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Eisenstein series and Poincaré series for mixed automorphic forms

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Abstract

Mixed automorphic forms generalize elliptic modular forms, and they occur naturally as holomorphic forms of the highest degree on families of abelian varieties parametrized by a Riemann surface. We construct generalized Eisenstein series and Poincaré series, and prove that they are mixed automorphic forms.

1. Introduction

Let E be an elliptic surface over a Riemann surface X (cf. [3]). Then the space of holomorphic two-forms on an elliptic surface is isomorphic to the space of cusp forms of weight three for the corresponding Fuchsian group $\Gamma \subset SL(2,\mathbb{R})$ that determines X if the monodromy representation is simply the inclusion map of Γ in $SL(2,\mathbb{R})$ (cf. [11]). However, when the monodromy representation is not the inclusion map, the holomorphic two-forms on the elliptic surface should be identified with mixed cusp forms whose automorphy factors involve the monodromy representation and the period map of the elliptic surface (see [2]). A geometric interpretation of such mixed automorphic forms of higher weights can be obtained by essentially taking the fiber product of a finite number of elliptic surfaces (cf. [4]). It is well-known that Eisenstein series and Poincaré series provide basic examples of elliptic modular forms (see e.g. [10]). The goal of this paper is to construct the analog of Eisenstein series and Poincaré series for mixed automorphic forms.

Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind acting on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations. Let $\chi : \Gamma \to SL(2,\mathbb{R})$ be a homomorphism, and let $\omega : \mathcal{H} \to \mathcal{H}$ be a holomorphic map such that $\omega(\gamma z) = \chi(\gamma)\omega(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. We assume that the inverse image of a parabolic subgroup of $\chi(\Gamma)$ under χ is parabolic. If $J : SL(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ is the automorphy factor defined by $J\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\big) = cz + d$, then a mixed automorphic (resp. cusp) form of type (p, q) is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = J(\gamma, z)^p J\big(\chi(\gamma), \omega(z)\big)^q f(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$ that is holomorphic (resp. vanishes) at the cusps of Γ. Various aspects of mixed automorphic forms of the above type have been investigated in a number of papers (see e.g. [1], [5], [8]). Mixed automorphic forms of several variables have also been studied in connection with holomorphic forms on families of abelian varieties (cf. [6], [7], [9]). In this paper we construct Eisenstein series and Poincaré series that are mixed automorphic forms in one variable of type $(2k + 2, 2m)$ for some nonnegative integers k and m.

2. Eisenstein series and Poincaré series

Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind acting on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations. Let $\chi : \Gamma \to SL(2,\mathbb{R})$ be a homomorphism, and let $\omega : \mathcal{H} \to \mathcal{H}$ be a holomorphic map satisfying

$$\omega(\gamma z) = \chi(\gamma)\omega(z) \tag{2.1}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. We assume that the inverse image of a parabolic subgroup of $\Gamma' = \chi(\Gamma)$ under χ is a parabolic subgroup of Γ so that the Γ-cusps and Γ'-cusps correspond. In addition, we also assume that $\operatorname{Im} \omega(z) \to \infty$ as $\operatorname{Im} z \to \infty$. Thus we can extend ω to a map

$$\mathcal{H} \cup \{\Gamma\text{-cusps}\} \to \mathcal{H}' \cup \{\Gamma'\text{-cusps}\},$$

which we also denote by ω, such that (2.1) holds for all $z \in \mathcal{H} \cup \{\Gamma\text{-cusps}\}$ and $\gamma \in \Gamma$. Let $J : SL(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ be the automorphy factor of $SL(2,\mathbb{R})$ defined by $J(g, w) = cw + d$ if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

and $w \in \mathcal{H}$. Thus we have

$$J(gg', z) = J(g, g'z)J(g', z)$$

for all $z \in \mathcal{H}$ and $g, g' \in G$. Given a function $f : \mathcal{H} \to \mathbb{C}$, an element γ in Γ, and nonnegative integers p and q, we define the function $f|_{(p,q)}\gamma : \mathcal{H} \to \mathbb{C}$ by

$$\big(f|_{(p,q)}\gamma\big)(z) = J(\gamma, z)^{-p}\, J\big(\chi(\gamma), \omega(z)\big)^{-q}\, f(\gamma z)$$

for all $z \in \mathcal{H}$.

