A STRUCTURE THEOREM FOR 2-HYPERGROUPOIDS WITH TOPOLOGICAL APPLICATIONS

by

P. CARRASCO, A.M. CEGARRA and J. OLMOS

ABSTRACT:

BRANDT [1] gave a structure theorem for connected groupoids G, which in a simplicial version stablishes that $K(\Pi_1(G,*), 1)$ is a strong deformation retract of G. The main object of this paper is to generalice Brandt's theorem to dimension tow: "If G is a Kan 1-connected 2-hypergroupoid, $K(\Pi_2(G,*), 2)$ is a strong deformation retract of G". Then, for a 1-connected topological space X and $*\epsilon X$ the homotopy 2-hypergroupoid of X is equivalent to the second homotopy group $\Pi_2(X,*)$ and the Hurewicz theorem is obtained as an elemental application.

Under the classical set theoretic notion of groupoid, i.e. a non empty category in which every arrow is an isomorphism, BRANDT in 1940 [1] gave a structure theorem for "Brandt groupoids", i.e. connected groupoids, in which he prove that they are characterized by a group G and a non empty set S in such a fashion that set of morphims of the groupoid is bijective to the set SxSxG. By the present terminology the Brandt's theorem simply says that every groupoid of Brandt is equivalent (as category) to a group, and this group is precisely, up to isomorphism, the group of automorphisms in the groupoid of any object chosen of the groupoid.

Thus, for instance, the fundamental groupoid of a path-connected topological space X is equivalent to the fundamental group $\Pi_1(X, *)$ for any $* \epsilon X$.

If for a simplicial set K we denote $\Delta^n_m(K)$ the set of open m-hors in dimension n, i.e. $\Delta^n_m(K) = \{ (x_0, \dots, x_{m-1}, -, x_{m+1}, \dots, x_n), x_i \epsilon K_{n-1}$, and such that $d_i(x_j) = d_{j-1}(x_i)$, for $i < j$, $i,j \neq m \}$ and $\Delta_n(K)$ is the n-th simplicial kernel, $\Delta_n(K) = \{ (x_0, \dots, x_n) / d_i(x_j) = d_{j-1}(x_i), i<j \}$, the concept of groupoid has

a simplicial version, that is: A groupoid is a simplicial set G such that $G_2 \cong \Delta_2^i(G)$, $i = 0, 1, 2$, by the canonical applications $x \mapsto (-, d_1(x), d_2(x))$, $x \longmapsto (d_0(x), -, d_2(x))$ and $x \longmapsto (d_0(x), d_1(x), -)$, and such that $G_m \cong \Delta_m(G)$, $m > 2$ by the canonical application $x \longmapsto (d_0(x), \ldots, d_m(x))$ (see [8] for example). In this versión G_0 corresponds to the set of objects of the groupoid, G_1 to the set of morphisms, $d_0: G_1 \longrightarrow G_0$ to the domain map, $d_1: G_1 \longrightarrow G_0$ to the range map and the composition $d_1: G_2 \longrightarrow G_1$ to the multiplication or composition in the groupoid. In this terminology, Brandt's theorem says that if the groupoid is connected ($\Pi_0(G)$ = coequalizer $(d_0, d_1) = *$), then the subgroup of automorphisms of $* \in G_0$, which is precisely the Eilenberg-MacLane space $K(\Pi_1(G, *), 1)$ is a deformation retract of G.

This simplicial version of groupoid has been generalized in the last years, when the n-dimensional problems in homological algebra were studied ([4] [5] [6] [7] [8]), by introducing the concept of n-hypergroupoid, $n \geqslant 1$; thus, for instance, a 2-hypergroupoid is a simplicial set G such that $G_3 \cong \Delta_3^i(G)$, $i = 0,1,2,3$, and $G_m \cong \Delta_m(G)$, $m > 3$, (in this case each composition $\Delta_3^i(G) \longrightarrow G_3 \longrightarrow G_2$ is a 3-ary operation satisfying certain identities). Analogously as a topological space X has associated a groupoid, the fundamental groupoid, X has also associated a 2-hypergroupoid which is precisely the Eilenberg-MacLane space $K(\Pi_2(X,*), 2)$, for $* \in X$.

The main object of this paper is to generalize the structure Brandt theorem to dimension two. Thus we obtain:

Teorema 1.1.– If G is a Kan 1-connected 2-hypergroupoid, then for every $* \in G_0$, $K(\Pi_2(G, *), 2)$ is a deformation retract of G.

