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A characterization of Valdivia compact spaces

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Abstract

We characterize Valdivia compact spaces K in terms of $\mathcal{C}(K)$ endowed with a topology introduced by M. Valdivia (1991). This generalizes R. Pol's characterization of Corson compact spaces. Further we study duality, products and open continuous images of Valdivia compact spaces. We prove in particular that the dual unit ball of $\mathcal{C}(K)$ is Valdivia whenever K is Valdivia and that the converse holds whenever K has a dense set of G_δ points. Another result is that any open continuous image of a Valdivia compact space with a dense set of G_δ points is again Valdivia.

1. Introduction

The class of Valdivia compact spaces plays an important role in study of various properties of Banach spaces. This class, which contains all Corson compact spaces, was studied from the functional-analytic point of view for example in [2], [3], [7], [14] and [15]. Topological properties of Valdivia compact spaces were investigated for example in [16] and [10]. In the present paper we give a characterization of Valdivia compact spaces which generalizes R. Pol's characterization of Corson compact spaces reproduced in [1, Chapter IV.3].

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First we give basic definitions and facts used in our paper. In Section 2 we prove our main result, which is a characterization of Valdivia compact spaces using a topology investigated in [15] and ideas of [1, Chapter IV.3]. An analogous theorem is proved for Valdivia compact spaces which are the dual unit ball of a Banach space.

In Section 3 we prove a theorem on relations of the Valdivia property of K, the dual unit ball of $\mathcal{C}(K)$, and the space of Radon probabilities on K, for K compact.

In Section 4 we study the class of those Valdivia compact spaces which have a dense set of G_δ points. Within this class we prove several results which are not known for general Valdivia compact spaces.

Let us start with basic definitions.

DEFINITION 1.1. Let K be a compact Hausdorff space.

(1) If Γ is a set, we put

$$\Sigma(\Gamma) = \big\{x \in \mathbb{R}^\Gamma \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is countable}\big\}.$$

(2) We say that $A \subset K$ is *a Σ-subset of* K if there is a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $A = h^{-1}(h(K) \cap \Sigma(\Gamma))$.

(3) K is called a *Corson compact space* if K is a Σ-subset of itself.

(4) K is called a *Valdivia compact space* if K has a dense Σ-subset.

(5) We say that K is a *super-Valdivia compact space* if each $x \in K$ is contained in some dense Σ-subset of K.

In the following definition we follow [1, Chapter IV.3], where the notion of primarily Lindelöf spaces is used to characterize Corson compact spaces.

DEFINITION 1.2.

(1) Let κ be an infinite cardinal. By D_κ we denote the discrete space of cardinality κ and L_κ will mean the *one-point Lindelöfication* of D_κ, i.e. $L_\kappa = D_\kappa \cup \{\infty\}$ such that each point of D_κ is isolated and neighborhoods of ∞ are formed by complements of countable subsets of D_κ.

(2) A topological space X is said to be *primarily Lindelöf* if X is a continuous image of a closed subspace of $(L_\kappa)^\omega$ for some infinite cardinal κ.

The following lemma is proved in [1, Proposition IV.3.4].

Lemma 1.3

(1) The class of primarily Lindelöf spaces is closed with respect to countable unions, countable products, closed subspaces and continuous images.

(2) Every primarily Lindelöf space is Lindelöf.

The following notation is inspired by [15], where a special case of these notions is strongly used.

Notation 1.4. Let K and X be topological spaces and $A \subset K$ be an arbitrary subset.

(1) $\mathcal{C}(K, X)$ will denote the set of all continuous functions of K to X.
(2) By τ_A we mean the weakest topology on $\mathcal{C}(K, X)$ such that the mapping $f \mapsto f(a)$ is continuous for every $a \in A$. We will use the shorter notation $\mathcal{C}_A(K, X)$ instead of $(\mathcal{C}(K, X), \tau_A)$.
(3) In case $A = K$ we use the standard notation τ_p and $\mathcal{C}_p(K, X)$ instead of τ_K and $\mathcal{C}_K(K, X)$.
(4) In case $X = \mathbb{R}$ we write $\mathcal{C}(K)$, $\mathcal{C}_A(K)$, $\mathcal{C}_p(K)$ instead of $\mathcal{C}(K, \mathbb{R})$, $\mathcal{C}_A(K, \mathbb{R})$, $\mathcal{C}_p(K, \mathbb{R})$, respectively.

It is clear that, in case $X = \mathbb{R}$, τ_A is a locally convex topology and that it is Hausdorff whenever A is dense in K. An analogous notion can be defined in case that K is the dual unit ball of a Banach space X.

Definition 1.5. Let X be a Banach space and $A \subset X^*$ be arbitrary. By w_A we denote the weakest topology on X such that the mapping $x \mapsto a(x)$ is continuous for every $a \in A$.

Now let us sum up some results on Σ-subsets from [10] which we will use later. To this end we recall some notions.

Definition 1.6.