Definition 2.1. Let p and q be nonnegative integers. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is said to be a *mixed automorphic form of type* (p, q) *associated to* Γ, ω *and* χ if f satisfies the following conditions:

(i) $f|_{(p,q)}\gamma = f$ for all $\gamma \in \Gamma$.

(ii) f is holomorphic at each Γ-cusp.

The function f is said to be a *mixed cusp form of type* (p, q) *associated to* Γ, ω *and* χ if (ii) is replaced by

(ii)$'$ f vanishes at each Γ-cusp (see [5] for details).

Remark 2.2. A mixed automorphic form of type $(p, 0)$ associated to Γ, ω and χ is a usual elliptic modular form of weight p for Γ. On the other hand, if ω and χ are the identity maps, then a mixed automorphic form of type (p, q) associated to Γ, ω and χ becomes a modular form of weight $p + q$ for Γ. Mixed automorphic forms of type (p, q) with p even can be identified with holomorphic forms of the highest degree on the fiber product of a finite number of elliptic surfaces (see e.g. [4]). Mixed automorphic forms can be extended to the case of several variables, and they are linked to holomorphic forms on families of more general abelian varieties, (cf. [6], [7], [8]).

Let s be a cusp of Γ with $\sigma s = \infty$ for some $\sigma \in SL(2, \mathbb{R})$. Then there is a parabolic element $\alpha \in \Gamma$ such that $\alpha s = s$. By our assumption on χ, $\chi(\alpha)$ is a parabolic element of $\chi(\Gamma)$, and hence there is a cusp s_χ of $\chi(\Gamma)$ and an element $\sigma_\chi \in SL(2, \mathbb{R})$ such that

$$\chi(\alpha)s_\chi = s_\chi, \quad \sigma_\chi s_\chi = \infty.$$

Given a function $f : \mathcal{H} \to \mathbb{C}$ and nonnegative integers k, m, we set

$$\big(f|_{(2k+2,2m)}\sigma^{-1}\big)(z) = J\big(\sigma^{-1}, z\big)^{-2k-2} J\big(\sigma_\chi^{-1}, \omega(z)\big)^{-2m} f\big(\sigma^{-1}z\big) \tag{2.2}$$

for all $z \in \mathcal{H}$. Let $\Gamma_s = \{\gamma \in \Gamma \mid \gamma s = s\}$ be the stabilizer of s in Γ, and let h be a positive real number such that

$$\sigma\Gamma_s\sigma^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n \;\middle|\; n \in \mathbb{Z} \right\}.$$

Then, for a nonnegative integer ν, we define the holomorphic function $\phi_\nu : \mathcal{H} \to \mathbb{C}$ associated to the cusp s by

$$\phi_\nu(z) = J(\sigma, z)^{-2k-2} J\big(\sigma_\chi, \omega(z)\big)^{-2m} \exp(2\pi i \nu \sigma z/h) \tag{2.3}$$

for all $z \in \mathcal{H}$.

Lemma 2.3

If s is a cusp of Γ, then the associated function ϕ_ν satisfies

$$\phi_\nu|_{(2k+2,2m)}\gamma = \phi_\nu$$

for all $\gamma \in \Gamma_s$.

Proof. For $z \in \mathcal{H}$ and $\gamma \in \Gamma_s$ we have

$$\begin{aligned} \phi_\nu(\gamma z) &= J(\sigma, \gamma z)^{-2k-2} J(\sigma_\chi, \omega(\gamma z))^{-2m} \exp(2\pi i \nu \sigma\gamma z/h) \\ &= J(\sigma, \gamma z)^{-2k-2} J(\sigma_\chi, \chi(\gamma)\omega(z))^{-2m} \exp(2\pi i \nu \sigma\gamma z/h) \\ &= J(\sigma\gamma, z)^{-2k-2} J(\gamma, z)^{2k+2} J(\sigma_\chi\chi(\gamma), \omega(z))^{-2m} \\ &\qquad \times J(\chi(\gamma), \omega(z))^{2m} \exp(2\pi i \nu (\sigma\gamma\sigma^{-1})\sigma z/h). \end{aligned}$$

