A CONSTRUCTIVE THEORY FOR THE EQUATIONS OF FLOWS WITH FREE BOUNDARIES (*)

BY

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INTRODUCTION

The theory of flows with free boundaries has been developed mainly in a non constructive way. The existence of flows is known in a large number of cases, but so far there is no sound method to compute them.

The first general proof of existence and uniqueness is due to A. WEINSTEIN [1], [2], [3], [4], [5], (**) who extended the method of continuity of Conformal Mapping to the determination of the flow of a plane jet issuing from a concave canal (See also G. HAMEL [7], [8], H. WEYL [9], K. FRIEDRICHS [10], J. LERAY and A. WEINSTEIN [11]). The idea consists in approaching the walls of the canal by polygonal walls, solving the problem for these, and proceeding to the limit. Unfortunately the existence of flows with polygonal walls is not attained in a constructive way and the method cannot be made the basis for effective computations.

The most general results were obtained by the french school headed by J. LERAY [12], [13], (J. KRAVTCHENKO [15], [16], A. OUDART [17], R. HURON [18], P. THERON [19]). However, these results are not intended to be constructive, and it is extremely doubtful whether they can be made so. A systematic use is made of Brouwer's notion of topological index of a continuous transformations as extended by J. SCHAUDER and J. LERAY [20] to Banach spaces, a powerful tool indeed but a non-constructive one.

The variational approach, first formulated in 1927 by D. RIABOUCHINSKY [21], which lately has lead to new proofs of existence and uni-

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queness for plane flows (See K. FRIEDRICHS [10], P. R. GARABEDIAN and D. C. SPENCER [23], P. R. GARABEDIAN and H. ROYDEN [23]) and to the first proof of existence for axially symmetric flows (P. R. GARABEDIAN, H. LEWY and M. SCHIFFER [24]), suggests, in principle, a way to compute a solution by succesive approximations, but it does not seem to be ready to support any computational scheme. The same should be said with regard to the variational theory of M. LAVRENTIEFF [25] (See also J. SERRIN [26], [27] and D. GILBARG [28]).

On the constructive side the results are by far more modest and restricted. N. NEKRASSOFF [29] seems to have been the first to prove the existence of the wake of a curved obstacle given before hand. He works with circular arcs and puts the problem into the form of a non linear integral equation (of the type later considered by A. HAMERSTEIN [30], see also A. QUARLERI [31], and A. WEINSTEIN [6]) which he tries to solve by iterations. These are shown to converge for arcs of a rather small angular extent. Nekrassoff method was later extended, with the same limitations as to the convergence, to more general obstacles by his disciples (N. ARJANNIKOFF [32], P. MIASNIKOFF and S. KALININE [33], J. SEKERJ - ZENKOWITCH [34]).

In 1934 C. SCHMIEDEN [35] tried a similar iteration process working on the Fourier coefficients of the unknown function, and made some computations (see also M. KOLSHER [36]) but failed to prove the convergence of his method.

Remarkably good approximate solutions for circular and elliptic arcs have been given by S. BRODETZKY [37], [38]. Putting aside all questions of existence and convergence he simply tries to approximate the obstacles by obstacles having at a finite number of points the same curvature as the given one. Analytically this amounts to satisfy the equation of the problem at a finite number of points only. Such procedure, however, appeared to be impracticable for more than three points and was left without a rigorous foundation. (For applications see L. ROSENHEAD [39], and Y. BERGMAN [40]).

A characteristic of these last methods is that they do not solve the hydrodynamical problem proper but one of the equations of which it consists. To make this clear we recall that the determination of a flow with free boundaries is equivalent to the determination of a function $\lambda(\sigma)$ and parameters $p_1, p_2, \ldots, p_n$ from equations of the form $F(\lambda) = 0$, $f_1(\lambda) = 0$, $f_2(\lambda) = 0, \ldots, f_m(\lambda) = 0$, where F is a non linear operator (transforming functions into functions) and $f_1, f_2, \ldots, f_m$ functionals (transforming functions into numbers) depending on the parameters.

The later methods attempt to solve the first equation ignoring the rest. In such a way λ is eliminated and the problem is reduced to the solution of a finite number of trascendental equations with a finite number of unknowns, which, if the number is small (≤ 2), can be numerically effected by inverse interpolation with not too great an effort.

In behalf of this point of view it should mentioned that equation $F(\lambda) = 0$ is the only one that keeps its form, while the others change a good deal from case to case, and so, that any result about the first equation has a universal value for problems with free boundaries.

The present paper is written in the same spirit and contains a constructive theory for the equation $F(\lambda) = 0$, which in our formulation takes the form

$$\lambda = \nu \varkappa (T \lambda) e^{-D\lambda}, \tag{1}$$

where ν and $\varkappa$ are given functions and T and D linear integral operators. The solution is attained in two stages: discretization and iteration. In the first, the equation is replaced by a finite number of equations (in the sense of BRODETZKY) which are solved by iteration in the second. Proof of convergence and bounds for the number of operations are given at each stage, thus making the method sound and practical. Most of the results of this paper apply also to the conformal mapping of the unit circle onto a domain bounded by a curve given by its intrinsic equation, for, as it had been shown somewhere else (G. BIRKHOFF, D. M. YOUNG and E. H. ZARANTONELLO [41]), such a mapping obeys also an equation of type (1).

I. THE EQUATIONS

§ 1. A typical case. Parametrization of Levi Civita. In this paper we shall deal only with plane, irrotational, steady motions of a non gravitating ideal fluid. In such case the conjugate of the velocity vector is an analytic function $\zeta(z)$ of the position z. Its integral $w(z) = \int \zeta(z)\,dz$, defined up to an additive constant, is the so called complex potential: the real part is the velocity potential and the imaginary part is the stream function. Streamlines are characterized as level lines of the imaginary part of the complex potential (Cf. MILNE - THOMPSON [42], Chaps. IV, VI). A free streamline is a streamline along which the velocity remains constant in modulus. Once $\zeta = \zeta(t)$ and $w = w(t)$ are given in terms of a parameter t running in the interior of a domain, the flow is determined by a simple quadrature: $z(t) = \int (1/\zeta(t)) \, (dw/dt) \, dt$. This we call a parametric representation of a flow.

In this § we shall derive the equations for a typical flow with free boundaries. This will illustrate, without the inconveniencies of « a most general treatment », the broad significance of equation (1).

Let us consider a free jet J of speed 1 and thickness d impinging on a curved barrier P, as in Fig. 1. J is divided by the barrier into two free jets J_1 and J_2. If d_1 and d_2 be their respective thicknesses, clearly

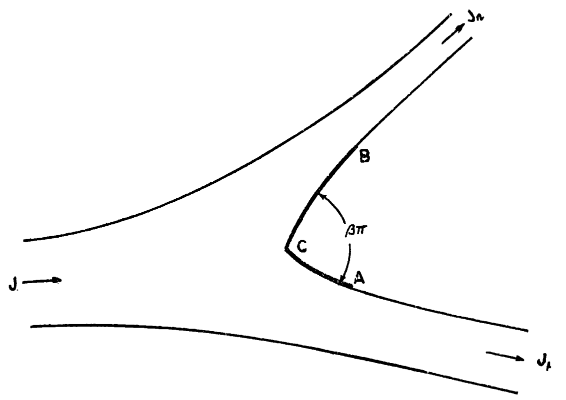

Fig. 1

by the equation of continuity, $d = d_1 + d_2$. On the obstacle there is a point C where the flow divides, it is the point of bifurcation of a streamline. We shall assume that the obstacle has a finite length and a continuously turning tangent, except perhaps for an angle of $\pi \beta$ radians at C; moreover the arcs AC and BC will be assumed to have a continuous curvature, end points inclusive, with a common value at each side of the vertex C.

Let ℓ be the arc length of the barrier measured from A in the (positive) sense leaving the interior of the flow on the left and φ the angle of the positive tangent with the impinging jet. Then, the intrinsic equation

$$\varphi = \varphi(\ell) \tag{1.2}$$

determines the barrier up to a translation (It is convenient to rotate coordinates so that the impinging jet is paralell to the positive x - axis).

Thus, according to the above hypothesis, $\varphi(t)$ is continuous except for a jump $\pi(1-\beta)$ at C, with a continuous derivative on each branche of the obstacle.

Following T. LEVI-CIVITA [43] we proceed now to represent the flow in terms of a parameter t varying on the interior of the unit semicircle $\Gamma : |t| < 1$, Im $t > 0$ (Fig. 2). By the Fundamental Theorem of Conformal Mapping there is exactly one, one-to-one transformation $z = f(t)$, mapping the domain occupied by the flow conformally onto Γ, such that the separation points A, B and the point at infinity of the oncoming jet correspond respectively to $t = 1, -1, 0$. In this way, the free streamline is mapped onto the real diameter and the barrier onto the semicircumference $t = e^{i\sigma}$, $0 \leq \sigma \leq \pi$ of Γ. The complex velocity ζ, which is analytic

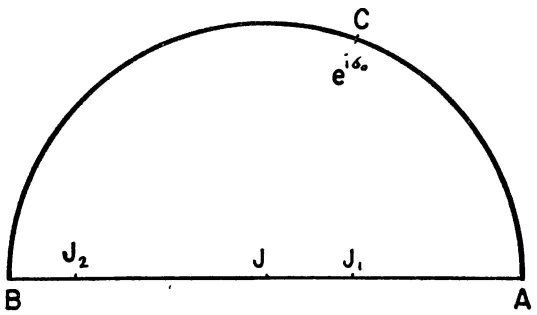

Fig. 2

and regular at interior points of the flow, becomes, through the transformation $z = f(t)$, an analytic and regular function of t on Γ. It has no zeros, except at the vertex $C\ (t_0 = e^{i\sigma_0})$. For an adequate formulation of the problem it appears convenient to represent the complex velocity in terms of a new function Ω as follows:

$$(1.3) \qquad \zeta = (tt_0^{-1} - 1)/(tt_0 - 1)^{\beta}\ e^{-i\Omega}(t), \qquad \Omega(t) = \theta(t) + i\,\tau(t).$$

According to this definition Ω is an analytic function, regular inside Γ. The first factor in (1.3) is intended to take care of the peculiar behavior of ζ at the vertex. Since we have made the jet horizontal, $\zeta_3 = 1$, and so by (1.3), $\Omega(0) = 0$. What is more important, since both ζ and $[(tt_0^{-1} - 1)/(tt_0 - 1)]^{\beta}$ have modulus one on the free boundary Im $t = 0$, $\Omega(t)$ *is real on the real diameter*. It follows, by SCHWARZ's Principle of Reflection, that $\Omega(t)$ can be extended as an analytic regular function to the unit circle.

We now pass to consider the behavior of $\Omega(t)$ along the fixed boundary $t = e^{i\sigma}$. On approaching the semicircumference, hence the barrier, we must have (if the velocity is continuous):

$$\lim_{t \to e^{i\sigma}} \arg (\zeta^{-1}) = \begin{cases} \varphi, & \pi \geq \sigma \geq \sigma_0, \\ \varphi - \pi, & \sigma_0 \geq \sigma \geq 0. \end{cases} \tag{1.4}$$

On the other hand, passing to the limit on the right of (1.3) we get:

$$\lim_{t \to e^{i\sigma}} \arg (\zeta^{-1}) = \begin{cases} \beta \tau_0 + \theta(\sigma) & \pi \geq \sigma \geq \sigma_0, \\ \beta(\sigma_0 - \pi) + \theta(\sigma) & \sigma_0 \geq \sigma \geq 0, \end{cases} \tag{1.5}$$

where for simplicity we have written $\theta(\sigma)$ for $\lim\limits_{t \to e^{i\sigma}} \theta(t)$. Comparing (1.4) and (1.5),

$$\theta(\sigma) = \varphi_1 - \frac{1}{2}\pi + \beta\left(\frac{1}{2}\pi - \sigma_0\right), \tag{1.6}$$

where

$$\varphi_1 = \begin{cases} \varphi + \frac{1}{2}\pi(1 - \beta) & \text{on } CB, \\ \varphi - \frac{1}{2}\pi(1 - \beta) & \text{on } AC. \end{cases} \tag{1.7}$$

$\varphi_1(l)$ is, as one immediately realizes, the angle of the positive tangent to the «straightened barrier» P_1, obtained by rotating AC and CB until they become perpendicular to the bisector of $< BCA$. $\varphi_1(l)$, unlike $\varphi(l)$, is continuous with continuous derivative in the whole interval of definition.

We can now prove that Ω has continuous boundary values. The arc length $l(\sigma)$ is an increasing continuous function of σ. Moreover, since the mapping $z = f(t)$ relates two domains whose boundaries have continuously turning tangents, except for a jump of $\pi(1 - \beta)$ radians, $l(\sigma)$ satisfies a Lipchitz condition of exponent arbitrarily close to minimum $(1, \beta)$(*). (For a detailed discussion see J. Kravtchenko [15], specially Chapter III). Hence, $\varphi_1(l(\sigma))$ as a composition of two Lipchitzian functions is itself Lipchitzian, and by (7) so is $\theta(\sigma)$. By Priwaloff Theorem [44], $\theta(\sigma)$ and $\tau(\sigma) = \lim\limits_{t \to e^{i\sigma}} \tau(t)$, as boundary values of conjugate harmonic functions, belong to the same Lipchitz class. Thus $\tau(\sigma)$ is also Lipchitzian. Summarizing:

(*) A function $l(\sigma)$ is said to satisfy Lipchitz condition with exponent γ in an interval (a, b) if there is a constant c such that for every couple of points σ_1 and σ_2 in (a, b), $|l(\sigma_1) - l(\sigma_2)| \leq c\,|\sigma_1 - \sigma_2|^{\gamma}$.

$\Omega(t)$ is an analytic function, regular in the unit circle $|t| < 1$, real on the real diameter, vanishing at the origing and having Lipchitzian boundary values.

We now consider the complex potential w. As a function of t, it is characterized by being an analytic function regular in the interior of Γ, having piecewise constant imaginary part on the boundary with jumps d, $-d_1$, $-d_2$ at the image points 0, t_1, t_2 of the jets J, J_1, J_2 respectively. Such a function can be immediately written as

$$w = -(d_1/\pi)\ln(T - T_1) - (d_2/\pi)\ln(T - T_2), \tag{1.8}$$

where $T = -(1/2)(t + t^{-1})$, $T_i = -(1/2)(t_i + t_i^{-1})$. At the dividing point $(T_0 = -(1/2)(t_0 + t_0^{-1}))$ the condition $dw/dT = 0$ has to be satisfied, so

$$d_1/(T_0 - T_1) + d_2/(T_0 - T_2) = 0. \tag{1.9}$$

On account of this relation and of $d = d_1 + d_2$, we get from (1,8)

$$dw/dT = -(d/\pi)(T - T_0)/((T - T_1)(T - T_2)). \tag{1.10}$$

Finally, from (1.3) and (1.10) the position z is readily computed by integrating $dz = \zeta^{-1} dw$.

In conclusion, *to every flow of the considered type corresponds a function* $\Omega(t)$ *and a set of parameters* β, d, $t_0 = e^{i\sigma_0}$, t_1, t_2 $(0 \leq \beta < 2,\ d > 0,\ 0 \leq \sigma_0 \leq \pi,\ -1 < t_1 < 0,\ 0 < t_2 < 1)$ *that describe it.*

§ 2. **The curvature equation.** This section is devoted to the determination of the conditions to impose upon the function $\Omega(t)$ and the parameters in order to obtain a flow past a given barrier. These will be reached by stating that the resulting obstacle has the proper intrinsic equation. To this purpose, it is better to introduce a new intrinsic equation, the equation $\varkappa_1 = \varkappa_1(\varphi_1)$ of the «straightened barrier», expressing the curvature as a function of the direction of the tangent. It is a single valued function only if the barrier has no inflexion.

The arc-length along the obstacle is obtained by setting $t = e^{i\sigma}$ and taking modulus in both sides of $dz = \zeta^{-1} dw$. On account of (1.3) and (1.10) we get (*)

$$dl = |dz| = \nu(\sigma)\, e^{-\tau(\sigma)}\, d\sigma, \tag{1.11}$$

(*) The existence of the derivative $dl(\sigma)/d\sigma$ and its continuity follows from the differentiability of $\varphi_1(l)$. Cf. J. Kravtchenko [15], pag. 163.

where

$$\nu(\sigma) = \frac{d}{\pi}\left|\frac{\sin(1/2)(\sigma+\sigma_0)}{\sin(1/2)(\sigma-\sigma_0)}\right|^{\beta} \frac{|\cos\sigma - \cos\sigma_0|}{|\cos\sigma - T_1)(\cos\sigma - T_2)|}\sin\sigma. \tag{1.12}$$

We can now compute the curvature of the «straightened barrier». By definition and (1.7)

$$\varkappa_1 = d\varphi_1/dl = d\theta/dl = \frac{\theta'(\sigma)}{\nu(\sigma)e^{-\tau(\sigma)}}. \tag{1.13}$$

Eliminating φ_1 with the aid of (1.7), this equation can be written in the form

$$\lambda(\sigma) = \nu(\sigma)\,\varkappa(\theta(\sigma))\,e^{-\tau(\sigma)}, \tag{1.14}$$

where

$$\lambda(\sigma) = -\theta'(\sigma), \tag{1.15}$$

$$\varkappa(\theta) = -\varkappa_1\left(\theta + \frac{\pi}{2} - \beta\left(\frac{\pi}{2} - \sigma_0\right)\right). \tag{1.16}$$

$\varkappa(\theta)$ can be considered as the curvature of P_1, *qua* function of the tangential direction measured from a new angular origin. It is positive if the obstacle is convex towards the fluid and negative if it is concave.