Then, for a 1-connected topological space X and $* \in X$, we obtain that the homotopy 2-hypergroupoid of X is equivalent to the second homotopy group $\Pi_2(X, *)$ (considered as a 2-hypergroupoid).

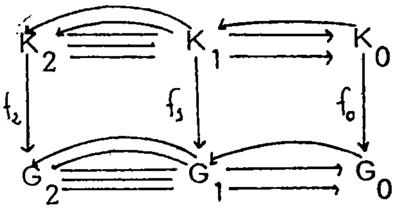
This fact implies the Hurewicz theorem ($H_1(X, A) = 0 = H^1(X, A)$, $H^2(X, A) = \text{Hom}(\Pi_2(X, *), A)$ and $H_2(X, A) = \Pi_2(X, *) \otimes A$, if X is 1-connected) as a elemental application with which we end the present work.

1.– THE BRANDT'S THEOREM FOR 2-HYPERGROUPOIDS.

The following lemma will be useful to proof the main theorem:

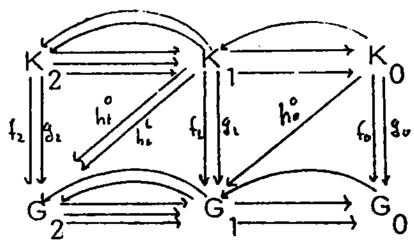
LEMMA 1.1.– Let G be a 2-Hypergroupoid, K a simplicial set.

i) A truncated simplicial map



has an extension, necessarily unique, to a simplicial map $f=(f_n)_{n \geqslant 0}: K \longrightarrow G$ if, and only if, for each 3-simplex $x \in K_3$ there exists $y \in G_3$ such that $d_i(y) = f_2 d_i(x)$. In this case $f_3(x) = y$ and f_m, $m>3$, is given by $f_m(x) = (f_{m-1} d_0(x), \dots, f_{m-1} d_m(x))$ which belongs to $\Delta_m(G) = G_m$.

ii) Given simplicial maps f,g: $K \longrightarrow G$, a truncated homotopy



has an extension, necessarily unique, to a homotopy h: $f \longrightarrow g$ if, and only if, for each $x \in K_2$ the following condition is satisfied: If $y_1 \in G_3$ is the element such that $d_0(y_1) = f_2(x)$, $d_2(y_1) = \overset{0}{h}_1 d_1(x)$ and $d_3(y_1) = \overset{0}{h}_1 d_2(x)$ (this unique y_1 exists since $(f_2(x), -, \overset{0}{h}_1 d_1(x), \overset{0}{h}_1 d_2(x)) \in \overset{1}{\Delta}_3(G) \cong G_3)$ and $y_2 \in G_3$ is the element such that $d_0(y_2) = \overset{0}{h}_1 d_0(x)$, $d_1(y_2) = d_1(y_1)$ and $d_3(y_2) = \overset{1}{h}_1 d_2(x)$ (which exists and is unique since $(\overset{0}{h}_1 d_0(x), d_1(y_1), -, \overset{1}{h}_1 d_2(x)) \in \overset{2}{\Delta}_3(G) \cong G_3)$, then the element $y_3 \in G_3$, the unique one such that $d_0(y_3) = \overset{1}{h}_1 d_0(x)$, $d_1(y_3) = \overset{1}{h}_1 d_1(x)$ and $d_2(y_3) = d_2(y_2)$, satisfies that $d_3(y_3) = g_3(x)$.

In this case the extended homotopy is such that $\overset{0}{h}_2(x) = y_1$, $\overset{1}{h}_2(x) = y_2$ and $\overset{2}{h}_2(x) = y_3$.

Proof

i) If (f_2, f_1, f_0) has an extension to a simplicial map f: $K \longrightarrow G$, then $d_i f_3(x) = f_2 d_i(x)$ and exists $y = f_3(x)$. Conversely, let us suppose that for each $x \in K_3$ there exists $y \in G_3$ such that $d_i(y) = f_2 d_i(x)$. y is necessarily unique since $G_3 \cong \overset{i}{\Delta}_3(G)$ and we define $f_3(x) = y$. Then (f_3, f_2, f_1, f_0) is a truncated simplicial map since by construction $d_i f_3(x) = f_2 d_i$ and also $f_3 s_j = s_j f_2$ because $d_i s_j f_2 = f_2 d_i s_j$.