Since $\sigma\gamma\sigma^{-1}$ and $\sigma_\chi\chi(\gamma)\sigma_\chi^{-1}$ stabilize ∞, we have

$$J\big(\sigma\gamma\sigma^{-1}, w\big) = J\big(\sigma_\chi\chi(\gamma)\sigma_\chi^{-1}, \sigma_\chi w\big) = 1$$

for all $w \in \mathcal{H}$, and hence we see that

$$\begin{aligned} J(\sigma\gamma, z) &= J\big(\sigma\gamma\sigma^{-1}, \sigma z\big) \cdot J(\sigma, z) = J(\sigma, z), \\ J\big(\sigma_\chi\chi(\gamma), \omega(z)\big) &= J\big(\sigma_\chi\chi(\gamma)\sigma_\chi^{-1}, \sigma_\chi\omega(z)\big) \cdot J\big(\sigma_\chi, \omega(z)\big) = J\big(\sigma_\chi, \omega(z)\big), \end{aligned}$$

and $\sigma\gamma z/h = (\sigma\gamma\sigma^{-1})\sigma z/h = \sigma z/h + d$ for some integer d. Thus we obtain

$$\begin{aligned} \phi_\nu(\gamma z) &= J(\sigma, z)^{-2k-2} J(\gamma, z)^{2k+2} \\ &\qquad \times J\big(\sigma_\chi, \omega(z)\big)^{-2m} J\big(\chi(\gamma), \omega(z)\big)^{2m} \exp(2\pi i \nu \sigma z/h) \\ &= J(\gamma, z)^{2k+2} J\big(\chi(\gamma), \omega(z)\big)^{2m} \phi_\nu(z), \end{aligned}$$

and therefore the lemma follows. □

Let s be a cusp of Γ as above, and set

$$P^{\nu}_{(2k+2,2m)}(z) = \sum_{\gamma \in \Gamma_s \backslash \Gamma} (\phi_\nu|_{(2k+2,2m)}\gamma)(z) \tag{2.4}$$

for all $z \in \mathcal{H}$. The convergence of this series will be proved in Section 3.

DEFINITION 2.4. The function $P^{\nu}_{(2k+2,2m)}(z)$ is called a *Poincaré series* for mixed automorphic forms if $\nu \geq 1$, and the function $P^{0}_{(2k+2,2m)}(z)$ is called an *Eisenstein series* for mixed automorphic forms.

3. Convergence and holomorphy

In this section, we show that the series in (2.4) defining the function $P^{\nu}_{(2k+2,2m)}(z)$ converges and is holomorphic on $\mathcal{H}$.

Lemma 3.1

Let $z_0 \in \mathcal{H}$, and let ε be a positive real number such that

$$N_{3\varepsilon} = \left\{ z \in \mathbb{C} \mid |z - z_0| \leq 3\varepsilon \right\} \subset \mathcal{H},$$

and let k and m be nonnegative integers. If ψ is a continuous function on $N_{3\varepsilon}$ that is holomorphic on the interior of $N_{3\varepsilon}$, then there exists a constant C such that

$$|\psi(z_1)| \leq C \int_{N_{3\varepsilon}} |\psi(z)| (Im\, z)^{k+1} (Im\, \omega(z))^m dV$$

for all $z_1 \in N_\varepsilon = \{z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon\}$, where $dV = dxdy/y^2$ with $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Proof. Let z_1 be an element of N_ε, and consider the Taylor expansion of $\psi(z)$ about z_1 of the form

$$\psi(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n.$$

We set $N'_\varepsilon = \{z \in \mathbb{C} \mid |z - z_1| < \varepsilon\}$. Then $N'_\varepsilon \subset N_{3\varepsilon}$, and we have

$$\int_{N'_\varepsilon} \psi(z) dxdy = \int_0^{2\pi} \int_0^{\varepsilon} \sum_{n=0}^{\infty} a_n r^{n+1} e^{in\theta} dr d\theta = \pi\varepsilon^2 a_0 = \pi\varepsilon^2 \psi(z_1).$$