$\theta(\sigma)$ and $\tau(\sigma)$ can be eliminated from (1.14) by expressing them in terms of λ. For $\theta(\sigma)$ such an expression is immediately derived from (1.15). By taking in account the fact $\theta(\sigma)$ is an even function with zero mean value in the interval $(0, \pi,)$ one gets,

$$\theta(\sigma) = \int_0^{\pi} T(\sigma, s)\,\lambda(s)\,ds, \tag{1.17}$$

where

$$T(\sigma, s) = (2/\pi)\sum_{j=1}^{\infty}\frac{\cos j\sigma \sin js}{j} = \begin{cases} -s/\pi, & 0 \leq s < \sigma \\ (1/2) - s/\pi, & s = \sigma \\ 1 - s/\pi, & \sigma \leq s \leq \pi. \end{cases} \tag{1.18}$$

As to $\tau(\sigma)$, it is given by a formula due to Dini [45] relating the boundary values of an harmonic function in the unit circle ($\tau = \operatorname{Im}\Omega(t)$) with its normal derivative ($-\lambda = \partial\tau/\partial n$) on the boundary :

$$\tau(\sigma) = \int_0^{\pi} D(\sigma, s)\,\lambda(s)\,ds, \tag{1.19}$$

where

$$(1.20)\quad D(\sigma, s) = (2/\pi)\sum_{j=1}^{\infty}\frac{\sin j\sigma \sin js}{j} = (1/2\pi)\ln\left[\frac{\operatorname{tang}\frac{1}{2}\sigma + \operatorname{tang}\frac{1}{2}s}{\operatorname{tang}\frac{1}{2}\sigma - \operatorname{tang}\frac{1}{2}s}\right]^2.$$

If the elimination of θ and τ is carried out by replacing them in (1.14) by their expressions (1.17) and (1.19), one obtains equation (1), where T and D stand for the transformations (1.17) and (1.19) respectively. (1) is, therefore, a necessary condition in order to have a barrier with the given intrinsic equation; moreover, by retracing our steps in the reverse order, it is found to be also sufficient. However, (1) does not say anything with regard to other geometrical aspects such as the location of the separation points, the orientation of the obstacle, etc. These have to be given by extra conditions. For instance, if the points A, B, C are fixed, the values of φ_1 at such points are then given and (1.6) yields three conditions, which in terms of λ are:

$$(1.21)\quad \int_0^{\pi} T(0, s)\,\lambda(s)\,ds = \varphi_1(A) - \frac{1}{2}\pi - \beta\left(\frac{1}{2}\pi - \sigma_0\right),$$

$$(1.22)\quad \int_0^{\pi} T(\sigma_0, s)\,\lambda(s)\,ds = \varphi_1(C) - \frac{1}{2}\pi - \beta\left(\frac{1}{2}\pi - \sigma_0\right),$$

$$(1.23)\quad \int_0^{\pi} T(\pi, s)\,\lambda(s)\,ds = \varphi_1(B) - \frac{1}{2}\pi - \beta\left(\frac{1}{2}\pi - \sigma_0\right).$$

For a separation « en proue » (convex free streamlines with finite curvature at the points of detachment), (1.21) and (1.23) have to be substituted by the conditions $\tau'(0) = \beta \tan\frac{1}{2}\sigma_0$, $\tau'(\pi) = -\beta \operatorname{cotan}\frac{1}{2}\sigma_0$ respectively, given by H. Villat [46]. These again can be expressed linearly in terms of λ as follows:

$$(1.24)\quad \frac{1}{\pi}\int_0^{\pi} \lambda(\sigma)\operatorname{cotan}\frac{1}{2}\sigma\,d\sigma = \beta\tan\frac{1}{2}\sigma_0,$$

$$(1.25)\quad \frac{1}{\pi}\int_0^{\pi} \lambda(\sigma)\tan\frac{1}{2}\sigma\,d\sigma = \beta\operatorname{cotan}\frac{1}{2}\sigma_0.$$

It is to be noticed, in accordance with what we said in the Introduction, that these conditions of geometry vary considerably from one case to another in their dependence upon the parameters of the problem.

But, putting aside such conditions, *the equation* (1) *is perfectly general and controls all flows occupying a simply connected domain bounded by a free streamline and a fixed wall with the stated regularity conditions* (*).

The form of the function $\nu(\sigma)$ depends only upon the geometrical (topological) type of the flow, while $\varkappa(\theta)$ is given with the obstacle. In other terms, different types of flows correspond to different ν's and different obstacles to different $\varkappa$'s.

From now on we shall abandon the hydrodynamical problem and deal exclusively with equation (1), or, what is the same, with equations (1.14), (1.17) and (1.19). The functions $\nu(\sigma)$ and $\varkappa(\theta)$, the data of the problem, will be restricted to the following ranges :

$\nu(\sigma)$ *is a p-integrable* ($p > 1$) *non-negative function defined in the interval* $(0, \pi)$ *and vanishing at the end points.*

$\varkappa(\theta)$ *is a bounded continuous function defined for all real values of* θ *with a bounded derivative.*

It is easy that to see the ν corresponding to the hydrodynamical problem we have discussed, defined by (1.12), is within the above range provided that $\beta < 2$. The same is true for most types of flows that have been described in the literature.

For simplicity we shall often write equation (1) in the form

$$\lambda = S\lambda. \tag{1.26}$$

§ 3. **The discrete equations.** Numerical computation are by essence discrete, that is, they deal only with a finite number of quantities at once, and so, to handle a continuous problem like the one set by the solution of equation (1), a previous discretization is required. In this regard we shall follow S. Brodetzky [31], and motivate our discretization of (1) in the idea of approaching the obstacle with varying obstacles whose intrinsic equations agree with the intrinsic equation of the given obstacle at a finite but increasing number of points. Thus, we are lead to satisfy (1.14) at a finite number of points. For practical and theoretical reasons, it appears to be convenient to take equally spaced points in the parameter plane t. We set

$$\lambda(\sigma_k^{(n)}) = \nu(\sigma_k^{(n)})\,\varkappa(\theta(\sigma_k^{(n)}))\,e^{-\tau}(\sigma_k^{(n)}),\quad \sigma_k^{(n)} = k\pi/(n+1),\quad k = 1, 2, \dots, n. \tag{1.27}$$

Clearly, θ and τ are related to λ by (1.17) and (1.19) respectively. Moreover, to make the problem determined, it is necessary to restrict the

(*) By a convenient departure from Levi-Civita's parametrization their validity can also be extended to some symetric flows with two free streamlines.

unknown function λ in (1.27) to an n-parameter family of functions. These we take to be the trigonometric polynomials of the form

$$\lambda(\sigma) = \sum_{j=1}^{n} a_j \sin j\sigma, \tag{1.28}$$

With this choice $\theta(\sigma)$ and $\tau(\sigma)$ become,

$$\theta(\sigma) = \sum_{j=1}^{n} \frac{a_j \cos j\sigma}{j}, \tag{1.29}$$

$$\tau(\sigma) = \sum_{j=1}^{n} \frac{a_j \sin j\sigma}{j}. \tag{1.30}$$

With such meaning fo rλ, θ and τ, (1.27) reduces to a set of n equations with the n unknowns $a_1, a_2, \ldots, a_n$. However, in these variables (1.27) looses entirely its similarity with (1.14) and a parallel treatment of both is impossible. To keep the resemblance one has to take $\lambda(\sigma_k^{(n)})$ as the unknowns themselves. $\theta(\sigma_k^{(n)})$ and $\tau(\sigma_k^{(n)})$ can be expressed in terms of $\lambda(\sigma_k^{(n)})$ by eliminating the a's with the help of the trigonometric identities (Cf. D. Jackson [47], pag. 114),

$$\sum_{k=1}^{n} \sin j\sigma_k^{(n)} \sin g\sigma_k^{(n)} = \frac{1}{2}(n+1)\,\delta_{j,g}, \qquad \delta_{j,g} = \begin{cases} 0, & j \neq g \\ 1, & j = g \end{cases} \tag{1.31}$$

as follows: setting $\sigma = \sigma_k^{(n)}$ in (1.28) and adding after multiplying by $\sin \sigma_k^{(n)}$, one gets, on account of (1.31),

$$a_j = \sum_{k=1}^{n} \lambda(\sigma_k^{(n)}) \frac{2 \sin j\sigma_k^{(n)}}{n+1}, \qquad j = 1, 2, \ldots, n, \tag{1.32}$$

which replaced in (1.29) and (1.30) give

$$\theta(\sigma_k^{(n)}) = \sum_{h=1}^{n} T_{k,h}^{(n)} \lambda(\sigma_h^{(n)}), \qquad k = 1, 2, \ldots, n, \tag{1.33}$$

$$T_{k,h}^{(n)} = \sum_{j=1}^{n} \frac{2 \cos j\sigma_k^{(n)} \sin j\sigma_h^{(n)}}{j(n+1)}, \qquad k, h = 1, 2, \ldots, n, \tag{1.34}$$

and

$$\tau(\sigma_k^{(n)}) = \sum_{h=1}^{n} D_{k,h}^{(n)} \lambda(\sigma_h^{(n)}), \qquad k = 1, 2, \ldots, n, \tag{1.35}$$

$$D_{k,h}^{(n)} = \sum_{j=1}^{n} \frac{2 \sin j\sigma_k^{(n)} \sin j\sigma_h^{(n)}}{j(n+1)}, \qquad k, h = 1, 2, \ldots, n. \tag{1.36}$$

Clearly (1.33) - (1.36) take the place of (1.17) - (1.20) of the continuous case. With their help (1.27) can be written in a form similar to (1),

$$(1.37) \qquad \lambda = \nu\, \varkappa\, (T^{(n)} \lambda)\, e^{-D^{(n)} \lambda}$$

where λ and ν stand for the functions $\lambda\,(\sigma_k^{(n)})$, $\nu\,(\sigma_k^{(n)})$ of the discrete variable $\sigma_k^{(n)}$, $k = 1, 2, \ldots, n$, and $T^{(n)}$, $D^{(n)}$ for the linear transformations (1.33) and (1.35) connected with the matrices $T_{k,h}^{(n)}$ and $D_{k,h}^{(n)}$ respectively. Parallel to (1.26) we shall write

$$(1.38) \qquad \lambda = S^{(n)} \lambda$$

where $S^{(n)}$ represents the operator $\nu\, \varkappa\, (T^{(n)} \lambda)\, e^{-D^{(n)} \lambda}$.
No confusion is to be expected from the fact that the same symbols λ and ν have one meaning in equations (1.37), (1.38) and another in (1), (1.26), their meaning will be always clear from the context.

In writing (1.27) we have assumed implicitly that none of the values $\nu\,(\sigma_k^{(n)})$ is infinite. To assure this and further questions of convergence, *we shall assume, in all questions relative to the discretization, that* $\nu\,(\sigma)$ *is continuous with a p-integrable derivative for some* $p > 1$. Such conditions are satisfied by (12) only if $\beta \leq 1$.

Two types of questions have to be examined with regard to the discrete equations (1.37): 1st. Existence and uniqueness, 2nd. Approximation of solutions of (1). The first belongs also to (1) and will be subjected to a parallel treatment in Section III. The second is the subject of Section IV, and it is to be understood as the convergence of the trigonometric polynomials (1.28) interpolating the solutions of (1.37) towards the solutions of (1).

II. THE TRANSFORMATIONS

This section will be devoted to the study of the various transformations intervening in equations (1) and (1.38) and will serve as a preparation to the following chapters. If the reader prefers, he can skip it and refer back to it whenever necessary. We shall systematically adhere to the following notation: the scalar product $\int_0^\pi x(\sigma)\, y(\sigma)\, d\sigma$ of two functions defined in the interval $(0, \pi)$ will be designated by (x, y), and the p-norm $\left\{ \int_0^\pi |x(\sigma)|^p\, d\sigma \right\}^{\frac{1}{p}}$ by $\| x \|_p$. L_p will represent the space of all p-integrable functions in $(0, \pi)$. If $x(\sigma_k^{(n)})$ and $y(\sigma_k^{(n)})$ are functions of a discrete variable $\sigma_k^{(n)} = k\pi/(n+1)$, $k = 1, 2, ..., n$, we shall write : $(x, y) = \sum_{k=1}^{n} x(\sigma_k^{(n)})\, y(\sigma_k^{(n)})\, \Delta\sigma_k^{(n)}$ and $\| x \|_p = \left\{ \sum_{k=1}^{n} x(\sigma_k^{(n)})^p \Delta\sigma_k^{(n)} \right\}^{\frac{1}{p}}$. Frequent use will be made of Hölder inequality $|(x, y)| \leq \| x\|_p \, \| y \|_q$ where p and q are positive numbers related by $(1/p) + (1/q) = 1$, and of Minkowski's inequality $\| x + y \|_p \leq \| x \|_p + \| y \|_p$. (Cf. Hardy, Littlewood and Polya [48], Chapters II and VI.) For brevity we shall usually write m and M for $g.l.b$ and $l.u.b$ respectively. We recall that $\lim_{p\to\infty} \| x \|_p = M\, \{x(\sigma)\}$. Finally, if for almost every σ, $x(\sigma) \leq y(\sigma)$, we write $x \leq y$; if in addition $x(\sigma)$ and $y(\sigma)$ are not equal almost everywhere we write $x < y$.

§ 1. **Properties of D and $D^{(n)}$.** Lemma 1. *a) For every $p > 1$, D is a linear transformation from the space L_p into the space of functions satisfying Lipchitz condition with exponent $1/q = 1 - 1/p$ and vanishing at $\sigma = 0$ and $\sigma = \pi$.*

b) There is a constant A_p depending on p only such that for every $\lambda \in L_p$,

$$|\tau(\sigma_1) - \tau\ (\sigma_2)| \leq A_p \,\| \lambda \|_p \, |\sigma_1 - \sigma_2|^{\frac{1}{q}}, \qquad \tau = D\lambda.$$

c) D is symmetric : $(D\lambda_1, \lambda_2) = (\lambda_1, D\lambda_2)$.

d) D is positive definite : $(D\lambda, \lambda) \geq 0$, $(D\lambda, \lambda) = 0$ implies $\lambda = 0$ almost everywhere.

e) D is order preserving : $\lambda_1 > \lambda_2$ implies $D\lambda_1 > D\lambda_2$.

Proof : *a)* and *b)*. For every σ, $D(\sigma, s)$ is a continuous function of s except at $s = \sigma$ where it has a logarithmic singularity. Therefore, as a function of s it belongs to all spaces L_q, $q > 1$. Hence if $\lambda \in L_p$ the scalar product $\tau(\sigma) = \int_0^{\pi} D(\sigma, s)\, \lambda(s)\, ds$ is, by Hölder inequality, defined and finite for every σ in $(0, \pi)$. Moreover, since $D(0, s) \equiv \equiv D(\pi, s) \equiv 0$, $\tau(0) = \tau(\pi) = 0$. Obviously the transformation $D\lambda$, so defined in L_p, is linear. We now prove that if $\lambda \in L_p$, $\tau = D\lambda$ is a continuous function. For that purpose we first observe that for every positive r, $|\sigma - s|^r D(\sigma, s) \leq M_r$, where M_r depends on r alone. This becomes plain if $D(\sigma, s)$ is written in the form $(1/2\pi) \ln \left[\dfrac{\sin \frac{1}{2}(\sigma + s)}{\sin \frac{1}{2}(\sigma - s)} \right]^2$.

It follows from this remark that for every $q > 1$, the integral $\left\{ \int_a^b | D(\sigma, s)|^q ds \right\}^{\frac{1}{q}}$ tends uniformly to zero as $|a - b| \to 0$. In fact, taking $r = 1/q^2$,

$$(2.1)\quad \left\{ \int_a^b |D(\sigma, s)|^q ds \right\}^{\frac{1}{q}} = \left\{ \int_a^b |\sigma - s|^{-\frac{1}{q}} [|\sigma - s|^{\frac{1}{q^2}} D(\sigma, s)]^q ds \right\}^{\frac{1}{q}} \leq$$

$$\leq M_{\frac{1}{q^2}} \left\{ \int_a^b |\sigma - s|^{-\frac{1}{q}} ds \right\}^{\frac{1}{q}} \leq M_{\frac{1}{q^2}} \left[2p\, |b - a|^{\frac{1}{p}} \right]^{\frac{1}{q}}.$$

Let σ_0 be a fixed point and σ a variable point in the interval $|\sigma - \sigma_0| < \delta$, where δ is a positive number. By definition and Hölder inequality,

$$(2.2)\quad |\tau(\sigma) - \tau(\sigma_0)| = \left| \int_0^\pi [D(\sigma, s) - D(\sigma, s)]\, \lambda(s)\, ds \right| \leq$$

$$\leq \|\lambda\|_p \left\{ \int_0 |D(\sigma, s) - D(\sigma_0, s)|^q ds \right\}^{\frac{1}{q}}.$$

If the interval $(0, \pi)$ is decomposed into three parts $(0, \sigma_0 - \delta)$, $(\sigma_0 - \delta, \sigma_0 + \delta)$, $(\sigma_0 + \delta, \pi)$, one gets, by Minkowski's inequality.

$$(2.3)\quad |\tau(\sigma) - \tau(\sigma_0)| =$$

$$= \left[\left(\int_0^{\sigma_0 - \delta} + \int_{\sigma_0 + \delta}^\pi \right) |D(\sigma, s) - D(\sigma_0, s)|^q ds \right]^{\frac{1}{q}} + \left[\int_{\sigma_0 - \delta}^{\sigma_0 + \delta} |D(\sigma, s)|^q ds \right]^{\frac{1}{q}} + \left[\int_{\sigma_0 - \delta}^{\sigma_0 + \delta} D(\sigma_0, s)^q ds \right]^{\frac{1}{q}}$$

Now, if $\sigma \to \sigma_0$, the first two integrals tend to zero because in the intervals of integration $|D(\sigma, s) - D(\sigma, s)|$ tends uniformly to zero, and the last two can be made beforehand arbitrarily small by a conveneint choice of δ, so $\tau(\sigma) \to \tau(\sigma_0)$, and the continuity of $\tau(\sigma)$ is proved. It remains to prove b).

The set of functions $\{\sin js\}$, $j = 1, 2, \ldots$, is a complete orthogonal set in the interval $(0, \pi)$ and so, to every integrable function λ it corresponds a unique development of the form (*)

$$(2.4)\quad \lambda \sim \sum_{j=1}^{\infty} a_j \sin j\sigma.$$

* The symbol $\sim$ means that the series on the rigth is the Fourier series of the function on the left.

A simple computation shows that if $\tau = D\lambda$,

$$\tau \sim \sum_{j=1}^{\infty} \frac{a_j \sin j\sigma}{j}. \tag{2.5}$$

Let now $\bar{\lambda}(\sigma)$ be the conjugate function of $\lambda(\sigma)$ (For the definition see A. ZYGMUND [49] pgs. 145 - 146). By a well known theorem of M. RIESZ [50]

$$\|\bar{\lambda}\|_p \leq A_p \|\lambda\|_p, \tag{2.6}$$

where A_p depends on p only. So $\bar{\lambda}(\sigma)$ is integrable, and its Fourier series is

$$\bar{\lambda} \sim \sum_{j=1}^{\infty} a_j \cos j\sigma. \tag{2.7}$$

Integrating both sides of (2.7) (termwise integration is permited, Cf. A. ZYGMUND [49], p. 16)

$$\int_0^{\sigma} \bar{\lambda}(s)\, ds \sim -\sum_{j=1}^{\infty} \frac{a_j \sin j\sigma}{j}. \tag{2.8}$$

Hence, τ and $-\int_0^{\sigma} \bar{\lambda}\, ds$ are continuous functions vanishing $\sigma = 0$, having the same Fourier series. Thus they are identical. Now by Hölder inequality and (2.3),

$$|\tau(\sigma_1) - \tau(\sigma_2)| = \tag{2.9}$$

$$= \left|\int_{\sigma_1}^{\sigma_2} \bar{\lambda}(s)\, ds\right| \leq |\sigma_1 - \sigma_2|^{\frac{1}{q}} \left[\int_{\sigma_1}^{\sigma_2} |\bar{\lambda}|\, ds\right]^{\frac{1}{q_1}} \leq |\sigma_1 - \sigma_2|^{\frac{1}{q}} \|\bar{\lambda}\|_p \leq A_p \|\lambda\|_p |\sigma_1 - \sigma_2|^{\frac{1}{q}}$$

which is b).

c) obvious from the symmetry of the kernel.

d) From (2.1), (2.2) and Parseval equation,

$$(\lambda, D\lambda) = (\lambda, \tau) = \frac{\pi}{2} \sum_{j=1}^{\infty} a_j^2 / j > 0 \tag{2.10}$$

unless all $a_j's$ are zero, in which case λ vanishes almost everywhere.

e) It is enough to prove that if $\lambda > 0$ then $D\lambda > 0$. This follows trivially from the positiveness of the kernel $D(\sigma, s)$ which in turn follows from the inequality $|x + 1/(x - 1)| > 1$ valid for positive x, by putting $x = \operatorname{tang} {}^{\sigma}/_{2}/\operatorname{tang} {}^{s}/_{2}$ in the definition (1.20) of $D(\sigma, s)$.