(1) A topological space X is called a *Fréchet-Urysohn* space (or shortly an *FU-space*) if, whenever $A \subset X$ and $x \in \overline{A}$, there exists a sequence of $x_n \in A$ with $x_n \to x$.
(2) Let X be a topological space and $A \subset X$. We say that A is *countably closed in* X if $\overline{C}^X \subset A$ for every $C \subset A$ countable.

Lemma 1.7

(1) [12, Theorem 2.1] and [10, Proposition 2.2] *The space* $\Sigma(\Gamma)$ *is a Fréchet-Urysohn space and is countably closed in* $\mathbb{R}^\Gamma$ *for every set* Γ. *In particular every* Σ*-subset of a compact space* K *is Fréchet-Urysohn and is countably closed in* K.
(2) [10, Proposition 2.5] *If* K *is a compact Hausdorff space and* $A \subset K$ *dense, then* A *is a* Σ*-subset of* K *if and only if* A *is homeomorphic to a coordinatewise bounded closed subset of some* $\Sigma(\Gamma)$ *and* $K = \beta A$.
(3) [10, Proposition 2.2] *If* K *is a compact Hausdorff space and* $G \subset K$ *a* G_δ *set, then* $G \cap A$ *is dense in* G *for every dense countably compact subset (in particular for every dense* Σ*-subset)* A *of* K.
(4) [10, Proposition 4.2] *Let* A_i *be a* Σ*-subset of* K_i *for* $i = 1, \dots, n$. *Then* $A_1 \times \cdots \times A_n$ *is a* Σ*-subset of* $K_1 \times \cdots \times K_n$.

The referee has pointed out that the assertion in the point (3) of the previous lemma also follows from an argument from pseudocompact spaces. As A is countably compact, it is also pseudocompact, and hence every zero set Z of any compactification of A meets A, i.e. $A \cap Z \neq \emptyset$.

2. A Pol-like characterization of Valdivia compacta

Our main result is the following theorem which generalizes Theorem IV.3.1 of [1].

Theorem 2.1

Let K be a compact Hausdorff space and A be a dense subset of K. Then the following two conditions are equivalent.

(1) A is a Σ-subset of K.

(2) A is countably compact and $\mathcal{C}_A(K)$ is primarily Lindelöf.

As a corollary we get that Theorem IV.3.1 of [1] holds also for Hausdorff completely regular countably compact spaces.

Corollary 2.2

Let X be a Hausdorff completely regular countably compact space. Then X is homeomorphic to a closed subset of some $\Sigma(\Gamma)$ if and only if $\mathcal{C}_p(X)$ is primarily Lindelöf.

Proof of Corollary 2.2. It suffices to observe that X is homeomorphic to a closed subset of some $\Sigma(\Gamma)$ if and only if X is a Σ-subset of βX (Lemma 1.7(2)), and that $\mathcal{C}_p(X)$ is homeomorphic to $\mathcal{C}_X(\beta X)$. Finally, use Theorem 2.1. □

An analogue of Theorem 2.1 holds also in the framework of Banach spaces.

Theorem 2.3

Let X be a Banach space and A be a weak-dense subset of B_{X^*}. Then the following two conditions are equivalent.*

(1) A is a symmetric convex Σ-subset of (B_{X^*}, w^*).

(2) A is weak-countably compact and (X, w_A) is primarily Lindelöf.*

The rest of Section 2 is devoted to proofs of these results. To prove Theorem 2.1 we need several auxiliary results analogic to those used to prove Theorem IV.3.1 in [1].

We start with the following lemma which can be proved in the same way as [1, Proposition IV.3.3].

Lemma 2.4

Let $\mathcal{P}$ be a class of topological spaces which is closed to continuous images and finite products. If K is a compact Hausdorff space, $A \subset K$ is arbitrary and there is $Y \subset \mathcal{C}_A(K,\{0,1\})$ which belongs to $\mathcal{P}$ and separates points of K, then $\mathcal{C}_A(K,\{0,1\})$ belongs to $\mathcal{P}_\sigma$ (i.e., it can be represented as a countable union of spaces from $\mathcal{P}$).

Lemma 2.5 (cf. [1, Lemma IV.3.7])

Let K be a zero-dimensional compact Hausdorff space and $A \subset K$ be arbitrary. Then $\mathcal{C}_A(K,[0,1])$ is a continuous image of $\mathcal{C}_A(K,\{0,1\}^\omega)$.

Proof. By [1, Lemma IV.3.6] there is a continuous map $\varphi : \{0,1\}^\omega \to [0,1]$ such that for every $f \in \mathcal{C}(K,[0,1])$ there exists $g_f \in \mathcal{C}(K,\{0,1\}^\omega)$ with $f = \varphi \circ g_f$. Let us define $F : \mathcal{C}(K,\{0,1\}^\omega) \to \mathcal{C}(K,[0,1])$ by putting $F(g) = \varphi \circ g$. By the choice of φ the mapping F is onto $\mathcal{C}(K,[0,1])$. Moreover, it is clear that F is $\tau_A \to \tau_A$ continuous. $\square$

Proposition 2.6

Let $K \subset \{0,1\}^\Gamma$ be a compact Hausdorff space such that the set $A = K \cap \Sigma(\Gamma)$ is dense in K. Then $\mathcal{C}_A(K)$ is primarily Lindelöf.

