An application of Tb theorem

In this section we give some lemmas as an application of Tb theorems which are needed for the proofs of Theorem 3 and 4.

Lemma 3.1

With the notation as in Theorem 3. Then

$$|D_lM_bE_jM_bE_kM_b\widetilde{D}_{k'}(x,y)|\le C\big(2^{(l-j)\epsilon'}\wedge 1\big)\big(2^{(k-k')\epsilon'}\wedge 1\big)\frac{2^{-(l\wedge j\wedge k\wedge k')\epsilon'}}{\big(2^{-(l\wedge j\wedge k\wedge k')}+\rho(x,y)\big)^{d+\epsilon'}}$$

for any $\epsilon', 0<\epsilon'<\epsilon$, $l,j,k,k'\in\mathbb{Z}$.

Proof. Lemma 3.1 is an easy consequence of the following estimates.

$$|E_k M_b \widetilde{D}_{k'}(x,y)| \le C(2^{(k-k')\epsilon'} \wedge 1)\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}},$$

$$|D_l M_b E_j(x,y)| \le C(2^{(l-j)\epsilon'} \wedge 1)\frac{2^{-(l\wedge j)\epsilon'}}{(2^{-(l\wedge j)}+\rho(x,y))^{d+\epsilon'}}.$$

Lemma 3.2

With the notation as in Lemma 2.1 and Theorem 3. Let $E_k^N = \sum_{j:|j-k|\le N} E_j$ *for* $j \in \mathbb{Z}$, *and define*

$$\dot{R}^b_{N,M}(f)(x) = \sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)} \times \int_{Q_\tau^{k,\nu}} M_b\left[E_k^N(x,y) - E_k^N(x,y_\tau^{k,\nu})\right](bE_k)(f)(y)d\mu(y),$$

where $y_\tau^{k,\nu}$ *is any point in* $Q_\tau^{k,\nu}$. *Suppose* $0<\epsilon\le\theta, M\in\mathbb{N}$, *is large enough, then there exist* $C>0$ *and* $\delta_1>0$ *independent of* M, N *and* f *such that*

$$\|\dot{R}^b_{N,M} f\|_{b\dot{F}^{\alpha,q}_p(X)} \le C2^{-M\delta_1}\|f\|_{b\dot{F}^{\alpha,q}_p(X)}$$

for $-\epsilon<\alpha<0$, $\frac{d}{d+\alpha+\epsilon}<p<\infty$, $\frac{d}{d+\alpha+\epsilon}<q\le\infty$.

Proof. Let $T = 2^{M\delta_1}M_{b^{-1}}\dot{R}^b_{N,M}$ and $K(x,y)$ is its kernel for some $\delta_1>0$. Similar to Lemma 5.24 in [10]. We can prove that for $0<\epsilon<\theta$ and some fixed N, there is a constant $C>0$ independent of M such that $K(x,y)$ satisfies (1.3) and (1.5), and $T^*(b)=0$, $T\in WBP$. We omit these details.

With this done, by Theorem 1, we know that T is bounded from $b\dot{F}^{\alpha,q}_p(X)$ to $b^{-1}\dot{F}^{\alpha,q}_p(X)$ for $-\epsilon<\alpha<0$, $\frac{d}{d+\alpha+\epsilon}<p<\infty, \frac{d}{d+\alpha+\epsilon}<q\le\infty$. Note that M_b is bounded from $b^{-1}\dot{F}^{\alpha,q}_p$ to $b\dot{F}^{\alpha,q}_p$ with an operator norm to be 1. This finishes the proof of this lemma.

Lemma 3.3

With the notation as in Theorem 3. $\dot{R}^b_N = \sum_{k=-\infty}^{\infty}\sum_{j:|j-k|>N} M_bE_jM_bE_k$ *for* $j\in\mathbb{Z}$. *Suppose* $0<\epsilon\le\theta, -\epsilon<\alpha<0, N\in\mathbb{N}$, N *is large enough. Then there exist* $C>0$ *and* $\delta_2>0$ *independent of* N *and* f *such that*

$$\|\dot{R}^b_N(f)\|_{b\dot{F}^{\alpha,q}_p(X)} \le C2^{-N\delta_2}\|f\|_{b\dot{F}^{\alpha,q}_p(X)} \tag{3.1}$$

for $\frac{d}{d+\alpha+\epsilon}<p<\infty$, $\frac{d}{d+\alpha+\epsilon}<q\le\infty$.