Lemma 2. *a)* $D^{(n)}$ *is a one-to-one linear transformation of the n-dimensional euclidean space* $E^{(n)}$ *onto itself.*

b) $(D^{(n)}\lambda, D^{(n)}\lambda) \leq (\lambda, \lambda)$.

c) $D^{(n)}$ *is symmetric:* $(D^{(n)}\lambda_1, \lambda_2) = (\lambda_1, D^{(n)}\lambda_2)$

d) $D^{(n)}$ *is positive definite;* $(D^{(n)}\lambda, \lambda) \geq 0$, $(D^{(n)}\lambda, \lambda) = 0$ *implies* $\lambda = 0$.

e) $D^{(n)}$ *is order preserving:* $\lambda_1 > \lambda_2$ *implies* $D^{(n)}\lambda_1 > D^{(n)}\lambda_2$.

Proof: *a)* The linearity of $D^{(n)}$ is part of the definition. To prove that it is one to one and that it transforms $E^{(n)}$ onto itself it is enough to show that $D^{(n)}\lambda = 0$ implies $\lambda = 0$. This follows from *d)* for which we give an independent proof.

b) Let $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and define, in accordance with (1.32),

$$a_j = \sum_{k=1}^{n} \lambda_k \frac{2 \sin j\sigma_k^{(n)}}{n+1}. \tag{2.11}$$

Then the polynomials

$$\lambda(\sigma) = \sum_{j=1}^{n} a_j \sin j\sigma, \qquad \tau(\sigma) = \sum_{j=1}^{n} \frac{a_j \sin j\sigma}{j}$$

take the values

$$\lambda(\sigma_k^{(n)}) = \lambda_k, \qquad \tau(\sigma_k^{(n)}) = \tau_k = \sum_{n=1}^{n} D_{k,h}^{(n)} \lambda_k.$$

Moreover

$$(\lambda, \lambda) = \sum_{k=1}^{n} \lambda_k^2 \Delta\sigma_k^{(n)} = \sum_{k=1}^{n} \lambda^2(\sigma_k^{(n)}) \Delta\sigma_k^{(n)} = \sum_{j,g=1}^{n} a_j a_g \sum_{k=1}^{n} \sin j\sigma_k^{(n)} \sin g\sigma_k^{(n)} \Delta\sigma_k^{(n)} \tag{2.12}$$

and by (1.31),

$$(\lambda, \lambda) = \frac{\pi}{2} \sum_{j=1}^{n} a_j^2. \tag{2.13}$$

Similarly

$$(D^{(n)}\lambda, D^{(n)}\lambda) = \frac{\pi}{2} \sum_{j=1}^{n} a_j^2/j^2, \tag{2.14}$$

and *b)* follows by comparison of the right hand sides of (2.10) and (2.11).

c) A trivial consequence of the symmetry of the matrix $D^{(n)}_{k,h}$.

d) Proceeding as in *b)*,

$$(2.15) \qquad (D^{(n)}\lambda, \lambda) = \sum_{k=1}^{n} \tau_k \lambda_k \Delta \sigma_k^{(n)} =$$

$$= \sum_{k=1}^{n} \tau(\sigma_k^{(n)}) \lambda \Delta \sigma_k^{(n)} = \sum_{j,g=1}^{n} \frac{a_j a_g}{j} \sum_{k=1}^{n} \sin j\sigma_k^{(n)} \sin g\sigma_k^{(n)} \Delta \sigma_k^{(n)} = \frac{\pi}{2} \sum_{j=1}^{n} a_j^2/j > 0.$$

Moreover, comparing (2.12) and (2.10)

$$(2.16) \qquad (D^{(n)}\lambda, \lambda) \leq (\lambda, \lambda) \leq n^2 (D^{(n)}\lambda, \lambda),$$

which trivially contains the last half of *d)*.

e) To prove this property we simply have to show that all the elements of the matrix $D^{(n)}_{k,h}$ are positive. On account of the symmetry $D^{(n)}_{k,h} = D^{(n)}_{h,k}$ we can restrict ourselves to the case $h < k$; moreover the relations $D^{(n)}_{k,h} = D^{(n)}_{k',h'}$ for $k + k' = n + 1$, $h + h' = n + 1$ allow the further restriction $k + h \leq n + 1$. Inserting the trigonometrical identity

$$(2.17) \quad 2\sin j\sigma_k^{(n)} \sin j\sigma_h^{(n)} = \cos j(\sigma_k^{(n)} - \sigma_h^{(n)}) - \cos j(\sigma_k^{(n)} + \sigma_h^{(n)}) = \Re(x_1^j - x_2^j),$$

where $x_1 = e^{i(\sigma_k^{(n)} - \sigma_h^{(n)})}$ and $x_2 = e^{i(\sigma_k^{(n)} + \sigma_h^{(n)})}$, into the definition (1,36) of $D^{(n)}_{k,h}$ we get

$$(2.18) \qquad D^{(n)}_{k,h} = \frac{1}{n+1} \Re \sum_{j=1}^{n} \left(\frac{x_1^j}{j} - \frac{x_2^j}{j} \right).$$

Now, for any $x \neq 1$,

$$(2.19) \quad \sum_{j=1}^{n} (x^j/j) = \int_0^x \sum_{j=1}^{n} x^{j-1}\,dx = \int_0^1 \frac{1-x^n}{1-x}\,dx + \int_1^x \frac{1-x^n}{1-x}\,dx.$$

If $x = e^{i\delta}$, a change of variables in the last integral gives

$$(2.20) \qquad \sum_{j=1}^{n} (x^j/j) = \int_0^1 \frac{1-x^n}{1-x}\,dx - \int_0^\delta \frac{e^{i\frac{1}{2}t} - e^{i\left(n+\frac{1}{2}\right)t}}{2\sin(t/2)}\,dt$$

which replaced in (2.15) yields

$$(2.21)\qquad D_{k,h}^{(n)} = (1/2\,(n+1))\,\Re \int_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} \frac{e^{it/2} - e^{i\left(n+\frac{1}{2}\right)t}}{\sin(t/2)}\,dt =$$

$$= (1/2\,(n+1)) \int_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} \frac{\cos(t/2) - \cos\left(n+\frac{1}{2}\right)t}{\sin(t/2)}\,dt =$$

$$= (1/2\,(n+1)) \int_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} \operatorname{cotg}(t/2)\,(1 - \cos(n+1)\,t)\,dt -$$

$$- \int_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} (1/2\,(n+1))\,\sin(n+1)\,t\,dt.$$

The last integral is equal to

$$(22.2)\qquad \left[\frac{\cos(n+1)\,t}{n+1}\right]_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} = (1/(n+1))\,[(-1)^{k+h} - (-1)^{k-h}] = 0.$$

Hence (2.18) becomes

$$(2.23)\qquad D_{k,h}^{(n)} = (1/(n+1)) \int_{\sigma_k^{(n)}-\sigma_h^{(n)}}^{\sigma_k^{(n)}+\sigma_h^{(n)}} \operatorname{cotg}(t/2)\,\sin^2((n+1)\,t/2)\,dt.$$

On account of the above restrictions, both the interval of integration and the integrand are positive and the positiveness of the elements of the matrix is proved.

Lemma 3. *a) For every $p > 1$, T is a linear transformation from the space L_p into the space of functions satisfying a Lipchitz condition of exponent $1/q = 1 - 1/p$ and having mean value zero.*

b) $|\theta(\sigma_1) - \theta(\sigma_2)| \leq \|\lambda\|_p\,|\sigma_1 - \sigma_2|^{\frac{1}{q}},\ \theta = T\lambda,$

$$|\theta(\sigma)| \leq \|\lambda\|_p\,\pi^{\frac{1}{q}}.$$

c) $(T\lambda, T\lambda) = (D\lambda, D\lambda).$

PROOF: *a)* and *b)*. $T(\sigma, s)$ is, for fixed σ, a continuous function of s except at $s = \sigma$ where it has a simple discontinuity. Moreover $|T(\sigma, s)| \leq 1$ and $\int_0^\pi T(\sigma, s)\, d\sigma \equiv 0$. Therefore, the transformation T is defined for every function in L_p, is linear and yields functions with mean value zero. Now, by definition and Hölder inequality,

$$(2.24) \qquad |\theta(\sigma_1) - \theta(\sigma_2)| = \left| \int_{\sigma_1}^{\sigma_2} \lambda(s)\, ds \right| \leq \| \lambda \|_p \, |\sigma_1 - \sigma_2|^{\frac{1}{q}}$$

$$|\theta(\sigma)| = \left| \int_0^\pi T(\sigma, s)\, \lambda(s)\, ds \right| \leq \int_0^\pi |\lambda(s)|\, ds \leq \| \lambda \|_p \, \pi^{\frac{1}{q}}.$$

From (2.6) it follows by direct computation

$$(2.25) \qquad T\lambda = \theta \sim \sum_{j=1}^{\infty} (a_j/j) \cos j\sigma.$$

A simple application of Plancherel Theorem to θ and τ (whose Fourier series are given by (2.25) and (2.7)) leads to *c)*.

LEMMA 4. *a)* *$T^{(n)}$ is a one to one linear transformation of $E^{(n)}$ onto itself.*

b) $(T^{(n)}\lambda, T^{(n)}\lambda) \leq (\lambda, \lambda)$.

c) $(T^{(n)}\lambda, T^{(n)}\lambda) = (D^{(n)}\lambda, D^{(n)}\lambda)$.

PROOF: *a)* Obvious except for the fact that the transformation is one to one. This follows from Lemma 2 and *c)* which is proved independently.

b) From Lemma 2, *b)*, and *c)*.

c) If $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and a_j are defined by (2.10), then the trigonometric polynomial $\theta(\sigma) = \sum_{j=1} (a_j/j) \cos j\sigma$ takes the values $\theta(\sigma_k^{(n)}) = \theta_k = \sum_{h=1}^{n} T_{k,h}^{(n)} \lambda_h$. Now,

$$(2.26) \quad (T^{(n)}\lambda, T^{(n)}\lambda) = \sum_{k=1}^{n} \theta(\sigma_k^{(n)})\, \theta(\sigma_k^{(n)})\, \Delta\sigma_k^{(n)} = \sum_{j,g=1}^{n} \frac{a_j a_g}{jg} \sum_{k=1}^{n} \cos j\sigma_k^{(n)} \cos g\sigma_k^{(n)}\, \Delta\sigma_k^{(n)}.$$

which, by virtue of the relations (Cf. D. JACKSON [47], page 114)

$$(2.27) \qquad \sum_{k=1}^{n} \cos j\sigma_k^{(n)} \cos g\sigma_k^{(n)}\, \Delta\sigma_k^{(n)} = \frac{\pi}{2}\, \delta_{j,g} \quad (\delta_{j,g} \text{ Kronecker symbol}),$$

becomes

$$(2.28)\qquad (T^{(n)}\lambda,\ T^{(n)}\lambda) = \frac{\pi}{2}\sum_{j=1}^{n} a_j^2/j^2,$$

which, in turn, is equal to $(D^{(n)}\lambda,\ D^{(n)}\lambda)$ by (2.11).

§ 2. **Bounds for** $|\Omega(e^{is})|$. Lemma 5. *If* $\lambda \in L_p$, $1 < p \leq 2$ *and* $\tau = D\lambda$, $\theta = T\lambda$, *then for every* r *between zero and* 1,

$$(2.29)\qquad \{\theta(\sigma)^2 + \tau(\sigma)^2\}^{\frac{1}{2}} \leq (\lambda,\ D\lambda)^{\frac{1}{2}}\left[\frac{2}{\pi}\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + C_p\|\lambda\|_p\left[\ln\frac{1}{r}\right]^{\frac{1}{q}},$$

where

$$(2.30)\qquad C_p = \frac{2^{2/p-1}}{\pi^{1/p}}\left\{\int_0^\infty \frac{(1-e^{-u})^p}{u}\,du\right\}^{\frac{1}{p}}.$$

Proof: Let $\sum_{j=1}^{\infty} a_j \sin j\sigma$ be the Fourier series of $\lambda(\sigma)$. Then, the analytic function

$$\Omega(t) = \sum_{j=1}^{\infty} a_j t^j/j = \frac{1}{\pi i}\int_0^\pi \ln\left\{\frac{1-te^{-is}}{1-te^{is}}\right\}\lambda(s)\,ds$$

takes on the boundary the values $\Omega(e^{i\sigma}) = \theta(\sigma) + i\tau(\sigma)$. Now if $t = re^{i\sigma}$ and $t_1 = e^{i\sigma}$,

$$(2.31)\qquad \Omega(e^{i\sigma}) = \Omega(t_1) = \Omega(t) + \Omega(t_1) - \Omega(t) =$$

$$= \sum_{j=1}^{\infty} a_j t^j/j + \frac{1}{\pi i}\int_0^\pi\left[\ln\frac{1-t_1e^{-is}}{1-t_1e^{is}} - \ln\frac{1-te^{-is}}{1-te^{is}}\right]\lambda(s)\,ds.$$

Applying Schwarz inequality to the first term on the right and Hölder inequality to the second, we get

$$(2.32)\qquad |\Omega(e^{is})| \leq \left\{\sum_{j=1}^{\infty} a_j^2/j\right\}^{\frac{1}{2}}\left\{\sum_{j=1}^{\infty}|t|^{2j}/j\right\}^{\frac{1}{2}} +$$

$$+ \|\lambda\|_p\frac{1}{\pi}\left\{\int_0^\pi\left[\ln\frac{1-t_1e^{-is}}{1-t_1e^{is}} - \ln\frac{1-te^{-is}}{1-te^{is}}\right]^q ds\right\}^{\frac{1}{q}}.$$

Replacing the first factor by (2.10) and the second by its value $\ln(1/(1-r^2))$,

$$|\Omega|(e^{is}) \leq (\lambda, D\lambda)^{\frac{1}{2}}\left[\frac{2}{\pi}\ln\frac{1}{1-r^2}\right]^{\frac{1}{2}} + \tag{2.33}$$

$$+ \|\lambda\|_p \frac{1}{\pi}\left\{\int_0^\pi \left|\ln\frac{1-t_1 e^{-is}}{1-t_1 e^{is}} - \ln\frac{1-te^{-is}}{1-te^{is}}\right|^q ds\right\}^{\frac{1}{q}}.$$

To obtain a bound for the last integral we shall make use of the Haussdorf - Young Theorem (Cf. A. Zygmund [49], page 190) which states that if $1 < p \leq 2$ and $f \sim \sum_{j=-\infty}^{+\infty} c_j e^{ij\sigma}$, then

$$(1/2\pi)\left\{\int_0^{2\pi} |f(\sigma)|^q d\sigma\right\}^{\frac{1}{q}} \leq \left\{\sum_{j=-\infty}^{+\infty} |c_j|^p\right\}^{\frac{1}{p}}. \tag{2.34}$$

In our case, the Fourier series of the function under the integral in (2.25) is $\sum_{j=1}^{\infty} \frac{t_1^j - t^j}{j} 2\sin j\sigma$, so

$$\left\{\int_0^\pi \left|\ln\frac{1-t_1 e^{-is}}{1-t_1 e^{is}} - \ln\frac{1-te^{-is}}{1-te^{is}}\right|^q ds\right\}^{\frac{1}{q}} \leq \pi^{\frac{1}{q}}\, 2^{\frac{1}{p}}\left\{\sum_{j=1}^{\infty}\left(\frac{1-r^j}{j}\right)^p\right\}^{\frac{1}{p}}. \tag{2.35}$$

Moreover by Cauchy integral criterion of convergence,

$$\sum_{j=1}^{\infty}\left(\frac{1-r^j}{j}\right)^p \leq \int_0^\infty \frac{(1-r^u)^p}{u}\,du = (\ln 1/r)^{p-1}\int_0^\infty \frac{(1-e^{-u})^p}{u}\,du \tag{2.36}$$

Which if replaced succesively in (2.35) and (2.33) yields (2.29) with r^2 in place of r. Putting $r = C_p \|\lambda\|_p / [(\lambda, D\lambda)^{\frac{1}{2}} + C_p\|\lambda\|_p]$ we obtain the following

Corollary (2.37) $\left\{\theta(\sigma)^2 + \tau(\sigma)^2\right\} \leq (\lambda, D\lambda)^{\frac{1}{2}}\left\{\frac{2}{\pi}\ln\left(1 + C_p\frac{\|\lambda\|_p}{(\lambda, D\lambda)^{\frac{1}{2}}}\right)\right\}^{\frac{1}{2}} +$

$$+ C_p\|\lambda\|_p\left\{\ln\left((1 + \frac{(\lambda, D\lambda)^{\frac{1}{2}}}{C_p\|\lambda\|_p}\right)\right\}^{\frac{1}{q}}.$$

An important consequence of this is that for $\lambda's$ with bounded p-norms, convergence of the $\lambda's$ with regard to the norm $(\lambda, D\lambda)$ is transformed into uniform convergence for $T\lambda$ and $D\lambda$. This fact could easily be obtained by non constructive arguments, the advantage of the above formula being that it gives an estimate of the error. In the discrete case we have a similar situation.