Finally, since $G_m \cong \Delta_m(G)$ for $m > 3$, the truncated map (f_3, f_2, f_1, f_0) determines an unique extension $(f_m)_{m \geqslant 0}$ by $f_m(x) = (f_{m-1} d_0(x), \ldots, f_{m-1} d_m(x)$ for $m > 3$.

ii) If the truncated homotopy $(h_i^k)_{0 \leqslant k \leqslant i \leqslant 2}$ has an extension to a homotopy $h: f \longrightarrow g$ one has that: 1) $d_0 h_2^0(x) = f_2(x)$, $d_2 h_2^0(x) = h_1^0 d_2(x)$ and $d_3 h_2^0(x) = h_1^0 d_2(x)$, so that $y_1 = h_2^0(x)$; 2) $d_0 h_2^1(x) = h_1^0 d_0(x)$, $d_1 h_2^1(x) = d_1 h_2^0(x)$, and $d_3 h_2^1(x) = h_1^1 d_2(x)$, then $y_2 = h_2^1(x)$; and 3) $d_0 h_2^2(x) = h_1^1 d_0(x)$, $d_1 h_2^2(x) = h_1^1 d_1(x)$ and $d_2 h_2^2(x) = d_2 h_2^1(x)$, consecuently $y_3 = h_2^2(x)$ and it is satisfied that $d_3(y_3) = d_3 h_2^2(x) = g_2(x)$.

Reciprocally, under the hypothesis it is plain to see that the maps h_2^k given by $h_2^0(x) = y_1$, $h_2^1(x) = y_2$ and $h_2^2(x) = y_3$ define a truncated homotopy which has an extension to a homotopy $h = (h_i^k)_{0 \leqslant k \leqslant i}$ given inductively by:

$$h_m^0(x) = (f_m(x), Y_1(x), h_{m-1}^0 d_1(x), \ldots\ldots, h_{m-1}^0 d_m(x)) \in G_{m+1} \cong \Delta_{m+1}(G)$$

$$h_m^1(x) = (h_{m-1}^0 d_0(x), Y_1(x), Y_2(x), \ldots\ldots, h_{m-1}^1 d_m(x)) \in G_{m+1}$$

. . . .

. . . .

$$h_m^{m-1}(x) = (h_{m-1}^{m-2} d_0(x), \ldots\ldots, h_{m-1}^{m-2} d_{m-2}(x), Y_{m-1}(x), Y_m(x), h_{m-1}^{m-1} d_m(x)) \in G_{m+1}$$

$$h_m^m(x) = (h_{m-1}^{m-1} d_0(x), \ldots\ldots\ldots, h_{m-1}^{m-1} d_{m-1}(x), Y_m(x), g_m(x)) \in G_{m+1}$$

where

$$Y_1(x) = (d_0 f_m(x), d_1 h_{m-1}^0 d_1(x), \ldots\ldots, d_1 h_{m-1}^0 d_m(x)) \in G_m \cong \Delta_m(G)$$

$$Y_2(x) = (d_1 h_{m-1}^0 d_0(x), d_1 h_{m-1}^0 d_1(x), d_2 h_{m-1}^1 d_2(x), \ldots\ldots, d_2 h_{m-1}^1 d_m(x)) \in G_m$$

. . . .

. . . .

$$Y_{m-1}(x) = (d_{m-2}h^{m-3}_{m-1}d_0(x), \ldots\ldots, d_{m-2}h^{m-3}_{m-1}d_{m-2}(x),\ d_{m-1}h^{m-2}_{m-1}d_{m-1}(x),$$

$$d_{m-1}h^{m-2}_{m-1}d_m(x))$$

$$Y_m(x) = (d_{m-1}h^{m-2}_{m-1}d_0(x), \ldots\ldots, d_{m-1}h^{m-2}_{m-1}d_{m-1}(x),\ d_m h^{m-1}_{m-1}d_m(x)) \in G_m$$