Proof. We can write

$$\dot{R}^b_N = \sum_{k=N}^{\infty}\sum_{j=-\infty}^{k-N} M_bE_jM_bE_k + \sum_{k=-\infty}^{\infty}\sum_{j=k+N}^{\infty} M_bE_jM_bE_k \doteq \dot{R}^{b,I}_N + \dot{R}^{b,II}_N.$$

Similar to the Lemma 3.2, using Theorem 1, there exist $C > 0$ and $\delta_2' > 0$ independent of N and f such that

$$\|\dot{R}_N^{b,I}(f)\|_{b\dot{F}_p^{\alpha,q}(X)} \le C2^{-N\delta_2'}\|f\|_{b\dot{F}_p^{\alpha,q}(X)}$$

for $-\epsilon < \alpha < 0$, $\frac{d}{d+\alpha+\epsilon} < p < \infty$, $\frac{d}{d+\alpha+\epsilon} < q \le \infty$.

To prove the inequalities (3.1), we only need to show: if $-\epsilon < \alpha < 0$, then there exist $C > 0$ and $\delta_2'' > 0$ independent of N and f such that

$$\|\dot{R}_N^{b,II}(f)\|_{b\dot{F}_p^{\alpha,q}(X)} \le C2^{-N\delta_2''}\|f\|_{b\dot{F}_p^{\alpha,q}(X)} \tag{3.2}$$

for $\frac{d}{d+\alpha+\epsilon} < p < \infty$, $\frac{d}{d+\alpha+\epsilon} < q \le \infty$.

Using the Lemma 2.1 and Theorem 2.2 in [5], for $f \in \widetilde{\mathcal{M}}_b(\beta,\gamma)$ we have

$$\begin{aligned}
&\|\dot{R}_N^{b,II}(f)\|_{b\dot{F}_p^{\alpha,q}(X)}\\
&\le C\Big\|\Big\{\sum_{l=-\infty}^{\infty}\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\Big[\inf_{z\in Q_\tau^{l,\nu}}\Big|\sum_{k=-\infty}^{\infty}\sum_{j=k+N}^{\infty}\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_\tau^{l,\nu})^{-\alpha/d}\\
&\quad\times\int_{Q_{\tau'}^{k',\nu'}} D_lM_bE_jM_bE_kM_b\widetilde{D}_{k'}M_b(z,y)d\mu(y)D_{k'}(f)(y_{\tau'}^{k',\nu'})\Big|\chi_{Q_\tau^{l,\nu}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)},
\end{aligned}$$

where $y_{\tau'}^{k',\nu'}$ are any point in $Q_{\tau'}^{k',\nu'}$.

By the Fefferman-Stein vector-valued maximal function inequality in [6], we obtain

$$\begin{aligned}
&\|\dot{R}_N^{b,II}(f)\|_{b\dot{F}_p^{\alpha,q}(X)}\\
&\le C\Big\|\Big\{\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{j=k+N}^{\infty}\sum_{k'=-\infty}^{\infty}\Big[2^{(l-k')\alpha}2^{-k'd}\big(2^{(l-j)\epsilon'}\wedge 1\big)\big(2^{(k-k')\epsilon'}\wedge 1\big)\\
&\quad\times 2^{(l\wedge j\wedge k\wedge k')d}2^{[k'-(l\wedge j\wedge k\wedge k')]d/r}\Big]^{q\wedge 1}\\
&\quad\times\Big[\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r\Big]^{q/r}\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\le C2^{N\alpha((q\wedge 1)/q)}\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\\
&\quad\times\Big[\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}\\
&\le C2^{N\alpha((q\wedge 1)/q)}\|f\|_{b\dot{F}_p^{\alpha,q}(X)}.
\end{aligned}$$

where we used the fact that if $-\epsilon' < \alpha < 0$, $\min(1,p,q) > r > \frac{d}{d+\epsilon'+\alpha}$, then

$$\begin{aligned}
&\sup_{l\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\sum_{j=k+N}^{\infty}\sum_{k'=-\infty}^{\infty}\Big[2^{(l-k')\alpha}2^{-k'd}\big(2^{(l-j)\epsilon'}\wedge 1\big)\big(2^{(k-k')\epsilon'}\wedge 1\big)\\
&\quad\times 2^{(l\wedge j\wedge k\wedge k')d}2^{[k'-(l\wedge j\wedge k\wedge k')]d/r}\Big] < \infty,
\end{aligned}$$

$$\Big\{\sup_{k'\in\mathbb{Z}}\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{j=k+N}^{\infty}\Big[2^{(l-k')\alpha}2^{-k'd}\big(2^{(l-j)\epsilon'}\wedge 1\big)\big(2^{(k-k')\epsilon'}\wedge 1\big)$$
$$\times\ 2^{(l\wedge j\wedge k\wedge k')d}2^{[k'-(l\wedge j\wedge k\wedge k')]d/r}\Big]^{q\wedge 1}\Big\}^{1/q}\le C2^{N\alpha((q\wedge 1)/q)}.$$

The Proof of (3.2) is completed with $\delta_2'' = -\alpha(\frac{q\wedge 1}{q})$.