Proof. Let us denote by L_Γ the 'one-point Lindelöfication' of the discrete space Γ (cf. Definition 1.2), and define $\psi : L_\Gamma \to \mathcal{C}(K,\{0,1\})$ by putting

$$\psi(\gamma) = \begin{cases} \pi_\gamma | K & \gamma \in \Gamma, \\ 0 & \gamma = \infty, \end{cases}$$

where π_γ is the projection of $\{0,1\}^\Gamma$ onto the γ-th coordinate. It is clear that ψ maps L_Γ to $\mathcal{C}(K,\{0,1\})$. Let us show that ψ is continuous to τ_A. Suppose that γ_ν is a net in L_Γ converging to some $\gamma \in L_\Gamma$. If $\gamma \in \Gamma$, then there is ν_0 such that for every $\nu > \nu_0$ we have $\gamma_\nu = \gamma$, and hence $\psi(\gamma_\nu) = \psi(\gamma)$. If $\gamma = \infty$, we will prove that $\psi(\gamma_\nu) \to 0$ in τ_A. Indeed, if $x \in A$ then $\operatorname{supp} x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$ is countable, thus $U = L_\Gamma \setminus \operatorname{supp} x$ is a neighborhood of ∞. Hence there is some

ν_0 such that $\gamma_\nu \in U$ for $\nu > \nu_0$. Therefore we have, for $\nu > \nu_0$, $\psi(\gamma_\nu)(x) = 0$, so $\psi(\gamma_\nu)(x) \to 0 = \psi(\gamma)(x)$.

By the definition $\psi(L_\Gamma)$ is primarily Lindelöf, an it is clear that it separates points of K. By Lemma 2.4 and Lemma 1.3 the space $\mathcal{C}_A(K, \{0,1\})$ is primarily Lindelöf, by Lemma 1.3 we get that $(\mathcal{C}_A(K, \{0,1\}))^\omega$ is primarily Lindelöf as well. Now it follows from the definition of product topology that $(\mathcal{C}_A(K, \{0,1\}))^\omega$ is homeomorphic to $\mathcal{C}_A(K, \{0,1\}^\omega)$. By Lemma 2.5 we get that $\mathcal{C}_A(K, [0,1])$ is primarily Lindelöf. Finally, the space $\mathcal{C}_A(K)$ is the union of the sequence $\mathcal{C}_A(K, [-n,n])$ for $n \in \mathbb{N}$. Thus it follows from Lemma 1.3 that $\mathcal{C}_A(K)$ is primarily Lindelöf. □

Definition 2.7. Let $\varphi : K \to L$ be a continuous surjection between compact Hausdorff spaces. By T_φ we denote the mapping $T_\varphi : \mathcal{C}(L) \to \mathcal{C}(K)$ defined by the formula $T_\varphi(g) = g \circ \varphi$.

It is well-known and easy to see that T_φ is an isometric embedding of $\mathcal{C}(L)$ into $\mathcal{C}(K)$.

Lemma 2.8

Let $\varphi : K \to L$ be a continuous surjection between two compact Hausdorff spaces and $f \in \mathcal{C}(K)$. Then $f \in T_\varphi(\mathcal{C}(L))$ if and only if f is constant on $\varphi^{-1}(l)$ for every $l \in L$.

Proof. The 'only if' part is obvious. Let us prove the 'if' part. Let f be constant on each $\varphi^{-1}(l)$. Then we can define a real function on L by putting

$$g(l) = f(k) \quad \text{if} \quad \varphi(k) = l\,.$$

Due to the assumption on f this definition is correct and we have $f = g \circ \varphi$. Now g is continuous as f is continuous and φ is a quotient map. This completes the proof. □

The formulation of the point (2) of the following lemma was suggested to the author by P. Holický.

Lemma 2.9

Let $\varphi : K \to L$ be a continuous surjection between two compact Hausdorff spaces, $A \subset K$ be arbitrary and $B = \varphi(A)$. Then the following assertions hold.

(1) T_φ is a $\tau_B \to \tau_A$ homeomorphism.
(2) Put $E = \{(x,y) \in K \times K \mid \varphi(x) = \varphi(y)\}$. If $E \cap (A \times A)$ is dense in E, then $T_\varphi(\mathcal{C}(L))$ is τ_A-closed in $\mathcal{C}(K)$.

Proof. (1) This is trivial.