1.2.– For a Kan simplicial set K(i.e. a simplicial set such that $(d_j): K_n \longrightarrow \Delta^i_n(K)$ is a apimorphism for each $0 \leqslant i \leqslant n$), it is said that two n-simplices x and x' of K are homotopic, written $x \sim x'$, if $d_i(x) = d_i(x')$, $0 \leqslant i \leqslant n$, and there exists $y \in K_{n+1}$ such that $d_n(y) = x$, $d_{n+1}(y) = x'$ and $d_i(y) = s_{n-1}d_i(x) = s_{n-1}d_i(x')$, $0 \leqslant i \leqslant n$. The simplex y is called a homotopy from x to x' and $\sim$ is a equivalence relation on the n-simplices of K, $n \geqslant 0$. For $* \in K_0$, if we denote $K_n(*) =$ $= \{ x \in K_n / d_i(x) = s_0^{n-1}(*)\ (= s_0 \overset{n-1)}{\ldots} s_0\ (*)) \}$ then the n-th homotopy group of X is $\Pi_n(K, *) = K_n(*)/\sim$, which are group for $n \geqslant 1$ and abelian groups if $n \geqslant 2$.

Let X a topologicla space and S(X) the (total) singular complex of X (i.e. $S_n(X)$, $n \geqslant 0$, is the set of singular n-simplices or continuous maps $\Delta^n \longrightarrow X$); S(X) is a Kan simplicial set and the homotopy groups of X coincide with the homotopy groups of S(X), i.e. $\Pi_n(X, *) = \Pi_n(S(X), *)$, $n \geqslant 0$.

We finally recall that for A an abelian group, the Eilenberg-MacLane complex K(A, 2) (which is a 2-hyperfroupoid) is given by: $K(A, 2)_0 = K(A, 2)_1 = *$, and $K(A, 2)_3 = A^3$, $d_i(a_0, a_1, a_2) = a$, $i = 0, 1, 2$; $d_3(a_0, a_1, a_2) = a_0 - a_1 + a_2$ and for $m > 3$ $K(A, 2)_m = \Delta_m(K(A, 2))$

$$K(A, 2): \ldots\ldots\ldots\ldots\ldots A^3 \rightrightarrows A \rightrightarrows * \rightrightarrows *$$

This complex, K(A, 2), is such that $\Pi_i\ (K(A, 2), *) = 0$ if $i \neq 2$ and $\Pi_2\ (K(A, 2), *) = A$. If B is another abelian group, the homology and cohomology of K(A, 2) with coefficients in B are such that $H^2(K(A, 2), B) = \mathrm{Hom}(A, B)$ $H_2(K(A, 2), B) = A \otimes B$ and $H^1(K(A, 2), B) = 0 = H_1(K(A, 2), B)$.

For more details see [9].

LEMMA 2.3.– Let G be a Kan 2-hypergroupoid and $* \in G_0$. Let $E_2(G, *)$ the sub-2-hypergroupoid of G over $*$ (i.e. the subcomplex of G which consists of those n-simplices whose faces on dimension 0 and 1, are $*$ and $s_0(*)$, respectively).

Then $E_2(G, *) = K(\Pi_2(G, *), 2)$; therefore one has a natural inclusion $i : K(\Pi_2(G, *), 2) \longrightarrow G$.

Proof.

It is clear that $E_2(G,*)_i = * = K(\Pi_2(G,*),2)_i$, $i=0,1$. Now let us note that $\Pi_2(G,*) = G_2(*) = \{x \in G_2/d_i(x) = s_0(*),\ i=0,1,2\}$ since if $y \in G_3$ is a homotopy from x to x′, x, x′ $\in G_2(*)$, then $d_i(y) = d_i s_2(x)$, $i = 0, 1, 2$, and as $G_3 \cong \overset{3}{\Delta}_3(G)$ we deduce that $y = s_2(x)$; consecuently $d_3(y) = d_3 s_2(x)$, that is $x = x'$. Thus $E_2(G,*)_2 = G_2(*) = \Pi_2(G,*) = K(\Pi_2(G,*),2)_2$.

Finally, $E_2(G,*)_3 \cong \overset{i}{\Delta}_3(E_2(G,*)) \cong \overset{i}{\Delta}_3(K(\Pi_2(G,*),2)) \cong K(\Pi_2(G,*),2)_3$ and using induction for $m > 3$ we have: $E_2(G,*)_m = \Delta_m(E_2(G,*)) = \Delta_m(K(\Pi_2(G,*),2)) = K(\Pi_2(G,*),2)_m$.

Now we can prove the structure theorem.