Combining estimates of $\dot{R}_N^{b,I}(f)$ and $\dot{R}_N^{b,II}(f)$, taking $\delta_2 = \min\{\delta_2', \delta_2''\}$, this shows (3.1). The proof of Lemma 3.3 is finished.

4. Proof of Theorem 3

First we give the proof of the upper bound in (1.16), namely

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}[2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_\tau^{k,\nu}}]^q\Big\}^{1/q}\Big\|_{L^p(X)}\le C\|f\|_{b\dot{F}_p^{\alpha,q}(X)} \quad (4.1)$$

for $-\epsilon<\alpha<0$ and $\frac{d}{d+\alpha+\epsilon}<p<\infty$, $\frac{d}{d+\alpha+\epsilon}<q\le\infty$.

To verify the (4.1), for $f\in\widetilde{\mathcal{M}}_b(\beta,\gamma)$ with β,γ satisfying (1.15), by Lemma 2.1, we deduce

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}[2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_\tau^{k,\nu}}]^q\Big\}^{1/q}\Big\|_{L^p(X)}$$
$$\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\Big[\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}2^{k\alpha}$$
$$\times\ m_{Q_\tau^{k,\nu}}\Big(\Big|\int_{Q_{\tau'}^{k',\nu'}}E_k(b\widetilde{D}_{k'}(y,\cdot)(\bullet)b(y)d\mu(y)\Big|\Big)|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_\tau^{k,\nu}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}$$
$$\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{k'=-\infty}^{\infty}2^{-k'd}2^{(k-k')\alpha}\big(2^{(k-k')\epsilon'}\wedge 1\big)2^{(k\wedge k')d}2^{[k'-(k\wedge k')]d/r}$$
$$\times\ \Big\{\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})\chi_{Q_{\tau'}^{k',\nu'}}|\Big)^r\Big\}^{1/r}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)}.$$

By the Hölder inequality for $q>1$ and $(a+b)^q\le a^q+b^q$ for $q\le 1$, the Fefferman-Stein vector-valued maximal function inequality in [6] and Theorem 2.2 in [5], we obtain

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}[2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_\tau^{k,\nu}}]^q\Big\}^{1/q}\Big\|_{L^p(X)}$$
$$\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}[2^{(k-k')\alpha}\big(2^{(k-k')\epsilon'}\wedge 1\big)2^{-k'd}2^{(k\wedge k')d}2^{[k'-(k\wedge k')]d/r}]^{q\wedge 1}$$

$$\Big[\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})\chi_{Q_{\tau'}^{k',\nu'}}|\Big)^r\Big]^{q/r}\Big\}^{1/q}\Big\|_{L^p(X)}$$

$$\le C\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\Big[\boldsymbol{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})\chi_{Q_{\tau'}^{k',\nu'}}|\Big)^r\Big]^{q/r}\Big\}^{1/q}\Big\|_{L^p(X)}$$

$$\le C\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}[\mu(Q_{\tau'}^{k',\nu'})^{-\alpha/d}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}]^q\Big\}^{1/q}\Big\|_{L^p(X)}$$

$$\le C\|f\|_{b\dot{F}_p^{\alpha,q}(X)}$$

where $\frac{d}{d+\alpha}<r<\min(1,p,q)$ and we used

$$\sup_{k\in\mathbb{Z}}\sum_{k'=-\infty}^{\infty}2^{(k-k')\alpha}(2^{(k-k')\epsilon'}\wedge 1)2^{-k'd}2^{(k\wedge k')d}2^{[k'-(k\wedge k')]d/r}\le C$$

and

$$\sup_{k'\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}\Big[2^{(k-k')\alpha}(2^{(k-k')\epsilon'}\wedge 1)2^{-k'd}2^{(k\wedge k')d}2^{[k'-(k\wedge k')]d/r}\Big]^{q\wedge 1}\le C.$$

This shows that (4.1) is true.

The other half proof of (1.16) will use a new decomposition with identity operator. More precisely, the identity operator $I=\dot{R}_N^b+\dot{T}_N^b$, where the kernel of $\dot{R}_N^b$ is

$$\dot{R}_N^b(x,y)=\sum_{k=-\infty}^{\infty}\sum_{j:|j-k|>N}\int_X b(x)E_j(x,z)b(z)E_k(z,y)d\mu(z).$$

Furthermore, for fixed large $N\in\mathbb{N}$ let $E_k^N=\sum_{j:|j-k|\le N}E_j$ and $\dot{T}_N^b=\dot{R}_{N,M}^b+\dot{T}_{N,M}^b$ where

$$\begin{aligned}\dot{R}_{N,M}^b(f)(x) &= \sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\int_{Q_\tau^{k,\nu}}\int_X b(x)\left[E_k^N(x,z)-E_k^N(x,y_\tau^{k,\nu})\right]\\ &\quad\times b(z)E_k(z,y)f(y)d\mu(z)d\mu(y).\end{aligned}$$