(2) Let $f_\nu \xrightarrow{\tau_A} f$, where $f_\nu \in T_\varphi(\mathcal{C}(L))$ and $f \in \mathcal{C}(K)$. By Lemma 2.8 it is enough to show that $f(x) = f(y)$ provided $\varphi(x) = \varphi(y)$. So let $\varphi(x) = \varphi(y)$, which means $(x, y) \in E$. By the assumptions there is a net $(x_\mu, y_\mu) \in E \cap (A \times A)$ converging to (x, y). As $f_\nu \in T_\varphi(\mathcal{C}(L))$, we have $f_\nu(x_\mu) = f_\nu(y_\mu)$ for every ν and μ. Since $x_\mu, y_\mu \in A$, we get $f_\nu(x_\mu) \xrightarrow{\nu} f(x_\mu)$ and $f_\nu(y_\mu) \xrightarrow{\nu} f(y_\mu)$, so $f(x_\mu) = f(y_\mu)$ for every μ. Now it follows from the continuity of f that $f(x) = f(y)$ which completes the proof. $\square$

Proposition 2.10

Let $K \subset [0,1]^\Gamma$ be compact such that the set $A = K \cap \Sigma(\Gamma)$ is dense in K. Then there is a compact space $L \subset \{0,1\}^{\Gamma'}$ with $B = L \cap \Sigma(\Gamma')$ dense in L and a continuous surjection $\varphi : L \to K$ such that $\mathcal{C}_A(K)$ is homeomorphic to $(T_\varphi(\mathcal{C}(K)), \tau_B)$ and the latter set is τ_B-closed in $\mathcal{C}(L)$.

Proof. Let $\psi : \{0,1\}^\omega \to [0,1]$ be a continuous surjection such that $\psi^{-1}(0) = \{0\}$. Such a mapping exists, one can take for example $\psi(x) = \sum\limits_{n\in\omega} \frac{x_n}{2^{n+1}}$. We can define the mapping $\Psi : (\{0,1\}^\omega)^\Gamma \to [0,1]^\Gamma$ by the formula $\Psi(x)(\gamma) = \psi(x(\gamma))$. It is clear that Ψ is a continuous surjection satisfying the condition

$$x \in \Sigma(\omega \times \Gamma) \Leftrightarrow \Psi(x) \in \Sigma(\Gamma). \tag{$*$}$$

Put $L = \Psi^{-1}(K)$, $B = L \cap \Sigma(\omega \times \Gamma)$ and $\varphi = \Psi|L$. By (*) we get that $A = \varphi(B)$, so $\mathcal{C}_A(K)$ is homeomorphic to $(T_\varphi(\mathcal{C}(K)), \tau_B)$ by Lemma 2.9(1). It remains to prove that B is dense in L and $(T_\varphi(\mathcal{C}(K)), \tau_B)$ is τ_B-closed in $\mathcal{C}(L)$. This will be proved when we show the following claim.

$$\begin{gathered} a, b \in L, \varphi(a) = \varphi(b) \Rightarrow \exists \text{ a net } (a_\nu, b_\nu) \in B \times B, \\ \varphi(a_\nu) = \varphi(b_\nu), (a_\nu, b_\nu) \to (a, b). \end{gathered} \tag{$**$}$$

Indeed, to show the density of B it suffices to take $a = b$, and to prove that $T_\varphi(\mathcal{C}(K))$ is τ_B-closed in $\mathcal{C}(L)$ it is enough to use Lemma 2.9(2). So let us prove (**).

Put $c = \varphi(a) = \varphi(b)$. Let $\mathcal{G}$ denote the family of all G_δ subsets of K containing c, ordered by the inverse inclusion. For any $G \in \mathcal{G}$ choose some $c_G \in A \cap G$. This is possible due to Lemma 1.7(3). The net c_G converges to c in a strong sense, i.e.

$$\forall \gamma \in \Gamma\ \exists G \in \mathcal{G}\ \forall H \in \mathcal{G}\ H \subset G\ c_H(\gamma) = c(\gamma). \tag{$***$}$$

Indeed, it suffices to take $G = \{x \in K \mid x(\gamma) = c(\gamma)\}$. Now we will construct a_G and b_G in the following manner. If $c_G(\gamma) = c(\gamma)$ put $a_G(\gamma) = a(\gamma)$ and $b_G(\gamma) = b(\gamma)$, otherwise choose $a_G(\gamma), b_G(\gamma) \in \psi^{-1}(c_G(\gamma))$ arbitrary. It is clear that $\varphi(a_G) = \varphi(b_G) = c_G$, by (*) we have $a_G, b_G \in B$ and it follows easily from (***) that $a_G \to a$ and $b_G \to b$. This completes the proof of (**). $\square$

Lemma 2.11

Let X be countably compact and $f : X \to \Sigma(\Gamma)$ be continuous. Then $f(X)$ is closed in $\Sigma(\Gamma)$. In particular f is homeomorphism whenever it is one-to-one.

Proof. As X is countably compact and f continuous, $f(X)$ is countably compact as well. If $x \in \overline{f(X)} \setminus f(X)$, then by Lemma 1.7(1) there is a sequence $x_n \in f(X)$ with $x_n \to x$. But then the infinite countable set $\{x_n \mid n \in \mathbb{N}\}$ has no accumulation point in $f(X)$ which contradicts the countable compactness of $f(X)$.