LEMMA 6. *If* $\lambda = \{ \lambda(\sigma_1^{(n)}), \lambda(\sigma_2^{(n)}), ..., \lambda(\sigma_n^{(n)}) \}$ *and* $\theta = T^{(n)} \lambda$, $\tau = D^{(n)} \lambda$, *then*

$$(2.38)\quad \{ \theta(\sigma_k^{(n)})^2 + \tau(\sigma_k^{(n)})^2 \}^{\frac{1}{2}} \leq (\lambda, D^{(n)} \lambda)^{\frac{1}{2}} \left[\frac{2}{\pi} \ln \frac{1}{1-r} \right]^{\frac{1}{2}} + C_p^* \|\lambda\|_p \left[\ln \frac{1}{r} \right]^{\frac{1}{q}},$$

where $C_p^* = B_p G_p$, ; B_p *depends on p only.*

PROOF : The proof follows the same line as the previous one. Let $\lambda(\sigma)$ be the trigonometric polynomial interpolating the values $\lambda(\sigma_k^{(n)})$ with coefficients (2,11). Then $\Omega(t) = \sum_{j=1}^{n} a_j t^j / j$ takes on the boundary the values $\Omega\left(e^{i\sigma_k^{(n)}}\right) = \theta(\sigma_k^{(n)}) + i\,\tau(\sigma_k^{(n)})$. Proceeding as before one obtains

$$(2.39)\quad \{ \theta(\sigma_k^{(n)})^2 + \tau(\sigma_k^{(n)})^2 \}^{\frac{1}{2}} \leq \left\{ \sum_{j=1}^{n} a_j^2/j \right\}^{\frac{1}{2}} \left[\ln \frac{1}{1-r} \right]^{\frac{1}{2}} + \\ + C_p \left\{ \int_0^{\pi} |\lambda(s)|^p \, ds \right\}^{\frac{1}{p}} \left\{ \ln \frac{1}{r} \right\}^{\frac{1}{q}}.$$

The sum in the first term on the right hand side is equal to $(2/\pi)\,(D^{(n)} \lambda, \lambda)^{\frac{1}{2}}$. As to the p-norm of the polynomial $\lambda(\sigma)$ can be majorized by the p-norm of the discrete function $\lambda(\sigma_k^{(n)})$ by using a result of J. MARCINKIEWICZ [51] (See also Section IV, Lemma 16) stating that

$$(2.40)\quad \left\{ \int_0^{\pi} |\lambda(s)|^p \, ds \right\}^{\frac{1}{p}} \leq B_p \left\{ \sum_{j=1}^{n} |\lambda(\sigma_k^{(n)})|^p \, \Delta \sigma_k^{(n)} \right\}^{\frac{1}{p}}.$$

(2.39) so majorized becomes (2.38).

§ 3. **The forms** $\chi_K(z)$. The range and rate of convergence of the iteration process for solving (1·1) will appear to depend upon the values of a special type of functionals applied to the data of the problem. These we shall now consider.

DEFINITION. *If* $K(\sigma, s)$ *is a symmetric kernel* $2\,q$*-integrable in the square* $0 \leq \sigma \leq \pi$, $0 \leq s \leq \pi$, *and* $z(\sigma)$ *is a function in* L_p, $p > 1$, $1/p + 1/q = 1$, *we define*

$$(2.41)\quad \chi_K(z) = \underset{(x,x)=1}{\mathrm{l.\,u.\,b.}} \int_0^{\pi} \int_0^{\pi} K(\sigma, s) \sqrt{|z(\sigma)|} \sqrt{|z(s)|} \; x(\sigma)\, x(s)\, d\sigma\, ds.$$

If $K = K_{k,h}$ is a symmetric matrix and $z = z(\sigma_k^{(n)})$ a function of a discrete variable, we set in a similar way

$$(2.42)\quad \chi_K(z) = \underset{(x,x)=1}{\mathrm{l.\,u.\,b.}} \sum_{k,h=1}^{n} K_{k,h} \sqrt{|z(\sigma_k^{(n)})|}\sqrt{|z(\sigma_h^{(n)})|}\, x(\sigma_k^{(n)})\, x(\sigma_h^{(n)})\, \Delta\sigma_k^n \Delta\sigma_h^n .$$

The reader will immediately recognize that $\chi_K(z)$ is the greatest eigenvalue of the operator K_z, corresponding to the kernel

$$\sqrt{|z(\sigma)|}\, K(\sigma, s)\sqrt{|z(s)|} \quad \text{(to the matrix } \sqrt{|z(\sigma_k^{(n)})|}\, K_{k,h} \sqrt{|z(\sigma_h^{(n)})|}).$$

This operator can be factored as follows $K_z = \sqrt{|z|}\, K \sqrt{|z|}$, where K indicates the operator corresponding to the kernel $K(\sigma, s)$ (the matrix $K_{k,h}$), and $\sqrt{|z|}$ the operation of multipliyng by the function $\sqrt{|z(\sigma)|}$.

LEMMA 7. *If K is definite non-negative,*

$$a)\quad 0 \le \chi_K(z) \le \begin{cases} \|z\|_p \left\{ \int_0^\pi \int_0^\pi |K(\sigma, s)|^{2q}\, d\sigma\, ds \right\}^{\frac{1}{2q}} & \text{for a kernel,} \\ \|z\|_p \left\{ \sum_{h,k=1}^{n} |K_{k,h}|^{2q}\, \Delta\sigma_k^{(n)}\, \Delta\sigma_h^{(n)} \right\}^{\frac{1}{2q}} & \text{for a matrix,} \end{cases}$$

$b)$ $\chi_K(\alpha z) = |\alpha|\, \chi_K(z)$,

$c)$ $\chi_K(z_1 + z_2) = \chi_K(z_1) + \chi_K(z_2)$,

$d)$ $\chi_K(\sqrt{z_1 z_2}) = \sqrt{\chi_K(z_1)\, \chi_K(z_2)}$,

$e)$ $|z_1| < |z_2|$ implies $\chi_K(z_1) \le \chi_K(z_2)$.

PROOF: We first show that the kernel $\sqrt{|z(\sigma)|}\, K(\sigma, s)\sqrt{|z(s)|}$ is square integrable. In fact, by Hölder inequality

$$(2.43)\qquad \left\{ \int_0^\pi \int_0^\pi \left[K(\sigma, s)\sqrt{|z(\sigma)|}\sqrt{|z(s)|}\right]^2 d\sigma\, ds \right\}^{\frac{1}{2}} \le$$
$$\le \|z\|_p \left\{ \int_0^\pi \int_0^\pi K(\sigma, s)^{2q}\, d\sigma\, ds \right\}^{\frac{1}{2q}}.$$

Similarly for a matrix. The second half of $a)$ is an immediate consequence of the above inequality. The first half follows from the fact that K being definite positive: $(K_z x, x) = (K\sqrt{|z|}\,x, \sqrt{|z|}\,x) \geqslant 0$.

$b)$ is an immediate consequence of the definition.

c) is somewhat involved and will require the proof of the following alternative definition of $\chi_K(z)$:

$$\chi_K(z) = \underset{(x,\,Kx)=1}{\text{l. u. b.}} \; (|z|\, Kx,\; Kx). \tag{2.44}$$

Let us assume first that z is positive. Then

$$\underset{(x,\,Kx)=1}{\text{l. u. b.}} \; (zKx, Kx) = \underset{(x/\sqrt{z},\; \sqrt{z}K\sqrt{z}x/\sqrt{z})=1}{\text{l. u. b.}} \; (\sqrt{z}K\sqrt{z}x/\sqrt{z}, \sqrt{z}K\sqrt{z}x/\sqrt{z}), \tag{2.45}$$

and putting $x/\sqrt{z} = y$

$$= \underset{(K_z y,\, y)=1}{\text{l. u. b.}} \; (K_z y,\; K_z y). \tag{2.46}$$

Now if φ_1, φ_2, ... is a complete orthonormal set of eigen-functions of K_z (Cf. P. Hammel [52]) corresponding to decreasing eigenvalues χ_1, χ_2, ..., every $y \in L_2$ can be uniquely expressed in the form

$$y = \sum_{i=1}^{\infty} a_i\,\varphi_i + y_1, \tag{2.47}$$

where y_1 is orthogonal to all the functions φ_i. Hence $K_z y_1 = 0$ and

$$K_z y = \sum_{i=1}^{\infty} a_i\,\chi_i\,\varphi_i, \tag{2.48}$$

$$K_z K_z y = \sum_{i=1}^{\infty} a_i\,\chi_i^2\,\varphi_i. \tag{2.49}$$

Performing the scalar products of (2.48) with itself and with (2.47)

$$(K_z y,\; K_z y) = \sum_{i=1}^{\infty} a_i^2\,\chi_i^2, \tag{2.50}$$

$$(y,\; K_z y) = \sum_{i=1}^{\infty} a_i^2\,\chi_i, \tag{2.51}$$

and by direct comparison of the right members

$$\underset{(y,\,K_z y)=1}{\text{l. u, b.}} \; (K_z y,\; K_z y) = \chi_1 = \chi_K(z). \tag{2.52}$$

Thus (2.44) is proved for positive z's. To extend it to non-negative z's let us put for a moment

$$\chi_K^{(1)}(z) = \underset{(x.\,Kx)=1}{\text{l. u. b.}} \; (zKx,\; Kx). \tag{2.53}$$

Putting $z = z_1 + z_2$ one gets

$$(2.54)\qquad \chi_K^{(1)}(z_1+z_2) = \underset{(x,Kx)=1}{\text{l.u.b.}} \; ((z_1+z_2)Kx,\, Kx) \leq$$

$$\leq \underset{(x,Kx)=1}{\text{l.u.b.}} \; (z_1 Kx,\, Kx) + \underset{(x,Kx)=1}{\text{l.u.b.}} \; (z_2 Kx,\, Kx) = \chi_K^{(1)}(z_1) + \chi_K^{(1)}(z_2).$$

From which it follows

$$(2.55)\qquad |\chi_K^{(1)}(z_1) - \chi_K^{(2)}(z_2)| \leq \chi_K^{(1)}(z_1 - z_2).$$

In particular if ε is a constant, $|\chi_K^{(1)}(z+\varepsilon) - \chi_K^{(1)}(z)| \leq \chi_K^{(1)}(\varepsilon) = \varepsilon\, \chi_K(1)$, and

$$(2.56)\qquad \chi_K^{(1)}(z) = \lim_{\varepsilon \to 0} \chi_K^{(1)}(z+\varepsilon).$$

On the other hand, by a similar argument

$$(2.57)\qquad [\chi_K((z_1+z_2)^2)]^{\frac{1}{2}} = \underset{(x,x)=1}{\text{l.u.b.}} \; (K(z_1+z_2)x,\, (z_1+z_2)x)^{\frac{1}{2}} \leq$$
$$\leq [\chi_K(z_1^2)]^{\frac{1}{2}} + \chi_K(z_2^2)]^{\frac{1}{2}},$$

so

$$(2.58)\qquad |\chi_K^{1/2}(z_1) - \chi_K^{1/2}(z_2)| \leq \chi_K^{1/2}((\sqrt{z_1} - \sqrt{z_2})^2),$$

$$(2.59)\qquad |\chi_K^{1/2}(z+\varepsilon) - \chi_K^{1/2}(z)| \leq \chi_K^{1/2}((\sqrt{z+\varepsilon} - \sqrt{z})^2).$$

By *a)* the term on the right tends to zero with ε, hence $\chi_K(z) = \lim \chi_K(z+\varepsilon)$. Therefore, the identity between $\chi_K^{(1)}(z)$ and $\chi_K(z)$ which was proved for positive z's, is, by continuity, extended to every non negative z, and so to every z. This identity being established, *c)* is nothing but (2.54).

d) results from a simply application of Schwarz inequality, as follows:

$$(2.60)\qquad \chi_K(\sqrt{|z_1|\,|z_2|}) = \underset{(x,x)=1}{\text{l.u.b.}} \int_0^\pi\!\!\int_0^\pi \{K(\sigma,s)\sqrt{|z_1(\sigma)|}\,\sqrt{|z_1(s)|}\,x(\sigma)\,x(s)\}^{\frac{1}{2}}$$
$$\{K(\sigma,s)\sqrt{|z_2(\sigma)|}\,\sqrt{|z_2(s)|}\,x(\sigma)\,x(s)\}^{\frac{1}{2}}\, d\sigma\, ds \leq$$
$$\leq \underset{(x,x)=1}{\text{l.u.b.}} \left\{ \int_0^\pi\!\!\int_0^\pi K(\sigma,s)\sqrt{|z_1(\sigma)|}\,\sqrt{|z_1(s)|}\,x(\sigma)\,x(s)\, d\sigma\, ds \right\}^{\frac{1}{2}}$$
$$\underset{(x,x)=1}{\text{l.u.b.}} \left\{ \int_0^\pi\!\!\int_0^\pi K(\sigma,s)\sqrt{|z_2(\sigma)|}\,\sqrt{|z_2(s)|}\,x(\sigma)\,x(s)\, d\sigma\, ds \right\} \leq \chi_K^{1/2}(z_1)\,\chi_K^{1/2}(z_2).$$

To prove *e)* we write the definition of $\chi_K(z)$ in the form

$$(2.61)\quad \chi_K(z) = \underset{(x,x)\le 1}{\text{l.u.b.}} \ (K\sqrt{|z|}\,x,\ \sqrt{|z|}\,x)^{\frac{1}{2}} = \underset{(y/\sqrt{|z|},\ y/\sqrt{|z|})\le 1}{\text{l.u.b.}} \ (Ky, y)^{\frac{1}{2}}.$$

Clearly, since the class of functions y such that $(y/\sqrt{|z|},\ y/\sqrt{|z|}) \le 1$, is larger the larger is $|z|$, $\chi_K(z)$ increases with $|z|$.

These results and definitions are now applied to obtain some important relations involving the operators T and D.

Lemma 8. *If z is a non-negative function and α, β are non-negative real numbers such that $\alpha + \beta = 1$, then*

$$(2.62)\qquad (\alpha zDx - \beta x,\ D(\alpha zDx - \beta x))^{\frac{1}{2}} \le \delta (x, Dx)^{\frac{1}{2}},$$

where D is the Dini's operator (see (1.19), (1.20)) *and*

$$(2.63)\quad \delta = \max\{|\alpha\chi_D(z) - \beta|,\ \beta\} = \begin{cases} \beta \text{ if } & 0 \le \alpha \le 2/(\chi_D(z)+2) \\ \alpha\chi_D(z) - \beta \text{ if } & 2/(\chi_D(z)+2) \le \alpha \le 1. \end{cases}$$

Proof: As in the proof of part *c)* of Lemma 7, we have

$$(2.64)\qquad \underset{(x,Dx)=1}{\text{l.u.b.}} \ (\alpha zDx - \beta x,\ D(\alpha zDx - \beta x))^{\frac{1}{2}} =$$

$$= \underset{(y,D_z y)=1}{\text{l.u.b.}} \ (\alpha D_z y - \beta y,\ D_z(\alpha D_z y - \beta y)),$$

so by (2.47), (2.48), (2.49) (K is now D),

$$(2.65)\quad (\alpha D_z y - \beta y,\ D_z(\alpha D_z y - \beta y))^{\frac{1}{2}} = \sum_{i=1}^{\infty} a_i^2 \chi_i (\alpha\chi_i - \beta)^2,$$

which compared with (2.51) leads to

$$(2.66)\quad (\alpha D_z y - \beta y,\ D_z(\alpha D_z y - \beta y))^{\frac{1}{2}} \le \underset{i}{\text{l.u.b.}} \ |\alpha\chi_i - \beta|\,(y, D_z y)^{\frac{1}{2}}.$$

Putting $y = \sqrt{|z|}\,x$ and noticing that, all the eigenvalues being positive, $\delta \geqslant \underset{i}{\text{l.u.b.}}\ |\alpha\chi_i - \beta|$, one obtains the desired result.

Lemma 9. (2.67) $(zTx, DzTx) \le [\chi_D(z)\,\chi_{\tilde D}(z)]\,(x, Dx)$,

where $\tilde{D}(\sigma, s) = (2/\pi) \sum_{j=1}^{\infty} \dfrac{\cos j\sigma \cos js}{j} = (1/2\pi) \ln \left[\dfrac{1}{2(\cos\sigma - \cos s)}\right]^2.$

Proof: We introduce three new operators in L_2: $\tilde{D}$, U, V, which we define by the way they act on Fourier series,

$$\tilde{D}(a_0/2 + \sum_{j=1}^{\infty} a_j \cos j\sigma) = \sum_{j=1}^{\infty} a_j \cos j\sigma/j, \tag{2.68}$$

$$U(a_0/2 + \sum_{j=1}^{\infty} a_j \cos j\sigma) = \sum_{j=1}^{\infty} a_j \cos j\sigma/\sqrt{j}, \tag{2.69}$$

$$V(\sum_{j=1}^{\infty} a_j \sin j\sigma) = \sum_{j=1}^{\infty} a \cos j\sigma/\sqrt{j}. \tag{2.70}$$

$\tilde{D}$ is the integral operator connected with the symmetric kernel $\tilde{D}(\sigma, s)$ define above. It is non negative definite and $\tilde{D}(\sigma, s)$ is integrable of any order. U and V are also bounded integral operators, but we shall not concern us with their kernels. We simply notice that their ranges are both dense in the same linear subspace of L_2 consisting of all functions with mean value zero (orthogonal to constants). D and T have, in terms of U and V, the factorizations: $\tilde{D} = UU$ and $T = UV$. Moreover, for every $x \in L_2$, $(x, Dx) = (Vx, Vx)$, $(x, \tilde{D}x) = (Ux, Ux)$. Therefore,

$$\underset{(x,Dx)=1}{\text{l.u.b.}} (zTx, DzTx) = \underset{(Vx,Vx)=1}{\text{l.u.b.}} (zUVx, DzUVx) \tag{2.71}$$

But, since when x runs in L_2, Vx and Ux describe spaces dense in each other, Vx can be replaced by Ux in the above equation, so

$$\underset{(x,Dx)=1}{\text{l.u.b.}} (zTx, DzTx) = \underset{(Ux,Ux)=1}{\text{l.u.b.}} (zUUx, DzUUx) = \underset{(x,\tilde{D}x)=1}{\text{l.u.b.}} (z\tilde{D}x, Dz\tilde{D}x). \tag{2.72}$$

Now by the positiveness of the kernel $D(\sigma, s)$ and by definition of $\chi_D(z)$,

$$\begin{aligned}(z\tilde{D}x, Dz\tilde{D}x) &\leq (\big||z|\tilde{D}x\big|, D\big||z|\,Dx\big|) = (\sqrt{|z|}\,D\sqrt{|z|}\sqrt{|z|}\,|\tilde{D}x|, \sqrt{|z|}\,|\tilde{D}x|) \leq \\ &\leq \chi_D(z)(\sqrt{|z|}\,\tilde{D}x, \sqrt{|z|}\,\tilde{D}x) = \chi_D(z)(|z|\,\tilde{D}x, \tilde{D}x).\end{aligned} \tag{2.73}$$

Thus, on account of (2.44),

$$\underset{(x,Dx)=1}{\text{l.u.b.}} (zTx, DzTx) \leq \chi_D(z) \underset{(x,\tilde{D}x)=1}{\text{l.u.b.}} (|z|\,\tilde{D}x, \tilde{D}x) = \chi_D(z)\,\chi_{\tilde{D}}(z), \tag{2.74}$$

which clearly implies the assertion of the Lemma.