Hence we obtain

$$\begin{aligned} |\psi(z_1)| &\leq (\pi\varepsilon^2)^{-1} \int_{N_{3\varepsilon}} |\psi(z)| dxdy \\ &= (\pi\varepsilon^2)^{-1} \int_{N_{3\varepsilon}} \frac{|\psi(z)|(\operatorname{Im} z)^{k+1}(\operatorname{Im}\omega(z))^m}{(\operatorname{Im} z)^{k-1}(\operatorname{Im}\omega(z))^m} dV \\ &\leq (\pi\varepsilon^2 C_1)^{-1} \int_{N_{3\varepsilon}} |\psi(z)|(\operatorname{Im} z)^{k+1}(\operatorname{Im}\omega(z))^m dV, \end{aligned}$$

where

$$C_1 = \inf\left\{(\operatorname{Im} z)^{k-1}(\operatorname{Im}\omega(z))^m \mid z \in N_{3\varepsilon}\right\}.$$

Thus the lemma follows by setting $C = (\pi\varepsilon^2 C_1)^{-1}$. □

If U is a connected open subset of $\mathcal{H}$, then we define the norm $\|\cdot\|_U$ on the space of holomorphic functions on U by

$$\|\psi\|_U = \int_U |\psi(z)|(\operatorname{Im} z)^{k+1}(\operatorname{Im}\omega(z))^m dV,$$

where ψ is a holomorphic function on U.

Lemma 3.2

Let $\{f_n\}$ be a Cauchy sequence of holomorphic functions on U with respect to the norm $\|\cdot\|_U$. Then the sequence $\{f_n\}$ converges absolutely to a holomorphic function on U uniformly on any compact subsets of U.

Proof. Let $\{f_n\}$ be a Cauchy sequence of holomorphic functions on an open set $U \subset \mathcal{H}$. Then by Lemma 3.1, for each $z \in U$, there is a constant C such that

$$\left|f_n(z) - f_m(z)\right| \leq C\left\|f_n - f_m\right\|_U$$

for all $n, m \geq 0$. Thus the sequence $\{f_n(z)\}$ of complex numbers is also a Cauchy sequence, and therefore it converges. We set $f(z) = \lim_{n\to\infty} f_n(z)$ for all $z \in U$. Let $z_0 \in U$, and choose $\delta > 0$ such that

$$N_{3\delta} = \left\{z \in \mathbb{C} \mid |z - z_0| \leq 3\delta\right\} \subset U.$$

Using Lemma 3.1 again, we have

$$\left|f_n(z) - f_m(z)\right| \leq C'\left\|f_n - f_m\right\|_U$$

for all $z \in N_\delta = \{z \in \mathbb{C} \mid |z - z_0| \leq \delta\}$. Given $\varepsilon > 0$, let N be a positive integer such that $\|f_n - f_m\|_U < \varepsilon/(2C')$ whenever $m, n > N$. For each $z \in N_\delta$, if we choose an integer $n' > N$ so that $|f_{n'}(z) - f(z)| < \varepsilon/2$, then we obtain

$$\left|f_n(z) - f(z)\right| \leq \left|f_n(z) - f_{n'}(z)\right| + \left|f_{n'}(z) - f(z)\right| < \varepsilon$$

for all $n > N$. Thus the sequence $\{f_n\}$ converges to f uniformly on N_δ and therefore on any compact subsets of U. Hence it follows that f is holomorphic function on U. □

Let ϕ_ν be as in (2.3), and let $\{s_1, \ldots, s_\mu\}$ be the set of all Γ-inequivalent cusps of Γ. We choose a neighborhoods U_i of s_i for each $i \in \{1, \ldots, \mu\}$. Then we have

$$\int_{\Gamma_0 \backslash \mathcal{H}'} |\phi_\nu(z)| (\operatorname{Im} z)^p (\operatorname{Im} \omega(z))^q dV < \infty , \tag{3.1}$$

where p and q are nonnegative integers and

$$\mathcal{H}' = \mathcal{H} - \bigcup_{i=1}^{\mu} \bigcup_{\gamma \in \Gamma} \gamma U_i. \tag{3.2}$$