Now suppose that f is one-to-one. If $F \subset X$ is closed, then F is countably compact and hence, by the previous paragraph, $f(F)$ is closed in $\Sigma(\Gamma)$. It follows that f is a closed mapping, so it is a homeomorphism. $\square$

Proposition 2.12

Let K be a Hausdorff completely regular topological space and $A \subset K$ be such that $\mathcal{C}_A(K)$ is primarily Lindelöf. Then there is a one-to-one continuous mapping of A into some $\Sigma(\Gamma)$.

Proof. It is proved in [1, Proposition IV.3.10] that, whenever X is primarily Lindelöf, there is a continuous linear one-to-one mapping of $\mathcal{C}_p(X)$ into some $\Sigma(\Gamma)$. Therefore under our assumptions there is $T : \mathcal{C}_p(\mathcal{C}_A(K)) \to \Sigma(\Gamma)$ continuous, linear and one-to-one. Consider the map $e : A \to \mathcal{C}_p(\mathcal{C}_A(K))$ defined by the formula $e(a)(f) = f(a)$. By the definition of τ_A we get that $e(a)$ is τ_A-continuous for every $a \in A$. Moreover, the mapping e is clearly continuous. Finally, e is one-to-one as K is Hausdorff completely regular. Now it suffices to observe that $T \circ e$ is a one-to-one continuous mapping of A into $\Sigma(\Gamma)$. $\square$

Proposition 2.13

Let K be a compact Hausdorff space and $A \subset K$ be a countably closed dense subset. If $\mathcal{C}_A(K)$ is Lindelöf, then $K = \beta A$.

Proof. Put $K' = \beta A$ and let $\psi : K' \to K$ be the continuous extension of the identity mapping of A. It is enough to show that $\psi(u) \neq \psi(v)$ whenever $u, v \in K' \setminus A$ are distinct.

Suppose that $u, v \in K' \setminus A$ are distinct such that $\psi(u) = \psi(v) = p$. Clearly $p \in K \setminus A$. Denote by $\mathcal{G}$ the set of all G_δ subsets of K containing p, ordered by the inverse inclusion. Choose U and V open neighborhoods of u and v, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. For every $G \in \mathcal{G}$ choose $u_G \in U \cap \psi^{-1}(G) \cap A$ and $v_G \in V \cap \psi^{-1}(G) \cap A$. This is possible by Lemma 1.7(3). Further put

$$W_G = \left\{ f \in \mathcal{C}(K) \mid |f(\psi(u_G)) - f(\psi(v_G))| < 1 \right\}, \qquad G \in \mathcal{G}\,.$$

Clearly each W_G is a τ_A-open set in $\mathcal{C}(K)$. Moreover these sets cover $\mathcal{C}(K)$. Indeed, it is enough to observe that $\psi(u_G) \to p$ and $\psi(v_G) \to p$ as well. As $\mathcal{C}_A(K)$ is Lindelöf, there is a sequence G_n, $n \in \mathbb{N}$ such that $\mathcal{C}(K) = \bigcup_{n\in\mathbb{N}} W_{G_n}$. Put $H = \{\psi(u_{G_n}) \mid n \in \mathbb{N}\}$ and $L = \{\psi(v_{G_n}) \mid n \in \mathbb{N}\}$. As A is countably closed, we have $\overline{H} \subset A$ and $\overline{L} \subset A$. Since $\psi^{-1}(H) \subset U$, $\psi^{-1}(L) \subset V$ and $\psi|A$ is a homeomorphism, we have $\overline{H} \cap \overline{L} = \emptyset$. So there is $f \in \mathcal{C}(K)$ with $f|L = 0$ and $f|H = 1$. Then f belongs to no W_{G_n}, which is a contradiction completing the proof. □

Proof of Theorem 2.1 (1) $\Rightarrow$ (2) Suppose that A is a dense Σ-subset of K. By Lemma 1.7(1) the set A is countably closed in K, and hence countably compact. It remains to show that $\mathcal{C}_A(K)$ is primarily Lindelöf. But it follows immediately from Proposition 2.10 and Proposition 2.6.

(2) $\Rightarrow$ (1) As K is Hausdorff completely regular, there is, by Proposition 2.12, a continuous one-to-one mapping $f : A \to \Sigma(\Gamma)$ for some set Γ. Since A is countably compact, it follows from Lemma 2.11 that f is a homeomorphism of A onto a closed subset of $\Sigma(\Gamma)$ which by countable compactness is coordinatewise bounded. By Proposition 2.13 we get $K = \beta A$. Now it is enough to use Lemma 1.7(2) to get that A is a Σ-subset of K. □

To prove Theorem 2.3 we need some more lemmas.

Lemma 2.14

Let X be a Banach space and $A \subset B_{X^*}$ be a dense convex symmetric set. Then (X, w_A) is homeomorphic to a closed subset of $\mathcal{C}_A(B_{X^*}, w^*)$.