THEOREM 1.4.– Let G be a 1-connected Kan 2-hypergroupoid; let $* \in G_0$. Then $K(\Pi_2(G,*),2)$ is a deformation retract of G.

Proof.

Firstly we define maps $\overset{0}{h}_0: G_0 \longrightarrow G_1$ and $\overset{0}{h}_1, \overset{1}{h}_1: G_1 \longrightarrow G_2$ as following: For $u \in G_0$, $\overset{0}{h}_0(u)$ is a homotopy from $*$ to u(i.e. an element of G_1 such that $d_0\overset{0}{h}_0(u) = *$ and $d_1\overset{0}{h}_0(u) = u$; this exists because G is connected). If $u = *$, $\overset{0}{h}_0(*)$ will be $s_0(*)$.

For $z \in G_1$, $\overset{1}{h}_1(x)$ is an element of G_2 such that $d_0\overset{1}{h}_1(z) = \overset{0}{h}_0 d_0(z)$ and $d_2\overset{1}{h}_1(z) = z$, this element exists since G is a Kan complex. If $z = s_0(*)$ we take $\overset{1}{h}_1(z)$ as $s_0^2(*)$. For each $z \in G_1$ we define $\overset{0}{h}_1(z)$ as a homotopy from $d_1\overset{1}{h}_1(z)$ to $\overset{0}{h}_0 d_1(z)$ (i.e. $\overset{0}{h}_1(z) \in G_2$ is such that $d_0\overset{0}{h}_1(z) = s_0(*)$, $d_1\overset{0}{h}_1(z) = d_1\overset{1}{h}_1(z)$ and $d_2\overset{0}{h}_1(z) = \overset{0}{h}_0 d_1(z)$; this exists because G is 1-connected). If $z = s_0(*)$, $\overset{0}{h}_1(z)$ will be $s_0^2(*)$.

Now we define r: $G \longrightarrow K(\Pi_2(G,*),2)$ as following: For each $x \in G_2$ we consider the element $(\overset{1}{h}_1 d_0(x), \overset{1}{h}_1 d_1(x), -, x) \in \overset{2}{\Delta}_3(G)$. As $\overset{2}{\Delta}_3(G) \cong G_3$ there exists an unique element $\overset{2}{h}_2(x) \in G_3$ such that $d_0\overset{2}{h}_2(x) = \overset{1}{h}_1 d_0(x)$, $d_1\overset{2}{h}_2(x) = \overset{1}{h}_1 d_1(x)$ and $d_3\overset{2}{h}_2(x) = x$. Let $X_2 = d_2\overset{2}{h}_2(x)$, then $(\overset{0}{h}_1 d_0(x), -, X_2, \overset{1}{h}_1 d_2(x)) \in \overset{1}{\Delta}_3(G) \cong G_3$ and there exists an unique element $\overset{1}{h}_2(x) \in G_3$ such that $d_0\overset{1}{h}_2(x) = \overset{0}{h}_1 d_0(x)$, $d_2\overset{1}{h}_2(x) = X_2$ and $d_3\overset{1}{h}_2(x) = \overset{1}{h}_1 d_2(x)$. Finally, let $X_1 = d_1\overset{1}{h}_2(x) \in G_2$; we define $r_2(x) \in K(\Pi_2(G,*),2)_2$ as $d_0\overset{0}{h}_2(x)$, where $\overset{0}{h}_2(x)$ is the unique element of G_3 such that $d_1\overset{0}{h}_2(x) = X_1$, $d_2\overset{0}{h}_2(x) = \overset{0}{h}_1 d_1(x)$ and $d_3\overset{0}{h}_2(x) = \overset{0}{h}_1 d_2(x)$.

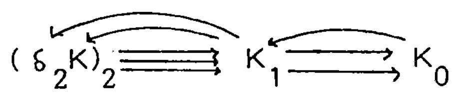
By using the lemman 1.1, we can deduce that r_2 is extended to a simplicial map r: $G \longrightarrow K(\Pi_2(G,*),2)$ and the maps $(\overset{i}{h}_j)$, $j = 0, 1$, $0 \leqslant i \leqslant j$ are extended to a homotopy h: $ir \longrightarrow 1_G$. Clearly $ri = 1_{K(\Pi_2(G,*),2)}$

2.– THE FUNDAMENTAL 2-HYPERGROPOID ASSOCIATED TO A TOPOLOGICAL SPACE.