In the discrete case, Lemmas 8 and 9 read:

LEMMA 10. *If $z\left(\sigma_k^{(n)}\right)$ is a non-negative function of the discrete variable $\sigma_k^{(n)}$ and α, β are non negative real numbers with $\alpha+\beta=1$, then,*

$$(2.75)\qquad (\alpha z D^{(n)} x - \beta x, D^{(n)}(\alpha z D^{(n)} x - \beta x))^{\frac{1}{2}} \leq \delta^{(n)}(x, D^{(n)} x)^{\frac{1}{2}},$$

$$(2.76)\qquad \delta^{(n)} = \max\left\{\left|\alpha \chi_D^{(n)}(z) - \beta\right|, \beta\right\}.$$

LEMMA 11. $(zT^{(n)} x, D^{(n)} zT^{(n)} x) \leq \chi_{D^{(n)}}(z)\, \chi_{\tilde{D}^{(n)}}(z)\,(x, D^{(n)} x)$,

where $$\tilde{D}_{k,h}^{(n)} = \sum_{j=1}^{n} 2\cos j\sigma_k^{(n)} \cos j\sigma_h^{(n)}/j\,(n+1).$$

§ 4. **Properties of S and $S^{(n)}$.** In the concave case, $\varkappa \leq 0$, a significant role is played by the functions of a real variable:

$$(2.77)\qquad H_q(x) = M\{|\varkappa|\}\left\{\int_0^\pi \left\{e^{h_p(\sigma)x}\right\}^q d\sigma\right\}^{\frac{1}{q}}, \qquad 1 < q \leq \infty$$

where $$h_p(\sigma) = \left\{\int_0^\pi D^p(\sigma, s)\, \nu^p(s)\, ds\right\}^{\frac{1}{p}}$$

for the continuous case, and

$$(2.78)\qquad H_q^{(n)}(x) = M\{|\varkappa|\}\left\{\sum_{k=1}^{n}\left[e^{h_p^{(n)}(\sigma_k^{(n)})x}\right]^q \Delta\sigma_k^{(n)}\right\}^{\frac{1}{q}},$$

where $$h_p^{(n)}\left(\sigma_k^{(n)}\right) = \left\{\sum_{h=1}^{n}\left[D_{k,h}\,\nu\left(\sigma_h^{(n)}\right)\right]^p \Delta\sigma_h^{(n)}\right\}^{\frac{1}{p}}$$

in the discrete case. X_p and $X_p^{(n)}$ will designate the greatest root of the equations:

$$(2.79)\qquad \begin{aligned} x &= H_q(x), \\ x &= H_q^{(n)}(x) \end{aligned}$$

respectively. If there is no root we put $X_p = \infty$ ($X_p^{(n)} = \infty$). The relevant facts for our investigation about these functions are summarized in the following Lemma.

LEMMA 12. *a) For every $q > 1$, $H_q(x)$ and $H_q^{(n)}(x)$ are positive, increasing and convex functions of x.*

b) (2.80)
$$\pi^{\frac{1}{q}} M\{|\varkappa|\} e^{\left\{\frac{1}{\pi}\int_0^{\pi} h_p(\sigma)\,d\sigma\right\} x} \leq H_p(x) \leq \pi^{\frac{1}{q}} M\{|\varkappa|\} e^{M\{h_p(\sigma)\} x}$$
$$\pi^{\frac{1}{q}} M\{|\varkappa|\} e^{\left\{\frac{1}{\pi}\sum_1^n h_p^{(n)}\left(\sigma_k^{(n)}\right) \Delta\sigma_k^{(n)}\right\} x} \leq H_p^{(n)}(x) \leq \pi^{\frac{1}{q}} M\{|\varkappa|\} e^{M\left\{h_p^{(n)}\left(\sigma_k^{(n)}\right)\right\} x}.$$

c) Equations (2.79) *have at most two solutions.*

PROOF: *a)* The only questionable part is the convexity. This is, however, by Minkowski's inequality, a direct consequence of the convexity of the exponential.

b) The right half is obtained by replacing $h_p(\sigma)$ by its maximum in the integral representation of $H_q(x)$. The left half is the result of applying the inequality

$$e^{\frac{1}{\pi}\int_0^{\pi} \ln|f(\sigma)|\,d\sigma} \leq \left\{\frac{1}{\pi}\int_0^{\pi} |f(\sigma(|^q d\sigma\right\}^{\frac{1}{q}} \tag{2.81}$$

(or its discrete analogous.) (Cf. HARDY, LITTLEWOOD and POLYA [48], page 138) to the function $e^{h_p(\sigma)}$.

c) The solutions of (2.79) can be interpreted as the intersections of a convex curve with a straight line. Naturally, there are at most two such intersections.

LEMMA 13. *If* $\nu \in L_p$, $p > 1$, *then for every* $q_0 > q = p/(p-1)$, $S|\lambda|$ *and* $S\lambda$ *are totally continuous operators in the space of all functions* $\lambda(\sigma)$ *such that* $\|\lambda/\nu\|_{q_0} < \infty$ *More precisely,* $(^1/_\nu) S|\lambda|$ and $(^1/_\nu) S\lambda$ *transform every bounded set into a family of uniformly bounded equicontinuous functions* (Cf. COURANT - HILBERT [53], page 49). *Moreover,* $S|\lambda|$ *maps the whole space into the sphere* $\|\lambda/\nu\|_{q_0} \leq M(|\varkappa|)$, *and* $S\lambda$ *the sphere* $\|\lambda/\nu\|_{q_0} \leq X_{q_0}$ *into itself.*

PROOF: By Hölder inequality

$$\|\lambda\|_u = \left\{\int_0^{\pi} |\lambda|^u d\sigma\right\}^{\frac{1}{u}} = \left\{\int_0^{\pi} |\lambda/\nu|^u |\nu|^u d\sigma\right\}^{\frac{1}{u}} \leq \tag{2.82}$$
$$\leq \left\{\int_0^{\pi} |\lambda/\nu|^{uq_1} d\sigma\right\}^{\frac{1}{uq_1}} \left\{\int_0^{\pi} |\nu|^{up_1} d\sigma\right\}^{\frac{1}{up_1}} = \|\lambda/\nu\|_{uq_1} \|\nu\|_{up_1}$$

for every $u > 0$, $p_1 > 1$, $q_1 > 1$ satisfying $(1/p_1)+(1/_1)=1$. Thus if u, p_1, q_1 are taken as to have $uq_1 = q_0$, $p_1 = p$, that is, $u = \dfrac{qq_0}{qq_0-(q_0-q)} > 1$, $q_1 = 1 + q_1/p$, the above inequality becomes, $\|\lambda\|_u \leq \|\lambda/\nu\|_{q_0} \|\nu\|_p$.

Hence, every bounded set with regard to the norm $\| \lambda/\nu \|_{q_0}$ is bounded with regard to $\| \lambda \|_u$, and by Lemmas 1, *b)*, 2, *b)*, D and T transform it into a family of uniformly bounded equicontinuous functions. Because of the continuity of $\varkappa$ and of the exponential, the same is true for the transformations $1/_\nu \, S \, |\lambda| = \varkappa (T \, |\lambda|) \, e^{-D|\lambda|}$ and $1/_\nu \, S \lambda = \varkappa \, (T \lambda) \, e^{-D(\lambda)}$. Since from every such set of functions, a uniformly convergent (a fortiori, convergent with regard to $\| \lambda/\nu \|_{q_0}$) subsequence can be extracted, the set is compact.

The operator D transforms non-negative functions into non negative ones, hence $1/_\nu \, S \, |\lambda| \leq M(|\varkappa|)$, $\| 1/_\nu \, S \, |\lambda| \|_{q_0} \leq M(|\varkappa|)$ and the assertion with regard to the range of $S \, |\lambda|$ is proved. As to the range of $S \lambda$ the situation is less simple. The statement of the Lemma is obvious is $X_q = \infty$, so we have to consider the case $X_q < \infty$ only. Applying Hölder inequality to the definition of $\tau = D \lambda$ we get

$$(2.83) \qquad |\tau| = \left| \int_0^\pi D(\sigma, s) \, \nu(s) \, \lambda(s)/\nu(s) \, ds \right| \leq$$

$$\leq \left\{ \int_0^\pi |D(\sigma, s) \, \nu(s)|^{p_0} ds \right\}^{\frac{1}{p_0}} \left\{ \int_0^\pi |\lambda(s)/\nu(s)|^{q_0} ds \right\}^{\frac{1}{q_0}} = h_{p_0}(\sigma) \, \| \lambda/\nu \|_{q_0},$$

so,

$$(2.84) \qquad \left| \frac{1}{\nu} \, S \lambda \right| = |\varkappa (T \lambda) \, e^{-D\lambda}| \leq M \{ |\varkappa| \} \, e^{|\tau|} \leq M \{ |\varkappa| \} \, e^{h_{p_0}(\sigma) \|\lambda/\nu\|_{q_0}},$$

and taking q_0-norms in both sides,

$$(2.85) \qquad \left\| \frac{1}{\nu} \, S \lambda \right\|_{q_0} \leq M \{ |\varkappa| \} \left\{ \int_0^\pi [e^{h_{p_0}(\sigma) \|\lambda/\nu\|_{q_0}}]^{q_0} d\sigma \right\}^{\frac{1}{q_0}} = H_{q_0}(\| \lambda/\nu \|_{q_0}).$$

Now, if $\| \lambda/\nu \|_{q_0} \leq X_{q_0}$ by the monotonicity of H_{q_0}, $H_{q_0}(\| \lambda/\nu \|_{q_0}) \leq \leq H_{q_0}(X_{q_0}) = X_{q_0}$, $\left\| \frac{1}{\nu} \, S \lambda \right\|_{q_0} \leq X_{q_0}$, and the proof is completed.

In the discrete case, the reader will find no difficulty in proving the following analogue of Lemma 13 :

LEMMA 14. *For every $q > 1$, the operador $S^{(n)} \, |\lambda|$ maps the whole space $E^{(n)}$ into the sphere $\| \lambda/\nu \|_q \leq M(|\varkappa|)$, and the operator $S^{(n)} \lambda$ maps the sphere $\| \lambda/\nu \|_q \leq X_q^{(n)}$ into itself.*

With these Lemmas we have reached the point where the existence of solutions of (1) and (1.37) can be almost immediately established in a large class of cases if we only give up to the requirement of constructivity. It seems worthwhile to digress from our main purpose to gather these early results.

THEOREM 1. *If* $\nu \in L_p$ *and* $\varkappa$ *is continuous, equation* (1) *has a solution for all non-negative* $\varkappa$'*s*; *it also has a solution for the non-positive* $\varkappa$'*s for which there is a* q_0 *with* $X_{q_0} < \infty$.

PROOF: If $\varkappa \geqq 0$, every solution of $\lambda = S\lambda$ is non-negative and so is a solution of $\lambda = S|\lambda|$, and reciprocally. By Lemma (13) the operator $S|\lambda|$ is totally continuous and maps the sphere $\| \lambda/\nu \|_{q_0} \leq M(|\varkappa|)$ into itself, so by the SCHAUDER-LERAY [20] extension of Brouwer Fixpoint Theorem, there is a fixed point under the transformation $S|\lambda|$, that is, a solution of $\lambda = S|\lambda|$. If $\varkappa \leq 0$, and $X_{q_0} < \infty$ the same argument applied to $S\lambda$ leads to the existence of a solution of $\lambda = S\lambda$. Similarly in the discrete case.

III. THE ITERATION PROCESS

Equations (1.26) and (1.38) appear in a form suitable for iteration. Such a method has actually been used to solve (1) and the numerical computations have shown that simple iteration is relatively weak and yields convergence for sufficiently flat obstacles only, that is, for sufficiently small ν's. Moreover if, in the covex case, instead of applying simple iteration one proceeds by averaged iteration (where a new iterate is the weighted average of the previous iterate and its transformed by S) the range of convergence appears to be considerably increased. Such is, for instance, the case with convex circular obstacles ($\varkappa$ = const. > 0) where one always gets convergence with averaged iteration.

In this section we shall try to provide a theoretical support for this numerical behaviour. We observe, in the first place, that simple iteration corresponds to the succesive application of the operator S while averaged iteration results from the repeated application of the averaged operator $S_\alpha = \alpha S + (1 - \alpha) I$, where I is the identity and α a positive number. It is clear that all equations $\lambda = S_\alpha \lambda$ have the same solutions, namely those of $\lambda = S\lambda$, and yet that the operators S_α may behave quite differently under iteration. It is, therefore, a natural question to ask which advantage can be taken from this situation to increase, by means of a judicious choice of α, the range of effectiveness of the iteration process. A partial answer to this question is to be found in Theorems 2 and 4 which, in the continuous and discrete case respectively, give intervals where α can be chosen to insure convergence together with bounds for the error commited after the n^{th} iteration.

§ 1. **Convex case, $\varkappa \geq 0$.** THEOREM 2. *Let $\varkappa(\theta)$ be a non-negative continuous function with bounded derivative, $\nu \in L_p$, and $M\{|\varkappa'(\theta)|\}[\chi_D(\nu)\chi_{\bar{D}}(\nu)]^{\frac{1}{2}} < 1$. Then for every α in the interval*

$$(3.1) \qquad 0 < \alpha < 2/[M\{\varkappa\}\chi_D(\nu) + M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu) + 1],$$

the successive averaged interates $\lambda_k = S_\alpha^k \lambda_0$ of a non-negative function λ_0, converge towards a solution λ of (1). More precisely, if $0 \leq \lambda_0 \leq M\{\varkappa\}$, then for every r between 0 and 1,

$$(3.2) \qquad |\lambda(\sigma) - \lambda_k(\sigma)| \leq \alpha\nu(\sigma)\left\{A\frac{\gamma^k - \beta^k}{\gamma - \beta}\left[\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + \right.$$
$$\left. + B\frac{1-\beta^k}{\alpha}\left[\ln\frac{1}{r}\right]^{\frac{1}{q}}\right\} + \beta^k[\lambda(\sigma) - \lambda_0(\sigma)],$$

where

$$(3.3) \qquad \begin{aligned} \beta &= 1 - \alpha, \\ \gamma &= \max\{|\alpha M\{\varkappa\}\chi_D(\nu) - \beta|, \beta\} + \alpha M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu), \\ A &= [(\pi/2)(M^2\{\varkappa\} + M^2\{|\varkappa'|\})(S_\alpha\lambda_0 - \lambda_0, D(S_\alpha\lambda_0 - \lambda_0))]^{\frac{1}{2}}. \\ B &= (M^2\{\varkappa\} + M^2\{|\varkappa'|\})^{\frac{1}{2}} 2C_p M\{\varkappa\}\|\nu\|_p. \ (*) \end{aligned}$$

The convergence is uniform in every closed interval where ν is bounded.

PROOF: Let us put $\theta_k = T\lambda_k$, $\tau_k = D\lambda_k$ and compute the difference between two consecutive iterates $\lambda_{k+1} - \lambda_k$,

$$(3.4) \qquad \begin{aligned} \lambda_{k+1} - \lambda_k &= S_\alpha\lambda_k - S_\alpha\lambda_{k-1} = \\ &= \alpha\nu[\varkappa(T\lambda_k)e^{-D\lambda_k} - \varkappa(T\lambda_{k-1})e^{-D\lambda_{k-1}}] + \beta(\lambda_k - \lambda_{k-1}). \end{aligned}$$

By simple algebraic manipulations this can be written as

$$(3.5) \qquad \delta_{k+1} = -\alpha\omega_k D\delta_k + \beta\delta_k + \alpha\varrho_k T\delta_k,$$

where

$$(3.6) \qquad \begin{aligned} \delta_k &= \lambda_k - \lambda_{k-1}, \\ \omega_k &= \nu\varkappa(\theta_k)(e^{-\tau_k} - e^{-\tau_{k-1}})/(-\tau_k + \tau_{k-1}), \\ \varrho_k &= \nu e^{-\tau_{k-1}}(\varkappa(\theta_k) - \varkappa(\theta_{k-1}))/(\theta_k - \theta_{k-1}). \end{aligned}$$

* C_p appearing in this formula has been defined in (2.30).