Proof. Consider the natural inclusion i of X into $\mathcal{C}(B_{X^*})$, i.e. $i(x)(f) = f(x)$, $f \in B_{X^*}$, $x \in X$. It is clear that i is $w_A \to \tau_A$ homeomorphism. Suppose that $x_\nu \in X$ and $h \in \mathcal{C}(B_{X^*})$ such that $i(x_\nu) \xrightarrow{\tau_A} h$. As $i(x_\nu)$ is affine and A convex symmetric, we get that $h|A$ is affine (i.e., $h(ta + (1-t)b) = th(a) + (1-t)h(b)$ whenever $a, b \in A$ and $t \in [0,1]$). Since A is dense and h continuous, it follows that h is affine on B_{X^*} (and, of course, $h(0) = 0$). And it is well-known that in this case $h \in i(X)$ (it follows from [5, Theorem V.5.6 and Theorem V.3.9]). This completes the proof. □

Lemma 2.15

Let X be a Hausdorff topological space and $M \subset X$ an arbitrary set. If $A \subset X$ and $B \subset X$ are both homeomorphic to a coordinatewise bounded closed subset of $\Sigma(\Gamma)$ for some Γ and $A \cap B \cap M$ is dense in M, then $A \cap M = B \cap M$.

Proof. Let $x \in A \cap M$. Then $x \in \overline{A \cap B \cap M}$, and hence there is a sequence $x_n \in A \cap B \cap M$ with $x_n \to x$ (since A is an FU-space by Lemma 1.7(1)). Now, $x_n \in B$ for each n, B is countably compact (by Lemma 1.7(1)), so $x \in B$. Thus $A \cap M \subset B \cap M$. The inverse inclusion can be proved by interchanging the roles of A and B. □

Lemma 2.16

Let X be a Banach space and $A \subset B_{X^*}$ be weak-dense and weak*-countably compact. If (X, w_A) is primarily Lindelöf, then A is a convex symmetric set which is homeomorphic to a closed coordinatewise bounded subset of some $\Sigma(\Gamma)$.*

Proof. By [1, Proposition IV.3.10] there is a continuous one-to-one linear mapping $T : \mathcal{C}_p(X, w_A) \to \Sigma(\Gamma)$ for some Γ. By the definition of w_A we get that $A \subset \mathcal{C}(X, w_A)$, and so by linearity of X it follows that $\operatorname{span} A \subset \mathcal{C}(X, w_A)$. As the weak* topology on $\operatorname{span} A$ coincides with the τ_p topology inherited from $\mathcal{C}_p(X, w_A)$, we have a continuous one-to-one linear mapping of $(\operatorname{span} A, w^*)$ into $\Sigma(\Gamma)$. Further, as A is countably compact, we get by Lemma 2.11 that A is homeomorphic to a closed subset of $\Sigma(\Gamma)$ which by countable compactness is coordinatewise bounded. Now we are going to prove that A is symmetric and absolutely convex. In fact, we show that $A = \operatorname{span} A \cap B_{X^*}$. Suppose that $g \in \operatorname{span} A \cap B_{X^*}$. Then $A \cup \{g\}$ is countably compact, and so homeomorphic to a coordinatewise bounded closed subset of $\Sigma(\Gamma)$ by Lemma 2.11. Now, as A is dense in B_{X^*}, it follows by Lemma 2.15 that $A \cup \{g\} = A$, i.e. $g \in A$, which was to be shown. □

Lemma 2.17

Let E be a locally convex space and $A \subset E$ be a convex symmetric set homeomorphic to a closed coordinatewise bounded subset of some $\Sigma(\Gamma)$. Then there is a vector space F endowed with a topology τ, $B \subset F$ convex symmetric, a linear isomorphism $L : \operatorname{span} B \to \operatorname{span} A$ such that the following hold.

(i) $L|B$ is a homeomorphism of B onto A.
(ii) $\beta B = \overline{B}$ and it is a convex symmetric set.
(iii) The vector operations ($+$ and $\cdot$) are continuous on $n\overline{B}$ for every $n \in \mathbb{N}$.
(iv) Every point of $\overline{B}$ has a neighborhood basis (in $\overline{B}$) consisting of convex sets.

Proof. Put $A_n = nA$ for $n \in \mathbb{N}$. Let $m < n$. By Lemma 1.7(2) we have that A_n is a Σ-subset of βA_n. It is clear that A_m is a Σ-subset of $\overline{A_m}^{\beta A_n}$, so $\beta A_m = \overline{A_m}^{\beta A_n}$ by Lemma 1.7(2). Hence we have $\beta A_m \subset \beta A_n$ in the natural sense. Therefore we can define

$$F = \bigcup_{n \in \mathbb{N}} \beta A_n \, .$$

We define a topology τ on F by the following formula.

$$U \in \tau \Leftrightarrow \forall n \in \mathbb{N} \; U \cap \beta A_n \text{ is open in } \beta A_n \, .$$

Now we are going to define operations on F.