Let K a Kan simplicial set, we will define a 2-hypergroupoid associated to K, $\delta_2 K$, as following:

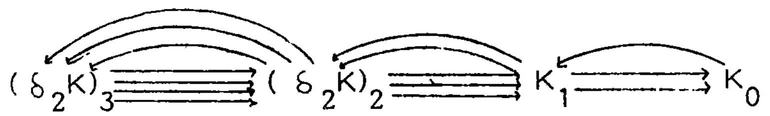
Let $(\delta_2 K)_m = K_m$ for m = 0, 1, and $(\delta_2 K)_2 = K_2/\sim$, where $\sim$ is the homotopy relation between 2-simplices considered in (1.1).

Since if $x \sim x'$ then $d_i(x) = d_i(x')$, it is induced a truncated simplicial set:



We consider $(\delta_2 K)_3 = \overset{3}{\Delta}_3((\delta_2 K)_2 \equiv K_1)$ and the canonical projections d_i: $(\delta_2 K)_3 \equiv (\delta_2 K)_2$, i = 0, 1, 2. For each $(z_0, z_1, z_2, -) \in (\delta_2 K)_3$, let $x_0, x_1, x_2 \in K_2$ the representatives of z_0, z_1 and z_2 respectively. Then $(x_0, x_1, x_2, -) \in \overset{3}{\Delta}_3(K)$ and since K is a Kan complex, there exists $y_3 \in K_3$ such that $d_i(y_3) = x_i$, i = 0, 1, 2. It is plain to see that the homotopy class of $d_3(y_3) = z_3$ is independent of the choice of y_3 and x_i, i = 0, 1, 2. Therefore we have a map d_3: $(\delta_2 K)_3 \longrightarrow (\delta_2 K)_2$ given by $d_3(z_0, z_1, z_2, -) = z_3$, according to the above.

We have now the truncated simplicial set:



where the degeneraby maps s_i: $(\delta_2 K)_2 \longrightarrow (\delta_2 K)_3$ are given by: $s_0(z) = (z, z, s_0 d_1(z), -)$, $s_1(z) = (s_0 d_0(z), z, z, -)$ and $s_2(z) = (s_1 d_0(z), s_1 d_1(z), z, -)$; we extend this truncated simplicial set to a simplicial set $\delta_2 K$ by iteration the simplicial kernels construction.

It is easy to see that $\delta_2 K$ is a Kan simplicial complex and it is also a 2-hypergroupoid. By construction, we have only to see that $(\delta_2 K)_3 \cong \overset{3}{\Delta}_3(\delta_2 K) \cong \overset{i}{\Delta}_3(\delta_2 K)$, i = 0, 1, 2. It is sufficient to see that if two elements $y, y' \in K_3$ are such that $d_i(y) \sim d_i(y')$ for each $i \neq j$, then $d_j(y) \sim d_j(y')$. In fact, let us suppose that $d_0(y) \sim d_0(y')$, $d_1(y) \sim d_1(y')$ and $d_2(y) \sim d_2(y')$, by the homotopies u_0, u_1 and u_2 respectively, then a homotopy u_3: $d_3(y) \sim d_3(y')$ is obtained as following: Since K is a Kan complex, there exists $x, x' \in K_4$ such that $d_i(x) = d_i(x') = u_i$, i = 0, 1, 2, $d_3(x) = y$ and $d_3(x') = y'$. As $(s_1 d_0(y'), s_1 d_1(y'), d_4(x), d_4(x'), -) \in \overset{4}{\Delta}_4(K)$, there exists $w \in K_4$ such that $d_0(w) = s_1 d_0(y')$, $d_1(w) =$

$= s_1 d_1(y')$, $d_2(w) = d_4(x)$ and $d_3(w) = d_4(x')$. Then $d_4(w) = u_3$ is a homotopy from $d_3(y)$ to $d_3(y')$ since $d_0(u_3) = d_3 s_1 d_0(y') = s_1 d_0 d_3(y')$, $d_1(u_3) = d_3 s_1 d_1(y') = s_1 d_1 d_3(y')$, $d_2(u_3) = d_3 d_4(x) = d_3(y)$ and $d_3(u_3) = d_3 d_4(x') = d_3(y')$.

***DEFINITION** 2.1.*– If K is a Kan simplicial set, the 2-hypergroupoid $\delta_2 K$, constructed above, is called the fundamental 2-hypergroupoid (or fundamental homotopy 2-hypergroupoid).