Since S_α transforms non-negative functions into non-negative ones, all the λ_k are non-negative and the following bounds hold for ω_k, ϱ_k,

$$0 \leq \omega_k \leq \nu M\{\varkappa\} \tag{3.7}$$

$$|\varrho_k| \leq \nu M\{|\varkappa'|\} \tag{3.8}$$

We now show that $(\delta_k, D\delta_k)$ is reduced at every iteration provided that α is taken within the indicated bounds. Since $(x, Dx)^{\frac{1}{2}}$ satisfies the triangle inequality,

$$\begin{aligned}(\delta_{k+1}, D\delta_{k+1})^{\frac{1}{2}} \leq (-\alpha\omega_k D\delta_k + \beta\delta_k, D(-\alpha\omega_k D\delta_k + \beta\delta_k))^{\frac{1}{2}} + \\ + \alpha(\varrho_k T\delta_k, D\varrho_k T\delta_k)^{\frac{1}{2}},\end{aligned} \tag{3.9}$$

and by Lemmas (8) and (9),

$$(\delta_{k+1}, D\delta_{k+1})^{\frac{1}{2}} \leq \gamma_k(\delta_k, D\delta_k)^{\frac{1}{2}}, \tag{3.10}$$

where, $\gamma_k = \max\{|\alpha\chi_D(\omega_k) - \beta|, \beta\} + \alpha\chi_D^{1/2}(\varrho_k)\chi_{\bar{D}}^{1/2}(\varrho_k)$. Because of the monotonicity of χ_D (Lemma 7, *e*)), the right hand side is increased if ω_k and ϱ_k are replaced by their bounds (3.7) and (3.8), so

$$\gamma_k \leq \gamma = \max\{|\alpha M\{\varkappa\}\chi_D(\nu) - \beta|, \beta\} + \alpha M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu). \tag{3.11}$$

A simple computation gives

$$\gamma = \begin{cases} \beta + \alpha M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu), \quad \text{for} \quad \alpha \leq 2/(M\{\varkappa\}\chi_D(\nu)+2); \\ \alpha[M\{\varkappa\}\chi_D(\nu) + M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu)] - \beta, \\ \text{for} \qquad 2/[M\{\varkappa\}\chi_D(\nu) + 2] \leq \alpha \leq 1, \end{cases} \tag{3.12}$$

showing that $\gamma < 1$ and that the minimum of γ is: $[M\{\varkappa\}\chi_D(\nu) + 2M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(v)]/[M\{\varkappa\}\chi_D(\nu) + 2]$, attained for

$$\alpha = 2/(M\{\varkappa\}\chi_D(\nu) + 2).$$

From (3.10) and (3.11) it follows

$$(\delta_k, D\delta_k)^{\frac{1}{2}} \leq \gamma^{k-1}(\delta_1, D\delta_1)^{\frac{1}{2}} \tag{3.13}$$

and by the triangle inequality, if $k > h$,

$$(3.14)\qquad (\lambda_k - \lambda_h, D(\lambda_k - \lambda_h))^{\frac{1}{2}} = (\sum_{i=h+1}^{k} \delta_i, D \sum_{i=h+1}^{k} \delta_i)^{\frac{1}{2}} \leq$$
$$\leq \sum_{i=h+1}^{k} (\delta_i, D\delta_i)^{\frac{1}{2}} \leq (\delta_1, D\delta_1)^{\frac{1}{2}} \sum_{i=h+1}^{k} \gamma^{i-1} \leq (\delta_1, D\delta_1)^{\frac{1}{2}} \frac{\gamma^h}{1-\gamma}.$$

Therefore $\{\lambda_k\}$ is a Cauchy sequence under the metric defined by the norm $(x, Dx)^{\frac{1}{2}}$. From this fact we shall deduce, with the help of Lemma 5, the uniform convergence of the sequence $\{\lambda_k\}$ towards a solution of (1.26). The λ_k all lie in the interval $(0, \nu(\sigma) M\{\varkappa\})$ and so $\|\lambda_k - \lambda_h\|_p \leq$ $\leq 2M\{\varkappa\}\|\nu\|_p$. Hence, by Lemma 5, and (3.14),

$$(3.15)\qquad \{[\theta_k(\sigma) - \theta_h(\sigma)]^2 + [\tau_k(\sigma) - \tau_h(\sigma)]^2\} \leq$$
$$\leq (\delta_1, D\delta_1)^{\frac{1}{2}} \frac{\gamma^h}{1-\gamma}\left[\frac{2}{\pi}\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + 2C_pM\{\varkappa\}\|\nu\|_p\left[\ln\frac{1}{r}\right]^{\frac{1}{q}},$$

which shows that the sequences $\theta_k(\sigma)$ and $\tau_k(\sigma)$ converge uniformly towards limits $\theta(\sigma)$ and $\tau(\sigma)$ respectively. Now if $\mu_k = \nu\varkappa(\theta_k)e^{-\tau_k}$,

$$(3.16)\qquad |\mu_k(\sigma) - \mu_h(\sigma)| = |\nu\varkappa(\theta_k)e^{-\tau_k} - \nu\varkappa(\theta_h)e^{-\tau_h}| \leq$$
$$\leq \nu(\sigma)[M\{\varkappa\}|\tau_k(\sigma) - \tau_h(\sigma)| + M\{|\varkappa'|\}|\theta_k(\sigma) - \theta_h(\sigma)|],$$

and by Schwarz inequality,

$$(3.17)\qquad |\mu_k(\sigma) - \mu_h(\sigma)| \leq \nu(\sigma)[M^2\{\varkappa\} + M^2\{|\varkappa'|\}]^{\frac{1}{2}}$$
$$\{[\theta_k(\sigma) - \theta_h(\sigma)]^2 + [\tau_k(\sigma) - \tau_h(\sigma)]^2\}^{\frac{1}{2}},$$

and $\mu_k(\sigma)$ converges towards a function $\lambda(\sigma)$ uniformly in any closed interval where $\nu(\sigma)$ is bounded. (3.15) and (3.16) yield, by letting $k \to \infty$,

$$(3.18)\qquad |\lambda(\sigma) - \mu_h(\sigma)| \leq \nu(\sigma)[M^2\{\varkappa\} + M^2\{|\varkappa'|\}]^{\frac{1}{2}}$$
$$\times\left\{(\delta_1, D\delta_1)\frac{\gamma^h}{1-\gamma}\left[\frac{2}{\pi}\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + 2C_pM\{\varkappa\}\|\nu\|_p\left[\ln\frac{1}{r}\right]^{\frac{1}{q}}\right\}.$$

For brevity, this we write as

$$(3.19)\qquad |\lambda(\sigma) - \mu_h(\sigma)| \leq \nu(\sigma)\left\{A\gamma^h\left[\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + B\left[\ln\frac{1}{r}\right]^{\frac{1}{q}}\right\}.$$

We now prove that $\lambda_k(\sigma)$ converges towards $\lambda(\sigma)$ in the same way as $\mu_k(\sigma)$. By definition

$$\begin{aligned} \lambda_k(\sigma) &= \alpha\,\mu_{k-1}(\sigma) + \beta\,\lambda_{k-1}(\sigma), \\ \lambda(\sigma) - \lambda_k(\sigma) &= \alpha\,[\lambda(\sigma) - \mu_{k-1}(\sigma)] + \beta\,[\lambda(\sigma) - \lambda_{k-1}(\sigma)]. \end{aligned} \tag{3.20}$$

Iterating,

$$\lambda(\sigma) - \lambda_h(\sigma) = \alpha \sum_{k=0}^{h-1} \beta^k\,[\lambda(\sigma) - \mu_{n-1-k}(\sigma)] + \beta^h\,[\lambda(\sigma) - \lambda_0(\sigma)]. \tag{3.21}$$

Taking modulus on both sides and applying (3.19),

$$\begin{aligned} |\lambda(\sigma) - \lambda_h(\sigma)| \le \alpha\,\nu(\sigma)\Big\{ A\,(\sum_{k=1}^{h-1} \beta^k \gamma^{n-1-k}) \Big[\ln \frac{1}{1-r}\Big]^{\frac{1}{2}} + \\ + B\,(\sum_{k=1}^{h-1} \beta^k) \Big[\ln \frac{1}{r}\Big]^{\frac{1}{q}} \Big\} + \beta^h\,\{\lambda(\sigma) - \lambda_0(\sigma)\}, \end{aligned} \tag{3.22}$$

that is,

$$\begin{aligned} |\lambda(\sigma) - \lambda_h(\sigma)| \le \alpha\,\nu(\sigma)\Big\{ A\,\frac{\gamma^h - \beta^h}{\gamma - \beta} \Big[\ln \frac{1}{1-r}\Big]^{\frac{1}{2}} + \\ + B\,\frac{1-\beta^h}{\alpha} \Big[\ln \frac{1}{r}\Big]^{\frac{1}{q}} \Big\} + \beta^h\,\{\lambda(\sigma) - \lambda_0(\sigma)\}. \end{aligned} \tag{3.23}$$

Since γ and β are both less than 1 and $\lambda(\sigma) \le M\{\varkappa\}\,\nu(\sigma)$, the sequence $\lambda_h(\sigma)$ converges, due to the arbitrariness of r, uniformly towards $\lambda(\sigma)$ in any closed interval where $\nu(\sigma)$ is bounded. Finally, since the operator S is continuous,

$$S_\alpha \lambda = \lim_{h \to \infty} S_\alpha \lambda_h = \lim_{h \to \infty} \lambda_{h+1} = \lambda \tag{3.24}$$

That is, $\alpha S \lambda + \beta \lambda = \lambda$, $S\lambda = \lambda$, and the Theorem is proved.

We shall now investigate a formula giving the variation of the solution of (1.26) when the data ν and $\varkappa$ are altered.

Theorem 3. Let
$$\lambda_1 = \nu_1 \varkappa_1 (T\lambda_1)\, e^{-D\lambda_1},$$
$$\lambda_2 = \nu_2 \varkappa_2 (T\lambda_2)\, e^{-D\lambda_2},$$

where $\nu_1, \nu_2 \in L_p$, $p > 1$ and $\varkappa_1, \varkappa_2$ are non-negative continuous functions with bounded first derivatives. If, $M\{|\varkappa_2'|\}\,\chi_D^{1/2}(\nu_2)\,\chi_{\bar{D}}^{1/2}(\nu_2) \le 1$, then

$$(3.25)\qquad (\lambda_1 - \lambda_2,\ D(\lambda_1 - \lambda_2))^{\frac{1}{2}} \leq$$
$$\leq \frac{M\{\varkappa_1\}\,((\nu_1 - \nu_2),\ D(\nu_1 - \nu_2))^{\frac{1}{2}} + M\{|\varkappa_1 - \varkappa_2|\}\,(\nu_2,\ D\nu_2)^{\frac{1}{2}}}{1 - M\{|\varkappa_2'|\}\,\chi_D^{1/2}(\nu_2)\,\chi_{\bar{D}}^{1/2}(\nu_2)}.$$

Proof: The difference between the first and second equation may be written in the form

$$(3.26)\qquad \delta = -\,\omega D\delta + \varrho T\delta + \varepsilon,$$

where,

$$(3.27)\qquad \begin{aligned} &\delta = \lambda_1 - \lambda_2, \quad \theta_1 = T\lambda_1, \quad \theta_2 = T\lambda_2, \quad \tau_1 = D\lambda_1, \quad \tau_2 = D\lambda_2, \\ &\omega = \nu_2\varkappa_2(\theta_1)\,(e^{-\tau_1} - e^{-\tau_2})/(-\,\tau_1 + \tau_2), \\ &\varrho = \nu_2 e^{-\tau_2}\,(\varkappa_2(\theta_1) - \varkappa_2(\theta_2))/(\theta_1 - \theta_2), \\ &\varepsilon = (\nu_1 - \nu_2)\,\varkappa_1(\theta_1)\,e^{-\tau_1} + \nu_2 e^{-\tau_1}\,(\varkappa_1(\theta_1) - \varkappa_2(\theta_1)). \end{aligned}$$

Let $\alpha + \beta = 1$, $0 < \alpha \leq 1$. Multiply (3.26) by α and add $\beta\delta$ to both members,

$$(3.28)\qquad \delta = -\,\alpha\omega D\delta + \alpha\varrho T\delta + \beta\delta + \alpha\varepsilon.$$

Now estimate $(\delta, D\delta)^{\frac{1}{2}}$ by using the triangle ineguality,

$$(3.29)\qquad (\delta,\ D\delta)^{\frac{1}{2}} \leq$$
$$\leq ((-\,\alpha\omega D\delta + \alpha\varrho T\delta + \beta\delta),\ D(-\,\alpha\omega D\delta + \alpha\varrho T\delta + \beta\delta))^{\frac{1}{2}} + \alpha(\varepsilon,\ D\varepsilon)^{\frac{1}{2}}.$$

As in Theorem 2, the firts term is not greater than $\gamma(\delta, D\delta)^{\frac{1}{2}}$ where $\gamma = \max\,[|\alpha\chi_D(\omega) - \beta|,\ \beta] + \alpha\chi_D^{1/2}(\varrho)\,\chi_{\bar{D}}^{1/2}(\varrho)$. From the definition of ϱ, it follows $|\varrho| \leq M\{|\varkappa_2'|\}\,\nu_2$, so by Lemma 7, *e)* and by Hypothesis, $\chi_D^{1/2}(\varrho)\,\chi_{\bar{D}}^{1/2}(\varrho) \leq M\{|\varkappa_2'|\}\,\chi_D^{1/2}(\nu_2)\,\chi_{\bar{D}}^{1/2}(\nu_2)$. Consequently, if α is taken to be $2/(\chi_D(\omega) + 2)$ then $\gamma = \dfrac{\chi_D(\omega) + 2\chi_D^{1/2}(\varrho)\,\chi_{\bar{D}}^{1/2}(\varrho)}{\chi_D(\omega) + 2} < 1$, and we can write

$$(3.30)\qquad (\delta,\ D\delta)^{\frac{1}{2}} \leq \gamma(\delta,\ D\delta)^{\frac{1}{2}} + \alpha(\varepsilon,\ D\varepsilon)^{\frac{1}{2}},$$
$$(\delta, D\delta)^{\frac{1}{2}} \leq \frac{\alpha}{1-\gamma}(\varepsilon, D\varepsilon)^{\frac{1}{2}} \leq \frac{(\varepsilon,\ D\varepsilon)^{\frac{1}{2}}}{1 - M\{|\varkappa_2'|\}\,\chi_D^{1/2}(\nu_2)\,\chi_{\bar{D}}^{1/2}(\nu_2)}.$$

Again, by the triangle inequality $(\varepsilon, D\varepsilon)^{\frac{1}{2}}$ can be split into two parts corresponding to each of the terms in the expression for ε. The first clearly does not exceed $M\{\varkappa_1\}(|\nu_1-\nu_2|, D|\nu_1-\nu_2|)^{\frac{1}{2}}$, and the second $M\{|\varkappa_1-\varkappa_2|\}(\nu_1, D\nu_2)^{\frac{1}{2}}$, so

$$(3.31)\quad (\varepsilon, D\varepsilon)^{\frac{1}{2}} \leq M\{\varkappa_1\}(|\nu_1-\nu_2|, D|\nu_1-\nu_2|)^{\frac{1}{2}} + M\{|\varkappa_1-\varkappa_2|\}(\nu, D\nu_2)^{\frac{1}{2}}.$$

Inserting this into (3.30) we obtain (3.25).

Letting $\nu_1 = \nu_2$, $\varkappa_1 = \varkappa_2$ we obtain as a Corolary the following result.

THEOREM 4. *Under the conditions* $\varkappa \geqslant 0$, $M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu) < 1$, *equation* (1) *admits one and only one solution.*

PROOF: If λ_1 and λ_2 are two solutions (1), the previous Theorem yields, $(\lambda_1-\lambda_2, D(\lambda_1-\lambda_2)) = 0$ which by Lemma 1, *d)* implies $\lambda_1(\sigma) = \lambda_2(\sigma)$ almost everywhere. But the functions λ_1/ν and λ_2/ν are continuous and so they must be identical.

The same formal proofs lead to the following discrete analogs to Theorems, 2, 3, 4:

THEOREM 5. *Let* $\varkappa(\theta)$ *be a non-negative continuous function with bounded derivative, and* $M\{|\varkappa'(\theta)|\}\chi_{D^{(n)}}^{1/2}(\nu)\chi_{\bar{D}^{(n)}}^{1/2}(\nu) < 1$. *Then for every* α *in the interval*

$$(3.32)\quad 0 < \alpha < 2/[M\{\varkappa\}\chi_{D^{(n)}}(\nu) + M\{|\varkappa'|\}\chi_{D^{(n)}}^{1/2}(\nu)\chi_{\bar{D}^{(n)}}^{1/2}(\nu) + 1]$$

the successive averaged iterates $\lambda_k = (S_\alpha^{(n)})^k\lambda_0$ *of a non-negative function* $\lambda_0(\sigma_h^{(n)})$ *converge towards a solution* $\lambda(\sigma_h^{(n)})$ *of* (1.37). *More precisely, if* $0 \leq \lambda_0(\sigma_h^{(n)}) \leq \nu(\sigma_h^{(n)})M\{\varkappa\}$, *then for every* r *between* 0 *and* 1,

$$(3.33)\quad |\lambda(\sigma_h^{(n)}) - \lambda_k(\sigma_h^{(n)})| \leq \nu(\sigma_h^{(n)})\left\{A^{(n)}\frac{\gamma^k-\beta^k}{\gamma-\alpha}\left[\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + \right.$$
$$\left. + B^{(n)}\frac{1-\beta^k}{\alpha}\left[\ln\frac{1}{r}\right]^{\frac{1}{q}}\right\} + \beta^k\{\lambda(\sigma_h^{(n)}) - \lambda_0(\sigma_h^{(n)})\}$$

where

$$(3.34)\quad \beta = 1-\alpha$$
$$\gamma = \max\{|\alpha M\{\varkappa\}\chi_{D^{(n)}}(\nu) - \beta|, \beta\} + \alpha M|\varkappa'|\}\chi_{D^{(n)}}^{1/2}(\nu)\chi_{\bar{D}^{(n)}}^{1/2}(\nu)$$
$$A^{(n)} = \left[\frac{\pi}{2}(M^2\{\varkappa\} + M^2\{|\varkappa'|\})(S_\alpha^{(n)}\lambda_0 - \lambda_0, D^{(n)}(S_\alpha^{(n)}\lambda_0 - \lambda_0))\right]^{\frac{1}{2}}$$
$$B^{(n)} = (M^2\{\varkappa\} + M^2\{|\varkappa'|\})^{\frac{1}{2}}\, 2\, C_p^* M\{\varkappa\}\|\nu\|_p.$$

THEOREM 5. *Let*
$$\lambda_1 = \nu_1 \varkappa_1 (T^{(n)} \lambda_1) e^{-D^{(n)}\lambda_1}$$
$$\lambda_2 = \nu_2 \varkappa_2 (T^{(n)} \lambda_2) e^{-D^{(n)}\lambda_2}$$

where $\varkappa_1$, $\varkappa_2$ *are non negative continuous functions with bounded first derivatives. If,* $M\{|\varkappa_2'|\} \chi_{D^{(n)}}^{1/2}(\nu_2) \chi_{\tilde{D}^{(n)}}^{1/2}(\nu_2) \leq 1$, *then,*

$$(\lambda_1 - \lambda_2, D^{(n)}(\lambda_1 - \lambda_2))^{\frac{1}{2}} \leq \tag{3.35}$$
$$\leq \frac{M\{\varkappa_1\}((\nu_1 - \nu_2), D^{(n)}(\nu_1 - \nu_2))^{\frac{1}{2}} + M\{|\varkappa_1 - \varkappa_2|\}(\nu_2, D^{(n)}\nu_2)^{\frac{1}{2}}}{2 - M\{|\varkappa_2'|\} \chi_{D^{(n)}}^{1/2}(\nu_2) \chi_{\tilde{D}^{(n)}}^{1/2}(\nu_2)}.$$

THEOREM 6. *Under the conditions* $\varkappa \geqslant 0$, $M\{|\varkappa'|\} \chi_{D^{(n)}}^{1/2}(\nu) \chi_{\tilde{D}^{(n)}}^{1/2}(\nu) < 1$, *equation* (1.37) *admits one and only one solution.*

§ **2. Concave case, $\varkappa \leq 0$.** In the concave case the situation is essentially more complicated because equation (1) not always has a solution. In the iteration process this roughly corresponds to the fact that the successive iterates do not remain bounded under any metric. Therefore, a choice of the data to prevent this from happening is required before entering into the convergence of the process. Such a choice will be based on Lemma 13, in which case we also know the existence of a solution (Theorem 1). Moreover, averaged iteration does not, in general, show any advantage and shall not be used. To substantiate our assertion that (1) does not always have a solution, we shall first give a simple and non trivial necessary condition for its solvability.

THEOREM 7. *If* $\varkappa \leq 0$, $N = (1/2)\int_0^\pi \ln [\nu(s)\, m\{|\varkappa|\}] \sin s\, ds \leq -1$, *is a necessary condition for the existence of a solution of* (1). *If the condition is fulfilled, then for all solutions of* (1), $-(1/2)\int_0^\pi \sin s\, \lambda(s)\, ds$ *lie between the smallest and largest root of the equation* $x = e^{x+N}$.