By Lemma 1.7(4) we get that $A_n \times A_n$ is a dense Σ-subset of $\beta A_n \times \beta A_n$, so $\beta(A_n \times A_n) = \beta A_n \times \beta A_n$ (this also follows directly from [8]). The mapping $A_n \times A_n \to A_{2n}$ assigning to a pair (x, y) its sum $x + y$ is continuous, hence we can extend it to a continuous mapping $\beta A_n \times \beta A_n \to \beta A_{2n}$. We shall denote this mapping by '$+_n$'. It is easy to check that '$+_n$' is an extension of '$+_m$' whenever $m < n$, so we have a mapping $+ : F \times F \to F$.

By Lemma 1.7(4) we get that $A_n \times [-n, n]$ is a dense Σ-subset of $\beta A_n \times [-n, n]$, so $\beta(A_n \times [-n, n]) = \beta A_n \times [-n, n]$. The mapping $A_n \times [-n, n] \to A_{n^2}$ assigning to a pair (x, t) its product $t \cdot x$ is continuous, hence we can extend it to a continuous mapping $\beta A_n \times [-n, n] \to \beta A_{n^2}$. We shall denote this mapping by '$\cdot_n$'. It is easy to check that '$\cdot_n$' is an extension of '$\cdot_m$' whenever $m < n$, so we have a mapping $\cdot : F \times \mathbb{R} \to F$.

It is clear that F together with just defined operations is a linear space. If we put $B = A$ and let L be the natural identity mapping, it is clear that conditions (i)–(iii) are satisfied. It remains to prove the condition (iv).

First let us prove that the point 0 has a neighborhood basis (in $n\overline{B} = \beta A_n$) consisting of convex sets for every n. It is clear that we can suppose without loss of generality that $n = 1$. Let U be a neighborhood of 0 in $\overline{B}$. By regularity (notice that $\overline{B}$ is compact) we can choose U', a neighborhood of 0 in $\overline{B}$, such that $\overline{U'} \subset U$. As $U' \cap B = U' \cap A$ is a neighborhood of 0 in A and E is locally convex, there is W, an open convex neighborhood of 0 in A with $W \subset U' \cap A$. Let V be open in $\overline{B}$ such that $V \cap A = W$. Then $\overline{V} = \overline{W}$ is a convex neighborhood of 0 with $\overline{V} \subset U$.

Now let $x \in \overline{B}$ be arbitrary and U be a neighborhood of x in $\overline{B}$. There is U', a neighborhood of x in $2\overline{B}$ with $U' \cap \overline{B} = U$. It follows from the condition (iii) that the set

$$W = \left\{y \in 2\overline{B} \mid y + x \in U'\right\}$$

is a neighborhood of 0 in the set

$$M = \left\{y \in 2\overline{B} \mid y + x \in 2\overline{B}\right\},$$

so there is a neighborhood W' of 0 in $2\overline{B}$ such that $M \cap W' = W$. By the previous paragraph there is a convex neighborhood V' of 0 in $2\overline{B}$ with $V' \subset W'$. As M is clearly convex, $V' \cap M$ is convex as well. Moreover, $x + (V' \cap M) \subset U'$, hence $G = (x + (V' \cap M)) \cap \overline{B} \subset U$. Clearly G is convex, so it remains to show that G is a neighborhood of x in $\overline{B}$. If it is not the case, there is a net $x_\nu \in \overline{B} \setminus G$ converging to x. Then $x_\nu - x \to 0$ and $x_\nu - x \notin V' \cap M$. But we have $x_\nu - x \in M$, so $x_\nu - x \notin V'$. But V' is a neighborhood of 0, so $x_\nu - x$ cannot converge to 0. This contradiction completes the proof. □

Lemma 2.18

Let X be a Banach space and $A \subset B_{X^*}$ be a convex symmetric weak-dense set homeomorphic to a closed coordinatewise bounded subset of some $\Sigma(\Gamma)$. If (X, w_A) is Lindelöf, then $B_{X^*} = \beta A$.*

Proof. Let F, B, L be as in Lemma 2.17, and $\psi : \overline{B} \to B_{X^*}$ be the continuous extension of $L|B$. It is clear that ψ is an affine mapping. We shall prove that ψ is one-to-one.