Theorem 3.3

The series in (2.4) *defining* $P^\nu_{(2k+2,2m)}(z)$ *converges absolutely on* $\mathcal{H}$ *and uniformly on compact subsets, and, in particular, the function* $P^\nu_{(2k+2,2m)}(z)$ *is holomorphic on* $\mathcal{H}$*.*

Proof. Let $s_1, \ldots, s_\mu$ be the Γ-inequivalent cusps of Γ as above, and let z_0 be an element of $\mathcal{H}$. We choose neighborhoods W of z_0 and U_i of s_i for $1 \leq i \leq \mu$ such that

$$\big\{\gamma \in \Gamma \mid \gamma W \cap W \neq \emptyset\big\} = \Gamma_{z_0}, \quad \gamma W \cap U_i = \emptyset \tag{3.3}$$

for all $\gamma \in \Gamma$ and $1 \leq i \leq \mu$, where Γ_{z_0} is the stabilizer of z_0 in Γ. Then, using (2.3) and

$$\operatorname{Im} \gamma w = |J(\gamma, w)|^{-2} \cdot \operatorname{Im} w, \quad \operatorname{Im} \omega(\gamma w) = |J(\chi\gamma, \omega(w))|^{-2} \cdot \operatorname{Im} \omega(w)$$

for $\gamma \in \Gamma$ and $w \in \mathcal{H}$, we have

$$\begin{aligned} \|P^\nu_{(2k+2,2m)}\|_W &= \int_W \Big| \sum_{\gamma \in \Gamma_s \backslash \Gamma} (\phi_\nu |_{(2k+2,2m)} \gamma)(z) \Big| (\operatorname{Im} z)^{k+1} (\operatorname{Im} \omega(z))^m dV \\ &\leq \int_W \sum_{\gamma \in \Gamma_s \backslash \Gamma} \big| (\phi_\nu |_{(2k+2,2m)} \gamma)(z) \big| (\operatorname{Im} z)^{k+1} (\operatorname{Im} \omega(z))^m dV \\ &= \sum_{\gamma \in \Gamma_s \backslash \Gamma} \int_W |\phi_\nu(\gamma z)| (\operatorname{Im} \gamma z)^{k+1} (\operatorname{Im} \omega(\gamma z))^m dV \\ &= \sum_{\gamma \in \Gamma_s \backslash \Gamma} \int_{\gamma W} |\phi_\nu(z)| (\operatorname{Im} z)^{k+1} (\operatorname{Im} \omega(z))^m dV. \end{aligned}$$

In order to estimate the number of terms in the above sum, let $\gamma' \in \Gamma$ and set

$$\Xi = \{\gamma \in \Gamma \mid \gamma''\gamma W \cap \gamma' W \neq \emptyset \quad \text{for some} \quad \gamma'' \in \Gamma_s\}.$$

Then by (3.3) we see that $\gamma' W \in \mathcal{H}'$ and

$$|\Gamma_s \backslash \Xi| \leq |\Gamma_s \backslash \Gamma_s \gamma' \Gamma_{z_0}| \leq |\Gamma_{z_0}|,$$

where $|\cdot|$ denotes the cardinality. Thus, using this and (3.1), we have

$$\sum_{\gamma \in \Gamma_s \backslash \Gamma} \int_{\gamma W} |\phi_\nu(z)| (\operatorname{Im} z)^{k+1} (\operatorname{Im} \omega(z))^m dV$$
$$\leq |\Gamma_{z_0}| \int_{\Gamma_s \backslash \mathcal{H}'} |\phi_\nu(z)| (\operatorname{Im} z)^{k+1} (\operatorname{Im} \omega(z))^m dV < \infty .$$

Hence we obtain $\|P^\nu_{(2k+2,2m)}\|_W < \infty$, and by Lemma 3.2 we see that $P^\nu_{(2k+2,2m)}(z)$ converges absolutely on W and uniformly on compact subsets of W. Thus it follows that the function $P^\nu_{(2k+2,2m)}(z)$ is holomorphic on W, and therefore is holomorphic on $\mathcal{H}$ as well. □

4. Cusp conditions

In this section, we show that the function $P^\nu_{(2k+2,2m)}(z)$ is holomorphic at each cusp for all nonnegative integers ν and that it vanishes at each cusp for $\nu > 0$.