PROOF: If $\varkappa \leq 0$, every solution of (1) is negative and $|\lambda| = \nu(\sigma)\, |\varkappa(T\lambda)| e^{D|\lambda|}$. So if $\theta = T\lambda$, $\tau = D\lambda$, then

$$\frac{1}{2}\int_0^\pi |\lambda(s)| \sin s\, ds = \frac{1}{2}\int_0^\pi \nu(s)\, |\varkappa(\theta(s))|\, e^{|\tau(s)|} \sin s\, ds.$$

But, the arithmetic mean being always greater than the geometric one, (HARDY, LITTLEWOOD and POLYA [48], page 137).

$$(3.36)\quad \frac{1}{2}\int_0^\pi \nu(s)\,|\varkappa(\theta(s))|\,e^{|\tau(s)|}\sin s\,ds \geqslant e^{\frac{1}{2}\int_0^\pi \{\ln[\nu(s)\,|\varkappa(\theta(s))|] + |\tau(s)|\}\sin s\,ds}$$

and since $\int_0^\pi |\lambda(s)|\sin s\,ds = \int_0^\pi |\tau(s)|\sin s\,ds,$

$$(3.37)\quad \frac{1}{2}\int_0^\pi |\lambda(s)|\sin s\,ds \geqslant e^{\frac{1}{2}\int_0^\pi \ln[\nu(s)\,|\varkappa(\theta(s))|]\sin s\,ds + \frac{1}{2}\int_0^\pi |\lambda(s)|\sin s\,ds} \geqslant$$

$$\geqslant e^{N + \frac{1}{2}\int_0^\pi |\lambda(s)|\sin s\,ds}.$$

So,

$$(3.38)\quad e^{-N} \geqslant \frac{e^{\frac{1}{2}\int_0^\pi |\lambda(s)|\sin s\,ds}}{\frac{1}{2}\int_0^\pi |\lambda(s)|\sin s\,ds} \geqslant \underset{0\leq x\leq\infty}{\text{g. l. b.}}\ \frac{e^x}{x} = e$$

and $N \leq -1$. Obviously, if $N \leq -1$, the equation $x = e^{x+N}$ has two rots and every x such that $x \geqslant e^{x+N}$ lies between then. The proof of the Theorem is, therefore, completed. We now prove the fundamental Theorem of Convergence for this case, which follows closely the main lines of Theorem 2.

THEOREM 8. *Let $\varkappa(\theta)$ be a non positive, continuous function with bounded derivative and $\nu \in L_p$, $p > 1$. If for some $q_0 > q$,*

$$(3.39)\quad \gamma = M\{|\varkappa|\}\chi_D(\nu^*) + M\{|\varkappa'|\}\chi_D^{1/2}(\nu^*)\chi_{\bar{D}}^{1/2}(\nu^*) < 1,$$
$$\nu^*(\sigma) = \nu(\sigma)\,e^{h\nu_0(\sigma)X_{q_0}},$$

then the iterates $\lambda_k = S^k\lambda_0$ of a function with $\|\lambda_0/\nu\|_{q_0} \leq X_{q_0}$ converge towards a solution λ of (1). Precisely, for every r, $0 < r < 1$,

$$(3.40)\quad |\lambda - \lambda_h| \leq \nu^*(\sigma)\left\{A\,\frac{\gamma^h}{1-\gamma}\left[\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + B\left[\ln\frac{1}{r}\right]^{\frac{1}{q}-\frac{1}{q_0}}\right\},$$

where

$$(3.41)\quad A = \left\{\frac{\pi}{2}\left(M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\right)\left(S\lambda_0 - \lambda_0,\ D(S\lambda_0 - \lambda_0)\right)^{\frac{1}{2}}\right\},$$
$$B = \left[M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\right]^{\frac{1}{2}}\,2\,C_{\frac{1}{1-\left(\frac{1}{q}-\frac{1}{q_0}\right)}}\,X_{q_0}\,\|\nu\|_p.$$

PROOF: Let us write $\theta_k = T\lambda_k$, $\tau_k = D\lambda_k$, then

$$(3.42)\quad \lambda_{k+1} - \lambda_k = S\lambda_k - S\lambda_{k-1} = \nu\varkappa(\theta_k)e^{-\tau_k} - \nu\varkappa(\theta_{k-1})e^{-\tau_{k-1}},$$

which, as in Theorem 2, can be written

$$(3.43)\quad \delta_{k+1} = -\omega_k D\delta_k + \varrho_k T\delta_k,$$

with the meanings given in (3.6). By Lemma 13 all iterates are in the sphere $\|\lambda/\nu\|_{q_0} \leq X_{q_0}$, and by (2.83), $|\tau_k| \leq h_{p_0}(\sigma)\,\|\lambda/\nu\|_{q_0} \leq h_{p_0}(\sigma)\,X_{q_0}$. This implies the bounds.

$$(3.44)\quad \begin{aligned} 0 \leq -\omega_k &\leq M\{|\varkappa|\}\,\nu(\sigma)\,e^{h_{p_0}(\sigma)X_{q_0}} = M\{|\varkappa|\}\,\nu^*(\sigma),\\ |\varrho_k| &\leq M\{|\varkappa'|\}\,\nu(\sigma)\,e^{h_{p_0}(\sigma)X_{q_0}} = M\{|\varkappa'|\}\,\nu^*(\sigma),\end{aligned}$$

Now by the triangle inequality and Lemmas 8 and 9

$$(3.45)\quad \begin{aligned}(\delta_{k+1}, D\delta_{k+1})^{\frac{1}{2}} &\leq (-\omega_k, D(-\omega_k D\delta_k))^{\frac{1}{2}} + \\ &+ (\varrho_k, D\varrho_k T\delta_k)^{\frac{1}{2}} \leq \gamma_k(\delta_k, D\delta_k)^{\frac{1}{2}},\end{aligned}$$

where $\gamma_k = \chi_D(\omega_k) + \chi_D^{1/2}(\varrho_k)\,\chi_{\bar{D}}^{1/2}(\varrho_k)$. By Lemma 7, *c)* and (2.82),

$$(3.46)\quad \gamma_k \leq \gamma < 1.$$

So multiplying equations (3.45) together,

$$(3.47)\quad (\delta_k, D\delta_k)^{\frac{1}{2}} \leq \gamma^{k-1}(\delta_1, D\delta_1)^{\frac{1}{2}},$$

and by triangle inequality, if $k > h$,

$$(3.48)\quad \begin{aligned}(\lambda_k - \lambda_h, D(\lambda_k - \lambda_h))^{\frac{1}{2}} &\leq (\sum_{i=h+1}^{k}\delta_i, D\sum_{i=h+1}^{k}\delta_i)^{\frac{1}{2}} \leq \sum_{i=h+1}^{k}(\delta_i, D\delta_i)^{\frac{1}{2}} \leq \\ &\leq (\delta_1, D\delta_1)^{\frac{1}{2}}\sum_{i=h+1}^{k}\gamma^{i-1} \leq (\delta_1, D\delta_1)^{\frac{1}{2}}\,\frac{\gamma^h}{1-\gamma}.\end{aligned}$$

As in Theorem 2, we shall prove that from this it follows the pointwise convergence, first of θ_k and τ_k and then of λ_k. We only have to recall that, as it was shown in Lemma 13, if $u = qq_0/(qq_0 - (q_0 - q))$ then $\|\lambda\|_u \leq \|\lambda/\nu\|_{q_0}\,\|\nu\|_p$ and apply Lemma 5 to $\lambda_h - \lambda_k$ with u in place of p,

$$(3.49)\quad \begin{aligned}&\{[\theta_k(\sigma) - \theta_h(\sigma)]^2 + [\tau_k(\sigma) - \tau_h(\sigma)]^2\}^{\frac{1}{2}} \leq \\ &\leq (\delta_1, D\delta_1)^{\frac{1}{2}}\,\frac{\gamma^h}{1-\gamma}\left[\frac{2}{\pi}\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + 2C_u X_{q_0}\|\nu\|_p\left[\ln\frac{1}{r}\right]^{1-\frac{1}{u}}\end{aligned}$$

which, since r is arbitrary imply that $\theta_k(\sigma)$ and $\tau_k(\sigma)$ converge uniformly towards two functions $\theta(\sigma)$ and $\tau(\sigma)$ respectively. We now prove that λ_k converges to $\lambda = \nu\varkappa(\theta)e^{-\tau}$. On account of $|\tau_k| \leq h_{p_0}(\sigma)X_{q_0}$ and by Schwarz inequality,

$$\begin{aligned}(3.50)\qquad |\lambda - \lambda_k| &= |\nu\varkappa(\theta)e^{-\tau} - \nu\varkappa(\theta_k)e^{-\tau_k}| \leq \\ &\leq \nu(\sigma)e^{h_{p_0}(\sigma)X_{q_0}}\{M\{|\varkappa|\}|\tau - \tau_k| + M\{|\varkappa'|\}|\theta - \theta_k|\} \leq \\ &\leq \nu^*(\sigma)\{M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\}^{\frac{1}{2}}\{|\theta - \theta_k|^2 + |\tau - \tau_k|^2\}^{\frac{1}{2}}.\end{aligned}$$

Letting $k \to \infty$ in (3.49) and replacing the result in (3.50), we get

$$\begin{aligned}(3.51)\qquad |\lambda - \lambda_k| &\leq \nu^*(\sigma)\left[M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\right]^{\frac{1}{2}} \\ &\times \left\{(\delta_1, D\delta_1)^{\frac{1}{2}}\frac{\gamma^k}{1-\gamma}\left[\frac{2}{\pi}\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + 2C_u X_{q_0}\|\nu\|_p\left[\ln\frac{1}{r}\right]^{1-\frac{1}{u}}\right\},\end{aligned}$$

and the convergence of λ_k towards λ is proved. Clearly the convergence is uniform in any closed interval where $\nu^*(\sigma)$ is bounded. Finally, it is plain from the continuity of the operator S that λ verifies $\lambda = S\lambda$.

In the concave case there is also an analogue of Theorem 3, but of a less precise form. We shall limit ourselves to the unicity theorem that follows from it.

Theorem 9. *Under the Hypothesis of Theorem* 8 *there is one and only one solution of* (1) *with* $\|\lambda/\nu\|_{g_0} \leq X_{q_0}$.

Proof: That there is a solution is the content of the previous Theorem; that there is no more than one is proved as follows: from $\lambda_1 = \nu\varkappa(T\lambda_1)e^{-D\lambda_1}$, $\lambda_2 = \nu\varkappa(T\lambda_2)e^{-D\lambda_2}$ follows by substraction $\delta = -\omega D\delta + \varrho D\delta$ with the meanings (3.27).

By Lemma (13), if $\|\lambda_i/\nu\|_{q_0} \leq X_{q_0}$, then $|\tau_1| \leq h_{p_0}(\sigma)X_{q_0}$ and

$$\begin{aligned}(3.52)\qquad 0 \leq -\omega &\leq M\{|\varkappa|\}\nu(\sigma)e^{h_{p_0}(\sigma)X_{q_0}}, \\ |\varrho| &\leq M\{|\varkappa'|\}\nu(\sigma)e^{h_{p_0}(\sigma)X_{q_0}}.\end{aligned}$$

Hence, as in the previous Theorem, $(\delta, D\delta)^{\frac{1}{2}} \leq \gamma(\delta, D\delta)^{\frac{1}{2}}$ where

$$(3.53)\qquad \gamma = M\{|\varkappa|\}\chi_D(\nu^*) + M\{|\varkappa'|\}\chi_D^{1/2}(\nu^*)\chi_{\bar{D}}^{1/2}(\nu^*).$$

Therefore $(\delta, D\delta) = 0$ and by Lemma 1, *d)*, $\lambda_1(\sigma) = \lambda_2(\sigma)$ almost everywhere, hence, everywhere, because λ_1/ν and λ_2/ν are continuous functions.

The conditions under which the last theorems hold, are rather involved and difficult to verify except perhaps when $p = \infty$. Moreover, they are presumably too stringent, as being based on inequalities valid for a broad range of values which in specific cases are likely to be very inaccurate. One suspects and the numerical computations confirm, that iteration converges in a much larger range. In the important case of symetry Theorem 8 can be considerably improved (though loosing most of its constructiveness) by showing that if $\varkappa(\theta)$ is negative and increasing, the iterates converge towards a solution of (1), if there is one. We shall say that equation (1) is symmetric when: $\nu(\sigma) = \nu(\pi - \sigma)$, $\varkappa(\theta) = \varkappa(-\theta)$; a symmetric solution is one for which $\lambda(\sigma) = \lambda(\pi - \sigma)$ Clearly, for a symmetric solution, $\tau(\sigma) = \tau(\pi - \sigma)$ and $\theta(\sigma) = -\theta(-\sigma)$, where $\tau = D\lambda$, $\theta = T\lambda$. This being clarified, we can enunciate the Theorem:

THEOREM 10. *Let equation* (1) *be symmertric and* $\varkappa(\theta)$ *a negative, non-decreasing function of* θ *for* $0 \leq \theta$. *Under these conditions, if* (1) *admits a symmetric solution, the iterates of the function identically zero converge decreasing towards a symmetric solution which, in addition, is the greatest of all solutions of* (1).

PROOF: In the first place we show that, in case of symmetry, the operator S is order preserving on negative symmetric functions, that is, $\lambda_1(\sigma) \leq \lambda_2(\sigma) \leq 0$ imply, $S\lambda_1 \leq S\lambda_2 \leq 0$. In fact, for symmetric functions the operator T becomes the integral $\int_0^{2/\pi}$ so

$$(3.54) \qquad \theta_1(\sigma) = \int_0^{\pi/2} \lambda_1(s)\,ds \leq \int_0^{\pi/2} \lambda_2(s)\,ds = \theta_2(\sigma) \quad \text{for} \quad 0 \leq \sigma \leq \frac{\pi}{2},$$

which, in account of the symetry and monotonicity of $\varkappa(\sigma)$, yields $\varkappa(\theta_1(\sigma)) \leq \varkappa(\theta_2(\sigma)) \leq 0$. By Lemma 1, *c)*, $\tau_1 = D\lambda_1 \leq D\lambda_2 = \tau_2 \leq 0$, so $e^{-\tau_1} \geqslant e^{-\tau_2} \geqslant 0$. Thus, since ν is positive, $S\lambda_1 = \nu\varkappa(\theta_1)e^{-\tau_1} \leq$ $\leq \nu\varkappa(\theta_1)e^{-\tau_2} = S\lambda_2$.

This being proved, let us assume now that $\lambda^*(\sigma)$ is a symmetric solution of (1). If $\lambda_0(\sigma)$ designates the function identically zero, $\lambda_0 \geqslant \lambda^*$, and so by applying S to this inequality

$$(3.55) \qquad 0 = \lambda_0 \geqslant S\lambda_0 \geqslant S\lambda^* = \lambda^*,$$

and reiterating the operation,

$$(3.56) \qquad 0 \geqslant \lambda_0 \geqslant S\lambda_0 \geqslant S^2\lambda_0 \geqslant \ldots \geqslant S^k\lambda_0 \geqslant \lambda^*.$$

Thus, $S^k \lambda_0$, as a bounded decreasing sequence, converges towards a function $\lambda \geqslant \lambda^*$. To see that λ is a solution of (1), we observe that all $S^k \lambda_0$ having a minorant λ^* (obviously integrable) we can by Lebesgue Theorem pass to the limit under the integrals $\theta_k = \int_0^{\pi/2} S^k \lambda_0 \, ds$, $\tau_k = \int_0^{\pi/2} D(\sigma, s) \, S^k \lambda_0 \, ds$, and conclude that $\lim \theta_k = T\lambda$ and $\lim \tau_k = D\lambda$, therefore, that $\lambda = \lim \nu \varkappa(\theta_k) \, e^{-\tau_k} = S\lambda$.

It is to be noticed that the Theorem just proved is not a constructive one, in the sense, that it does not provide a bound for the number of iterations required to obtain a solution with a given accuracy. For completeness we state the corresponding results for the discrete case:

THEOREM 11. *Let $\varkappa(\theta)$ be a non positive continuous function with a bounded derivative. If for some $q > q_2$*

$$(3.57) \quad \gamma^{(n)} = M\{|\varkappa|\} \chi_{D^{(n)}}(\nu^*) + M\{|\varkappa'|\} \chi_{D^{(n)}}^{1/2}(\nu^*) \chi_{\tilde{D}^{(n)}}^{1/2}(\nu^*) < 1,$$
$$\nu^*(\sigma_h^{(n)}) = \nu(\sigma_h^{(d)}) \, e^{h_{p_0}^{(n)}(\sigma_h^{(n)}) X_{q_0}^{(n)}},$$

then the iterates $[S^{(n)}]^k \lambda_0 = \lambda_k$ of a function $\lambda_0(\sigma_h^{(n)})$ with $\| \lambda_0/\nu \|_{q_0} < X_{q_0}^{(n)}$ converge towards a solution λ of (1.37), namely, for every r, $0 < r < 1$,

$$(3.58) \quad |\lambda(\sigma_h^{(n)}) - \lambda_k(\sigma_h^{(n)})| \leq \nu^*(\sigma_h^{(n)}) \left\{ A^{(n)} \frac{(\gamma^{(n)})^k}{1-\gamma^{(n)}} \left[\ln \frac{1}{1-r}\right]^{\frac{1}{2}} + B^{(n)} \left[\ln \frac{1}{r}\right]^{\frac{1}{q}-\frac{1}{q_0}} \right\}$$

where

$$(3.59) \quad A^{(n)} = \left\{ \left[M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\right] \frac{\pi}{2} \left(S^{(n)} \lambda_0 - \lambda_0, \, D^{(n)}(S^{(n)} \lambda_0 - \lambda_0)\right) \right\}^{\frac{1}{2}}$$
$$B^{(n)} = \left[M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}\right]^{\frac{1}{2}} 2\, C^*_{\frac{1}{1-\left(\frac{1}{q}-\frac{1}{q_0}\right)}} X_{q_0}^{(n)} \|\nu\|_p .$$

THEOREM 12. *Under the Hypothesis of Theorem 11 there is one and only one solution of (1.37) with $\| \lambda/\nu \|_{q_0} \leq X_{q_0}^{(n)}$.*

THEOREM 13. *Let equation (1.37) be symmetric and $\varkappa(\theta)$ be a non positive, non decreasing function of θ, $0 \leq \theta$. Then if (1.37) admits a solution, the iterates under the operator $S^{(n)}$ of the function identically zero converge decreasing towards a symmetric solution vhich, in addition, is the largest of all solutions of (1.37).*

IV. CONVERGENCE OF DISCRETIZATION

In this last section we shall investigate how the solutions of equation (1) can be approximated by the solutions of equations (1.37). As it stands this problem does not have a clear sense, because the solutions of (1), being functions of a continuous variable, cannot be compared with the solutions of (1.37) which are functions of a discrete variable. This is inmediately obviated by extending the discrete solutions by means of trigonometric interpolation to the continuous interval $(0, \pi)$ (see section I, § 3), thus reducing the problem to determine how the interpolating polynomials approximate the solutions of (1).