Using Lemmas 3.2 and 3.3, choosing N and M are large enough, such that $C2^{-N\delta_1}+C2^{-M\delta_2}<1$, we show that $\lim_{N\to\infty}\lim_{M\to\infty}\dot{T}_{N,M}=I$ in the norm of $b\dot{F}_p^{\alpha,q}(X)$ for the range of α,p and q indicated in (1.16) of Theorem 3. Moreover, $\dot{T}_{N,M}^{-1}$ exists and is bounded on $b\dot{F}_p^{\alpha,q}(X)$ for the same range of α,p and q. Therefore, to finish the proof of Theorem 3, we only need to verify following claims. For $-\epsilon<\alpha<0$ and some fixed large enough N,M, if $\frac{d}{d+\alpha+\epsilon}<p<\infty$, $\frac{d}{d+\alpha+\epsilon}<q\le\infty$, then

$$\|\dot{T}_{N,M}(f)\|_{b\dot{F}_p^{\alpha,q}(X)}\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}[2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_\tau^{k,\nu}}]^q\Big\}^{1/q}\Big\|_{L^p(X)}. \tag{4.2}$$

From the Lemma 2.2, the Hölder inequality for $q > 1$ and $(a+b)^q \le a^q + b^q$ for $q \le 1$, and the Fefferman-Stein vector-valued maximal function inequality in [6], it follows that

$$\begin{aligned}
&\|\dot{T}_{N,M}(f)\|_{b\dot{F}_p^{\alpha,q}(X)} \\
&\le C\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(\tau',k',M)}\Big[\sum_{k=-\infty}^{\infty}(2^{(k'-k)\epsilon'}\wedge 1)\inf_{x\in Q_{\tau'}^{k',\nu'}}\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y_\tau^{k,\nu}))^{d+\epsilon'}} \\
&\quad\times 2^{(k'-k)\alpha}2^{-kd}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_{\tau'}^{k',\nu'}}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)} \\
&\le C\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\Big[\sum_{k=-\infty}^{\infty}(2^{(k'-k)\epsilon'}\wedge 1)2^{(k'-k)\alpha}2^{-kd}2^{(k\wedge k')d}2^{[k-(k\wedge k')]d/r} \\
&\quad\times\Big(\boldsymbol{M}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\mu(Q_\tau^{k,\nu})^{-\alpha r/d}\big[m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\big]^r\chi_{Q_\tau^{k,\nu}}\Big]\Big)^{1/r}\Big]^q\Big\}^{1/q}\Big\|_{L^p(X)} \\
&\le C\Big\|\Big\{\sum_{k'=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\big[(2^{(k'-k)\epsilon'}\wedge 1)2^{(k'-k)\alpha}2^{-kd}2^{(k\wedge k')d}2^{[k-(k\wedge k')]d/r}\big]^{q\wedge 1} \\
&\quad\times\Big(\boldsymbol{M}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\mu(Q_\tau^{k,\nu})^{-\alpha r/d}\big[m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\big]^r\chi_{Q_\tau^{k,\nu}}\Big]\Big)^{q/r}\Big\}^{1/q}\Big\|_{L^p(X)} \\
&\le C\Big\|\Big\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)}\big[2^{k\alpha}m_{Q_\tau^{k,\nu}}(|E_k(f)(y)|)\chi_{Q_\tau^{k,\nu}}\big]^q\Big\}^{1/q}\Big\|_{L^p(X)},
\end{aligned}$$

where $\frac{d}{d+\alpha+\epsilon'} < r < \min(1,p,q)$ and we used

$$\sup_{k'\in\mathbb{Z}}\sum_{k=-\infty}^{\infty}(2^{(k'-k)\epsilon'}\wedge 1)2^{(k'-k)\alpha}2^{-kd}2^{(k\wedge k')d}2^{[k-(k\wedge k')]d/r} \le C$$

in third inequality and

$$\sup_{k\in\mathbb{Z}}\sum_{k'=-\infty}^{\infty}\Big[(2^{(k'-k)\epsilon'}\wedge 1)2^{(k'-k)\alpha}2^{-kd}2^{(k\wedge k')d}2^{[k-(k\wedge k')]d/r}\Big]^{q\wedge 1} \le C$$

in fourth inequality, the arbitrariness of $y_\tau^{k,\nu}$ in penultimate inequality. Therefore, (4.2) is true. This finishes the proof of (1.16) in Theorem 3.

The proof of (1.17) is similar to (1.16), we omit these details. This finishes the proof of Theorem 3.

As mentioned above, the proof of Theorem 4 cannot be obtained by the duality argument which was used in [10]. The idea of the proof, however, is similar to Theorem 3 with necessary modifications. We leave the details to the reader.

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