Lemma 4.1

Let s' be a cusp of Γ such that $\sigma' s' = \infty$ with $\sigma' \in SL(2,\mathbb{R})$, and let $\sigma'_\chi \in SL(2,\mathbb{R})$ be an element with $\sigma'_\chi \omega(s) = \infty$. Using the notation in (2.2), the function ϕ_ν given in (2.3) satisfies the following conditions.

(i) If s' is not Γ-equivalent to s, then there exist positive real numbers M and λ such that

$$\left|(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1})(z)\right| \leq M|z|^{-2k-2} \tag{4.1}$$

whenever $\operatorname{Im} z > \lambda$.

(ii) If s' is Γ-equivalent to s, then there exist positive real numbers M and λ such that

$$\left|(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1})(z)\right| \leq M \tag{4.2}$$

whenever $\operatorname{Im} z > \lambda$. If in addition $\nu > 0$, then we have

$$(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1})(z) \to 0 \tag{4.3}$$

as $\operatorname{Im} z \to \infty$.

Proof. Using (2.2) and (2.3), for $z \in \mathcal{H}$ we have

$$\begin{aligned}\big(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1}\big)(z) = J(\sigma'^{-1},z)^{-2k-2}&J\big(\sigma_\chi'^{-1},\omega(z)\big)^{-2m}\\ &\times J\big(\sigma,\sigma'^{-1}z\big)^{-2k-2}J\big(\sigma_\chi,\omega(\sigma'^{-1}z)\big)^{-2m}\\ &\times \exp(2\pi i\nu\sigma\sigma'^{-1}/h)\,.\end{aligned}$$

If $\sigma\sigma'^{-1} = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and if $\operatorname{Im} z > 2|d|/|c|$, then we have

$$\begin{aligned}\big|J(\sigma,\sigma'^{-1}z)\cdot J(\sigma'^{-1},z)\big| &= \big|J(\sigma\sigma'^{-1},z)\big| = |cz+d|\\ &\geq |c||z| - |d| \geq |c||z| - (|c|/2)\operatorname{Im} z\\ &= |c||z| - (|c|/2)|z| = |c||z|/2\,.\end{aligned}$$

On the other hand, if $\sigma_\chi'^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right)$ and $\sigma_\chi = \left(\begin{smallmatrix} a'' & b'' \\ c'' & d''\end{smallmatrix}\right)$, then we obtain

$$|J(\sigma_\chi'^{-1},\omega(z))||J(\sigma_\chi,\omega(\sigma'^{-1}z))| = |c'\omega(z)+d'||c''\omega(\sigma'^{-1}z)+d''|\,.$$

Since $\operatorname{Im}\omega(z) \to \infty$ and $\omega(\sigma'^{-1}z) \to \omega(s')$ as $\operatorname{Im} z \to \infty$, there exist real numbers $A, \lambda' > 0$ such that

$$|J(\sigma_\chi'^{-1},\omega(z))||J(\sigma_\chi,\omega(\sigma'^{-1}z))| \geq A$$

whenever $\operatorname{Im} z > \lambda'$. We set $\lambda = \max(\lambda', 2|d|/|c|)$. Then, whenever $\operatorname{Im} z > \lambda$, we have

$$\big|(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1})(z)\big| \leq (|c||z|/2)^{-2k-2}A^{-2m}\exp(-2\pi\nu\sigma\sigma'(\operatorname{Im} z)/h)\,.$$

Thus (4.1) holds for $M = (|c|/2)^{-2k-2}A^{-2m}\exp(-2\pi\nu\sigma\sigma'\lambda/h)$, and therefore (i) follows. As for (ii), if s' is equivalent to s, we may assume that $\sigma = \sigma'$. Thus we have

$$\begin{aligned}\big(\phi_\nu|_{(2k+2,2m)}\sigma'^{-1}\big)(z) = J(1,z)^{-2k-2}&J\big(\sigma_\chi^{-1},\omega(z)\big)^{-2m}\\ &\times J\big(\sigma_\chi,\omega(\sigma^{-1}z)\big)^{-2m}\exp(2\pi i\nu z/h)\,.\end{aligned}$$

Since $J(1,z) = 1$, we obtain (4.2) by arguing as in the case of (i). □

Theorem 4.2

Let s_0 be a cusp of Γ. Then the function $P^\nu_{(2k+2,2m)}(z)$ is holomorphic at s_0 for all nonnegative integers ν. Furthermore, $P^\nu_{(2k+2,2m)}(z)$ vanishes at s_0 if $\nu > 0$.