If X is a topological space, it is defined the fundamental 2-hypergroupoid of X, $\delta_2 X$, as $\delta_2 X = \delta_2 S(X)$

From the theorem 1.4 we have:

***THEOREM** 2.2.*– If X is a 1-connected topological space, the fundamental 2-hypergoupoid of X is equivalent, in a natural way, to the second homotopy group of X $\Pi_2(X, *)$ (where $\Pi_2(X, *)$ is considered as a 2-hypergroupoid, that is $K(\Pi_2(X, *), 2)$).

A elemental propierty of $\delta_2 X$ is the following:

***PROPOSITION** 2.3.*– If X is a topological space and A is an abelian group, then:

$$H^1(X,A) \cong H^1(\delta_2 X,A) \text{ and } H^2(X,A) \cong H^2(\delta_2 X,A)$$

in a natural way.

Proof.

By the definition of $\delta_2(X)$ it is plain to see that $H^1(X,A) = H^1(\delta_2 X, A)$. To proof the second isomorphism it is sufficient to see that every normalized 2-cocycle is factorized through $(\delta_2 X)_2$ (since the coborders are equal); in fact, let $f \in Hom(S_2 X,A)$ a normalized cocycle and let $x,x' \in S_2 X$ such that $x \sim x'$, then there exists $y \in (S_3 X)_3$ such that $d_3(y) = x$, $d_2(y) = x'$ and $d_i(y) = s_1 d_i(x)$, $i = 0, 1$. Then, since f is a mormalized 2-cocycle, one has that $\Sigma_{i=0}^{3}(-1)^i f d_i = 0$ and therefore $\Sigma_{i=0}^{3}(-1)^i f d_i(y) = 0$, so that we deduce that $f(x) = f(x')$ and therefore f is factoriced through $(\delta_2 X)_2$. Thus $H^2(X,A) \cong H^2(\delta_2 X,A)$.

Finally, let us see that the theorem 2.2 implies the Hurewicz theorem on dimension two.

COROLLARY 2.4.– If X is a 1-connected topological space, then one has natural isomorphisms (which are induced by the Hurewicz morphism):

$$H^2(X,A) \cong \mathrm{Hom}(\Pi_2(X, *), A) \qquad H_2(X,A) \cong \Pi_2(X, *) \otimes A$$

also $H^1(X,A) = 0 = H_1(X,A)$, for every abelian group A.

Proof.

Since X is 1-connected, the canonical inclusion i: $K(\Pi_2(X, *), 2) \longrightarrow \delta_2 S(X)$ (see lemma 1.3) is a homotopy equivalence and the corollary is a consequence of the proposition 2.3 and of the elemental homological propierties of K(A,n) (see 1.1).

REFERENCES

[1] Brandt, H.– Uber die Axiome des Gruppoids, Vierteljscher, Naturforsch Ges, Zurich 85, Beiblat, Cfestschrift Rudolph Fueter, 95-104 (1940).

[2] Brayton, G.– Homotopy theory. An introduction to Algebraic Topology. Academic Press (1975).

[3] Bruck, R.H.– A survey of binary systems. Ergebrisse del Mathematik und Ihrer Grenzgebiete 20, Springer Verlag, Berlin (1958).

[4] Cegarra, A.M., Aznar, E.– An exact sequence in the first variable for the torsor cohomology. The 2-dimensional theory of obstruction. J.P.A.A. (por aparecer) (1984).

[5] Cegarra, A.M., Bullejos, M.– n-cocicles non abéliens. C.R. Acad. Sc. Paris, 2981, nº 17, 401-404, (1984).

[6] Cegarra, A.M., Bullejos, M.– High dimensional obstruction theory in algebraic categories. Preprint (1985).

[7] Duskin, J.– Simplicial methods and the interpretation of triple cohomology. Memoirs of the A.M.S. Vol. 3, Issue 2 163 (1975).

[8] Glenn, P.– Realization of cohomology classes in arbitraires exact categories J.P.A.A. Vol. 25, 1, 33-107, (1982).

[9] May, J.P.– Simplicial objects in algebraic topology. Princenton New Jersey. (1967).

P. Carrasco, A.M. Cegarra
Dpto. de Algebra y Fundamentos
Universidad de Granada
Granada

J. Olmos de Cara
Instituto Julio Rodríguez
Granada