§ 1. Auxiliary lemmas from the theory of trigonometric interpolation. The main tool in dealing with the above problem will be a Lemma on trigonometric interpolation we originally developed for this specific purpose [54] which we shall state here without proof.

LEMMA 15. *Let $f(\sigma)$ be a periodic function of period 2π with absolutely continuous derivatives up to the r-1st order and let $P(\sigma)$ be a trigonometric polynomial of degree n coinciding with $f(\sigma)$ a more than $2n$ equidistant points modulo 2π. Then for every $p, 1 < p < \infty$,*

$$a)\quad \left\{\int_0^{2\pi}\left|\frac{d^r P(\sigma)}{d\sigma^r}\right|^p d\sigma\right\}^{\frac{1}{p}} \leq B_{p,r}\left\{\int_0^{2\pi}\left|\frac{d^r f(\sigma)}{d\sigma^r}\right|^p d\sigma\right\}^{\frac{1}{p}},$$

$$b)\quad |f(\sigma)-P(\sigma)| \leq \left(\frac{1}{\pi}\right)^{r-\frac{1}{p}} M_{r,p}\left\{\int_0^{2\pi}\left|\frac{d^r f(\sigma)}{d\sigma^r}\right|^p d\sigma\right\}^{\frac{1}{p}},$$

$$c)\quad \left\{\int_0^{2\pi}|f(\sigma)-P(\sigma)|^p d\sigma\right\}^{\frac{1}{p}} \leq \left(\frac{1}{\pi}\right)^r N_{r,p}\left\{\int_0^{2\pi}\left|\frac{d^r f(\sigma)}{d\sigma^r}\right|^p d\sigma\right\}^{\frac{1}{p}},$$

where $B_{p,r}$, $M_{p,r}$ and $N_{p,r}$ depend on their subindices only.

We shall also need the following result of J. MARCINKIEWICZ [51] of which we have already made use.

LEMMA 16. *For every trigonometric polynomial of degree n, $P(\sigma)$, and m $(m > 2n)$ equidistant points modulo 2π, $\sigma_1, \sigma_2, \ldots, \sigma_m$,*

$$\left\{\int_0^{2\pi}|P(\sigma)|^p d\sigma\right\}^{\frac{1}{p}} \leq B_p\left\{\sum_{k=1}^{m}|P(\sigma_k)|^p \Delta\sigma_k\right\}^{\frac{1}{p}},$$

where B_p is a positive number depending on p only.

§ **2. Convergence of discretization. Convex case, $\varkappa \geqq 0$.** THEOREM 14. *Let $\varkappa(\theta)$ be a non-negative continuous function with a bounded derivative and $\nu(\sigma)$ a non-negative continuous function vanishing at $\sigma = 0$, $\sigma = \pi$, and having a p-integrable ($p > 1$) first derivative. Then, the class of all trigonometric polynomials $\lambda_n(\sigma) = \sum_{j=1}^{n} a_j^{(n)} \sin j\sigma$, $n = 1, 2, \ldots$, whose values $\lambda_n(\sigma_k^{(n)})$ at the points $\sigma_k^{(n)} = k\pi/(n+1)$, $k = 1, 2, \ldots, n$ satisfy equation (37) form a uniformly bounded family of equicontinous functions all whose limits are solutions of (1). If (1) has only one solution, the above polynomials tend uniformly, as n goes to infinity, to such a solution.*

PROOF: For every $\lambda_n(\sigma)$ we define the polynomials $\theta_n(\sigma) = \sum_{j=1}^{n} (a_j^{(n)}/j) \cos j\sigma$, $\tau_n(\sigma) = \sum_{j=1}^{n} (a_j^{(n)}/j) \sin j\sigma$, and the function

$$(4.1) \quad \mu_n(\sigma) = \begin{cases} \nu(\sigma)\varkappa(\theta_n(\sigma))\, e^{-\tau_n(\sigma)} & 0 \leq \sigma \leq \pi \\ -\mu_n(-\sigma) & -\pi \leq \sigma \leq 0 \\ \mu_n(\sigma + 2k\pi) & k = 0, \pm 1, \pm 2, \ldots \end{cases}$$

The μ_n are clearly non negative, continuous and odd functions of σ. With their help the hypothesis that $\lambda_n(\sigma_k^{(n)})$ satisfy equation (1.37) can be written as: $\lambda_n(\sigma_k^{(n)}) = \mu_n(\sigma_k^{(n)})$ (See Section I, § 3). Thus, $\lambda_n(\sigma)$ are trigonometric polynomials of order n interpolating the functions $\mu_n(\sigma)$ at the $2n+2$ points $\sigma_k^{(n)} = k\pi/(n+1)$, $k = 0, \pm 1, \pm 2, \ldots, \pm n, n+1$. The proof of the Theorem shall be attained by successively proving that $\{\theta_n\}$, $\{\tau_n\}$, $\{\mu_n\}$ and $\}\lambda_n\}$ are uniformly bounded families of equicontinuous functions.

The μ_n being non-negative functions, $\lambda_n(\sigma_k^{(n)}) = \mu_n(\sigma_k^{(n)}) \geqq 0$ and so, since $D^{(n)}$ is order preserving, $\tau_n(\sigma_k^{(n)}) \geqq 0$. Therefore

$$(4.2) \quad 0 \leq \lambda_n(\sigma_k^{(n)}) = \mu_n(\sigma_k^{(n)}) \leq \nu(\sigma_k^{(n)})\varkappa(\theta(\sigma_k^{(n)})) \leq M\{\varkappa\}\,\nu(\sigma_k^{(n)}).$$

Applying Lemma 16, to $\lambda_n(\sigma)$ and taking account of the fact that the $\lambda_n(\sigma)$ are odd functions,

$$(4.3) \quad \|\lambda_n\|_p \leq B_p \left\{ \sum_{k=1}^{n} |\lambda_n(\sigma_k^{(n)})|^p \Delta\sigma_k^{(n)} \right\}^{\frac{1}{p}} \leq B_p M\{\varkappa\}\, M\{\nu\}\, \pi^{\frac{1}{p}}.$$

Now by Lemma 1 *b)* and Lemma 3, *b)*,

$$(4.4) \quad |\tau_n(\sigma') - \tau_n(\sigma'')| \leq \pi^{\frac{1}{p}} A_p B_p M\{\varkappa\}\, M\{\nu\}\, |\sigma' - \sigma''|^{\frac{1}{q}},$$

$$(4.5) \quad |\theta_n(\sigma') - \theta_n(\sigma'')| \leq \pi^{\frac{1}{p}} \quad B_p M\{\varkappa\}\, M\{\nu\}\, |\sigma' - \sigma''|^{\frac{1}{q}},$$

which show the equicontinuity of θ_n and τ_n. By setting $\sigma'' = 0$ in (4.4) and by Lemma 3, *b)* we get

$$|\tau_n(\sigma)| \leq A_p B_p M\{\varkappa\} M\{\nu\}, \tag{4.6}$$

$$|\theta_n(\sigma)| \leq B_p M\{\varkappa\} M\{\nu\}, \tag{4.7}$$

and the uniform boundedness of τ_n and θ_n is assured. In the interval $(0, \pi)$ we have,

$$\mu_n' = e^{-\tau_n}[\nu' \varkappa(\theta_n) + \nu\,\theta_n' \varkappa'(\theta_n) - \nu\varkappa(\theta_n)\tau_n']. \tag{4,8}$$

By Minkowski inequality the p-norm of μ_n' is not greater than the sum o the p-norms of the terms on the right of (4.8). Hence, taking p-norms and replacing $e^{-\tau_n}$, $\varkappa(\theta_n)$ and $\varkappa'(\theta_n)$ by their bounds,

$$\|\mu_n'\|_p \leq e^{M\{\tau_n\}}[M\{\varkappa\}\|\nu'\|_p + M\{\nu\} M\{\varkappa'\}\|\theta_n'\|_p + \\ + M\{\nu\} M\{\varkappa\}\|\tau_n'\|_p]. \tag{4.9}$$

The three terms in the brackets are uniformly bounded ; the first does not depend on n and is bounded by hypothesis, the second because $\theta_n' = -\lambda_n$ and (4.3), and the third by virtue of Riesz Theorem : $\|\tau_n'\|_p \leq A_p \|\lambda_n\|_p$ (τ_n' and λ_n are conjugate functions) and by (4.3). So

$$\|\mu_n'\|_p \leq N, \tag{4.10}$$

where

$$N = e^{A_p B_p M\{\nu\} M\{\varkappa\}}\Big[M\{\varkappa\}\|\nu'\|_p + M^2\{\nu\} M\{|\varkappa'|\} M\{\varkappa\} B_p \pi^{\frac{1}{p}} + \\ + M^2\{\nu\} M^2\{\varkappa\} A_p B_p \pi^{\frac{1}{p}}\Big]. \tag{4.11}$$

By Lemma 15 *a)*, applied with $r = 1$, to the function μ_n and its interpolating polynomial λ_n (both functions being odd the interval of integration can be restricted to $(0, \pi)$),

$$\|\lambda_n'\|_p \leq B_{p,1}\|\mu_n'\|_p \leq B_{p,1} N, \tag{4.12}$$

and by Lemma 15, *b)*

$$|\mu_n - \lambda_n| \leq \left(\frac{1}{n}\right)^{\frac{1}{q}} M_{1,p} N. \tag{4.13}$$

An application of Hölder inequality yields, from (4,9) and (4.12),

$$(4.14) \qquad |\mu_n(\sigma') - \mu_n(\sigma'')| \leq N \, |\sigma' - \sigma''|^{\frac{1}{q}}$$

$$(4.15) \qquad |\lambda_n(\sigma') - \lambda_n(\sigma'')| \leq N \, B_{p,1} \, |\sigma' - \sigma''|^{\frac{1}{q}}$$

and the equicontinuity of $\{\mu_n\}$ and $\{\lambda_n\}$ is proved. Moreover, since they vanish at $\sigma = 0$ they are uniformly bounded. To complete the proof let us take a subsequence $\lambda_{n_m}(\sigma)$ convergent to a function $\lambda(\sigma)$. On account of the equicontinuity the convergence is uniform (Cf. Courant - Hilbert [53], page 50), and By Lemma 1, *b)* and Lemma 3, *b)*, the sequences $\theta_{n_m} = T\lambda_{n_m}$ and $\tau_{n_m} = D\lambda_{n_m}$ converge uniformly to $\theta = T\lambda$ and $\tau = D\lambda$ respectively. By the continuity of the exponential and of $\varkappa$, $\mu_{n_m} = \nu\varkappa(\theta_{n_m}) e^{-\tau_{n_m}}$ converges to $\nu\varkappa(\theta) e^{-\tau}$. But by (4.13) λ_{n_m} and μ_{n_m} converge to the same limit, so $\lambda = \nu\varkappa(\theta) e^{-\tau}$ and λ is a solution of (1). Finally, if (1) has at most one solution, there is at most a limit function of the family $\{\lambda_n\}$ and the sequence λ_n is itself convergent.

The above Theorem is not constructive, but if the conditions of Theorem 4 assuring the uniqueness of the solution are added to its Hypothesis, then bounds for the degree of approximation can be actually given.

Theorem 15. *Under the Hypothesis of Theorem* 14, *if*

$$M\{|\varkappa'|\} \, \chi_D^{1/2}(\nu) \, \chi_{\tilde{D}}^{1/2}(\nu) < 1$$

then equation (1) *admits one solution* $\lambda(\sigma)$ *only (Theorem* 4*), and for any trigonometric polynomial of order* n, $\lambda_n(\sigma)$ *whose values at the points* $\sigma_k^{(n)} = k\pi/(n+1)$ *satisfy equation* (1.37) *and any* r, $0 < r < 1$,

$$(4.16) \quad |\lambda(\sigma) - \lambda_n(\sigma)| \leq \nu(\sigma)\left\{Q_1 \frac{1}{n}\left[\ln\frac{1}{1-r}\right]^{\frac{1}{2}} + Q_2\left[\ln\frac{1}{r}\right]^{\frac{1}{q}}\right\} + Q_3\left(\frac{1}{n}\right)^{\frac{1}{q}},$$

where

$$(4.17) \begin{cases} Q_1 = \sqrt{\dfrac{2}{\pi}} \, \dfrac{[M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}]^{\frac{1}{2}}}{1 - M\{|\varkappa|\} \, \chi_D^{1/2}(\nu) \, \chi_{\tilde{D}}^{1/2}(\nu)} \left\{\displaystyle\int_0^\pi \int_0^\pi |D(\sigma, s)|^q \, d\sigma \, ds\right\}^{\frac{1}{2q}} N_{1,p} \, N, \\ Q_2 = [M^2\{|\varkappa|\} + M^2\{|\varkappa'|\}]^{\frac{1}{2}} \, C_2 \left[M\{|\varkappa|\} \, \|\nu\|_p + M\{|\varkappa|\} \, M\{\nu\} \, B_p \, \pi^{\frac{1}{p}}\right], \\ Q_3 = M_{1,p} N. \end{cases}$$

C_2, $N_{1,p}$, $M_{1,p}$, are the constants defined in Lemma 5 *and Lemma* 15 *and N is given by* (4.11).

PROOF: With the same notations as in Theorem 15.

$$\lambda - \lambda_n = \lambda - \mu_n + \mu_n - \lambda_n = \nu\varkappa(\theta)e^{-\tau} - \nu\varkappa(\theta_n)e^{-\tau_n} + \mu_n - \lambda_n, \tag{4.18}$$

which can be written

$$\delta_n = -\omega_n D\delta_n + \varrho_n T\delta_n + \varepsilon_n, \tag{4.19}$$

where

$$\begin{aligned} \delta_n &= \lambda - \lambda_n \\ \varepsilon_n &= \mu_n - \lambda_n \\ \omega_n &= \nu\varkappa(\theta)\frac{e^{-\tau} - e^{-\tau_n}}{-\tau + \tau_n} \\ \varrho_n &= \nu\frac{\varkappa(\theta) - \varkappa(\theta_n)}{\theta - \theta_n}. \end{aligned} \tag{4.20}$$

As in Theorem 3, from this we derive (see formula (3.30))

$$(\delta_n, D\delta_n)^{\frac{1}{2}} \leq \frac{(\varepsilon_n, D\varepsilon_n)^{\frac{1}{2}}}{1 - M\{|\varkappa'|\}\chi_D^{1/2}(\nu)\chi_{\bar{D}}^{1/2}(\nu)}. \tag{4.21}$$

Moreover by Hölder inequality,

$$\begin{aligned} (\varepsilon_n, D\varepsilon_n)^{\frac{1}{2}} &= \left\{\int_0^\pi\int_0^\pi D(\sigma, s)\,\varepsilon_n(s)\,\varepsilon_n(\sigma)\,ds\,d\sigma\right\}^{\frac{1}{2}} \\ &\leq \left\{\int_0^\pi\int_0^\pi |D(\sigma, s)|^q\,d\sigma\,ds\right\}^{\frac{1}{2q}}\|\varepsilon_n\|_p, \end{aligned} \tag{4.22}$$

and by Lemma 15, *c)* with $r = 1$, applied to λ_n and μ_n,

$$\|\varepsilon_n\|_p \leq \frac{1}{n}N_{1,p}\|\mu_n'\|_p. \tag{4.23}$$

Replacing in (4.21), and on account of (4.10),

$$(\lambda - \lambda_n, D(\lambda - \lambda_n))^{\frac{1}{2}} \leq Q_0\frac{1}{n}, \tag{4.24}$$

where

$$(4.25)\qquad Q_0 = \frac{1}{1 - M\{|\varkappa'|\}\, \chi_D^{1/2}(\nu)\, \chi_{\bar D}^{1/2}(\nu)} \left\{ \int_0^\pi \int_0^\pi |D(\sigma, s)|^q \, ds\, d\sigma \right\}^{\frac{1}{2q}} N_{1,p} N.$$

Clearly $\|\lambda\|_p \leq M\{\varkappa\}\|\nu\|_p$, so by Minkowski inequality and (4.3) $\|\lambda - \lambda_n\|_p \leq M\{\varkappa\}\|\nu\|_p + M\{\varkappa\}\, M\{\nu\}\, B_p \pi^{\frac{1}{p}}$. Hence, by Lemma 5, for every r, $0 < r < 1$,

$$(4.26)\qquad \{(\theta - \theta_n)^2 + (\tau - \tau_n)^2\}^{\frac{1}{2}} \leq Q_0 \frac{1}{n} \left[\frac{2}{\pi} \ln \frac{1}{1 - r}\right]^{\frac{1}{2}} +$$
$$+ C_2 \left[M\{\varkappa\}\|\nu\|_p + M\{\varkappa\}\, M\{\nu\}\, B_p \pi^{\frac{1}{p}}\right] \left[\ln \frac{1}{r}\right]^{\frac{1}{q}}.$$

Now

$$(4.27)\qquad |\lambda - \lambda_n| \leq |\lambda - \mu_n| + |\mu_n - \lambda_n|.$$

The first term on the right can be majorized by using Schwarz inequality, as follows,

$$(4.22)\qquad |\lambda - \mu_n| \leq \nu(\sigma)\, [M^2\{\varkappa\} + M^2\{\varkappa'\}]^{\frac{1}{2}}\, [|\theta - \theta_n|^2 + |\tau - \tau_n|^2]^{\frac{1}{2}},$$

and the second by (4.13). So on account of (4.26),

$$(4.89)\qquad |\lambda - \lambda_n| \leq \nu(\sigma) \left\{ Q_1 \frac{1}{u} \left[\ln \frac{1}{1 - r}\right]^{\frac{1}{2}} + Q_2 \left[\ln \frac{1}{r}\right]^{\frac{1}{q}} \right\} + Q_3 \left(\frac{1}{n}\right)^{\frac{1}{q}},$$

where Q_1, Q_2, and Q_3 have the values (4.17).

§ **3. Concave case, $\varkappa \leq 0$.** The concave case presents here, as in the iteration process, new difficulties. Theorem 14 very likely holds also for $\varkappa \leq 0$, but we have not been able to prove it. The main obstacle is to find a priori bounds for the values of the interpolating polynomials $\lambda_n(\sigma)$ from the mere fact that they satisfy equation (1.37). Once this is settled, the proof proceeds, as the reader can easily verify, as in the convex case. Such is, for instance, the case when the solutions $\lambda_n(\sigma_k^{(n)})$ are obtained by iteration of the null function under the condition of Theorem 11, in which case Theorem 15 admits also a natural generalization. The details are left to the reader.

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