Proof. Let $\Gamma_{s_0} \subset \Gamma$ be the stabilizer of the cusp s_0, and let $\{\delta\}$ be a complete set of representatives of $\Gamma_s\backslash\Gamma/\Gamma_{s_0}$. Given δ, let $\{\eta\}$ be a complete set of representatives of $\delta^{-1}\Gamma_s\delta \cap \Gamma_{s_0}\backslash\Gamma_{s_0}$, so that we have $\Gamma = \coprod_{\delta,\eta} \Gamma_s\delta\eta$. We set

$$\phi_{\nu,\delta}(z) = \sum_{\eta} (\phi_\nu|_{(2k+2,2m)}\delta\eta)(z)$$

for all $z \in \mathcal{H}$. Then we have

$$P^{\nu}_{(2k+2,2m)}(z) = \sum_{\delta}\sum_{\eta} (\phi_\nu|_{(2k+2,2m)}\delta\eta)(z) = \sum_{\delta} \phi_{\nu,\delta}(z)\,.$$

By Theorem 3.3 there is a neighborhood U of s_0 in $\mathcal{H}$ such that $P^{\nu}_{(2k+2,2m)}(z)$ converges uniformly on any compact subset of U. Hence, if $\sigma_0 s_0 = \infty$ with $\sigma_0 \in SL(2,\mathbb{R})$, then the function

$$P^{\nu}_{(2k+2,2m)}|_{(2k+2,2m)}\sigma_0^{-1} = \sum_{\delta} \phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$$

converges uniformly on any compact subset of $\{z \in \mathcal{H} \mid \operatorname{Im} z > d\}$ for some positive real number d. Therefore it suffices to show that each $\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$ is holomorphic at ∞ and that it has zero at ∞ if $\nu > 0$. First, suppose that δs_0 is not a cusp of Γ_s. Then $\delta^{-1}\Gamma_s\delta \cap \Gamma_{s_0}$ coincides with $\{1\}$ or $\{\pm 1\}$, and hence we have

$$\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1} = C \cdot \sum_{\eta \in \Gamma_{s_0}} \left(\phi_\nu|_{(2k+2,2m)}\delta\sigma_0^{-1}\sigma_0\eta\sigma_0^{-1}\right)$$

with $C = 1$ or $1/2$, respectively. Applying (4.1) for $s = \delta s_0$, $\sigma = \sigma_0\delta^{-1}$, we obtain

$$\left|(\phi_\nu|_{(2k+2,2m)}\delta\sigma_0^{-1})(z)\right| \le M|z|^{-2k-2}$$

for all z with $\operatorname{Im} z > \lambda$ for some $M, \lambda > 0$. Thus we obtain

$$\left|(\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1})(z)\right| \le 2M \sum_{\alpha \in \mathbb{Z}} |z + \alpha b|^{-2k-2}, \tag{4.4}$$

where b is a positive real number such that

$$\sigma_0\Gamma_{s_0}\sigma_0^{-1} \cdot \{\pm 1\} = \left\{\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{\alpha} \,\middle|\, \alpha \in \mathbb{Z}\right\}.$$

By comparing the series on the right hand side of (4.4) with the series $\sum_{\alpha\in\mathbb{Z}} \alpha^{-2k-2}$, we see that it converges uniformly on any compact subset of the domain $\operatorname{Im} z > \lambda$. Hence it follows that $\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$ is holomorphic at ∞. Furthermore, $\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$ vanishes at ∞ because the right hand side of (4.4) approaches zero as $z \to \infty$. Next, suppose δs_0 is a cusp of Γ_s. Then $\delta^{-1}\Gamma_s\delta \cap \Gamma_{s_0}$ is a subgroup of Γ_{s_0} of finite index; hence the sum on the right hand side of

$$\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1} = \sum_{\eta} \left(\phi_\nu|_{(2k+2,2m)}\delta\sigma_0^{-1}\sigma_0\eta\sigma_0^{-1}\right),$$

where the summation is over $\eta \in \delta^{-1}\Gamma_s\delta \cap \Gamma_{s_0}\backslash\Gamma_{s_0}$, is a finite sum. Using (4.2) for $s = \delta s_0$ and $\sigma = \sigma_0\delta^{-1}$, for each δ we obtain

$$\left|(\phi_\nu|_{(2k+2,2m)}\delta\sigma_0^{-1})(z)\right| \le M$$

for all $\operatorname{Im} z > \lambda$ for some $M, \lambda > 0$. For each $\eta \in \Gamma_{s_0}$ we have

$$\sigma_0\eta\sigma_0^{-1} = \pm\begin{pmatrix} 1 & \beta b \\ 0 & 1 \end{pmatrix}$$

for some $\beta \in \mathbb{Z}$; hence we have

$$\left|(\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1})(z)\right| \le M$$

for all $\operatorname{Im} z > \lambda$, and it follows that $\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$ is holomorphic at ∞. Furthermore, if $\nu > 0$, then by (4.3) we have

$$\left(\phi_\nu|_{(2k+2,2m)}\delta\sigma_0^{-1}\right)(z) \to 0$$

as $\operatorname{Im} z \to \infty$; hence we see that $\phi_{\nu,\delta}|_{(2k+2,2m)}\sigma_0^{-1}$ vanishes at ∞. $\square$

Theorem 4.3

The Eisenstein series $P^0_{(2k+2,2m)}(z)$ is a mixed automorphic form and the Poincaré series $P^\nu_{(2k+2,2m)}(z)$ is a mixed cusp form for Γ of type $(2k+2, 2m)$.

Proof. Using the relations

$$J(\gamma, \gamma' z) = J(\gamma', z)^{-1} J(\gamma\gamma', z),$$
$$J\big(\chi(\gamma), \chi(\gamma')\omega(z)\big) = J\big(\chi(\gamma'), \omega(z)\big)^{-1} J\big(\chi(\gamma\gamma'), \omega(z)\big)$$

for $\gamma, \gamma' \in \Gamma$ and $z \in \mathcal{H}$, we obtain

$$\begin{aligned} P^{\nu}_{(2k+2,2m)}(\gamma' z) &= \sum_{\gamma \in \Gamma_s \backslash \Gamma} (\phi_\nu|_{(2k+2,2m)}\gamma)(\gamma' z) \\ &= \sum_{\gamma \in \Gamma_s \backslash \Gamma} J(\gamma, \gamma' z)^{-2k-2} J\big(\chi(\gamma), \omega(\gamma' z)\big)^{-2m} \phi_\nu(\gamma\gamma' z) \\ &= J(\gamma', z)^{2k+2} J\big(\chi(\gamma'), \omega(z)\big)^{2m} \\ &\quad \times \sum_{\gamma \in \Gamma_s \backslash \Gamma} J(\gamma\gamma', z)^{-2k-2} J\big(\chi(\gamma\gamma'), \omega(z)\big)^{-2m} \phi_\nu(\gamma\gamma' z) \\ &= J(\gamma', z)^{2k+2} J\big(\chi(\gamma'), \omega(z)\big)^{2m} P^{\nu}_{(2k+2,2m)}(z) \end{aligned}$$

for all $\gamma' \in \Gamma$ and $z \in \mathcal{H}$; hence $P^{\nu}_{(2k+2,2m)}$ satisfies the condition (i) in Definition 2.1. Therefore the theorem follows from the cusp conditions given in Theorem 4.2. □

Remark 4.4. If ω and χ are the identity maps, then $P^{0}_{(2k+2,2m)}(z)$ and $P^{\nu}_{(2k+2,2m)}(z)$ for $\nu > 0$ are the Eisenstein series and the Poincaré series, respectively, for elliptic modular forms for Γ of weight $2(k+m+1)$. Poincaré series were also considered in [5] for mixed cusp forms of type $(2, 2m)$.

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