We proceed similarly as in the proof of Proposition 2.13. Suppose that there are $u, v \in \overline{B} \setminus B$ distinct with $\psi(u) = \psi(v) = p \in B_{X^*} \setminus A$. Let U and V be convex neighborhoods of u and v, respectively, with $\overline{U} \cap \overline{V} = \emptyset$. Denote by $\mathcal{G}$ the set of all weak* G_δ subsets of B_{X^*} containing p, ordered by the inverse inclusion. For every $G \in \mathcal{G}$ choose $u_G \in U \cap \psi^{-1}(G) \cap B$ and $v_G \in V \cap \psi^{-1}(G) \cap B$. This is possible by Lemma 1.7(3). Further put

$$W_G = \left\{x \in X \mid |\psi(u_G)(x) - \psi(v_G)(x)| < 1\right\}, \qquad G \in \mathcal{G}.$$

Clearly each W_G is a w_A-open set in X. Moreover these sets cover X. Indeed, it is enough to observe that $\psi(u_G)\xrightarrow{w^*}p$ and $\psi(v_G)\xrightarrow{w^*}p$ as well. As (X, w_A) is Lindelöf, there is a sequence G_n, $n \in \mathbb{N}$ such that $X = \bigcup_{n\in\mathbb{N}} W_{G_n}$. Put $H_1 = \{\psi(u_{G_n}) \mid n \in \mathbb{N}\}$ and $H_2 = \{\psi(v_{G_n}) \mid n \in \mathbb{N}\}$. As A is convex and weak*-countably closed, we have $\overline{\operatorname{conv} H_1}^{w^*} \subset A$ and $\overline{\operatorname{conv} H_2}^{w^*} \subset A$. Since $\psi^{-1}(H_1) \subset U$, $\psi^{-1}(H_2) \subset V$ and $\psi|B$ is an affine homeomorphism, we have $\overline{\operatorname{conv} H_1}^{w^*} \cap \overline{\operatorname{conv} H_2}^{w^*} = \emptyset$. It remains to use Hahn-Banach theorem to get $x \in X$ such that

$$\sup\left\{f(x) \mid f \in \overline{\operatorname{conv} H_1}^{w^*}\right\} + 1 < \inf\left\{f(x) \mid f \in \overline{\operatorname{conv} H_2}^{w^*}\right\}.$$

Then x belongs to no W_{G_n}, which is a contradiction completing the proof. □

Proof of Theorem 2.3 (1) ⇒ (2) Let $A \subset (B_{X^*}, w^*)$ be a convex symmetric dense Σ-subset. By Lemma 1.7(1) we get that A is countably compact. By Theorem 2.1 the space $\mathcal{C}_A(B_{X^*}, w^*)$ is primarily Lindelöf. It follows from Lemma 2.14 that (X, w_A) is homeomorphic to a closed subset of $\mathcal{C}_A(B_{X^*}, w^*)$, so it is primarily Lindelöf by Lemma 1.3.

(2) ⇒ (1) By Lemma 2.16 we have that A is convex, symmetric and homeomorphic to a coordinatewise bounded closed subset of some $\Sigma(\Gamma)$. It follows from Lemma 2.18 that $B_{X^*} = \beta A$. It remains to use Lemma 1.7(2) to get that A is a Σ-subset of B_{X^*}. □

Remarks 2.19. (1) It is easy to check that neither of the two assumptions in the condition (2) of Theorem 2.1 (or Theorem 2.3) can be omitted. Indeed, notice that $\mathcal{C}_B(K)$ is primarily Lindelöf whenever $\mathcal{C}_A(K)$ is primarily Lindelöf and $B \subset A$; and that K is countably compact but need not be a Σ-subset of itself.

(2) In Theorem 2.1 we characterize dense Σ-subsets of K. It is natural to ask whether we can characterize in a similar way dense sets which are contained in a Σ-subset. Of course, $\mathcal{C}_A(K)$ is primarily Lindelöf, whenever A is contained in a dense Σ-subset of K. But the converse implication does not hold. Put $K = \beta\mathbb{N}$ and $A = \mathbb{N}$. Then $\mathcal{C}_A(K)$ is homeomorphic to an F_σ subset of $\mathbb{R}^{\mathbb{N}}$. The latter space is a separable completely metrizable space, and so it is primarily Lindelöf by a classical theorem [11, Theorem 7.9]. Hence $\mathcal{C}_A(K)$ is primarily Lindelöf by Lemma 1.3. However, A is contained in no dense Σ-subset of K, as K is not Valdivia (it follows e.g. from [4, Theorem II.7.10] and [14, Corollary], or it can be proved by an elementary argument).

Theorem 2.1 has the following consequence on continuous images of Σ-subsets.

Theorem 2.20

Let $\varphi : K \to L$ be a continuous surjection between compact Hausdorff spaces, $A \subset K$ be a dense Σ-subset and $B = \varphi(A)$. Then the following assertions are equivalent.

(1) B is a Σ-subset of L.
(2) $T_\varphi(\mathcal{C}(L))$ is τ_A-closed in $\mathcal{C}(K)$.
(3) $L = \beta B$ and $\varphi|A$ is a quotient mapping of A onto B.

Proof. (1) $\Rightarrow$ (3) If B is a Σ-subset of L, then $L = \beta B$ by Lemma 1.7(2). Moreover, B is homeomorphic to a closed coordinatewise bounded subset of $\Sigma(\Gamma)$ for some set Γ, hence it follows from Lemma 2.11 that $\varphi|A$ is closed, and therefore a quotient mapping.

(3) $\Rightarrow$ (2) Let $f_\nu \in T_\varphi(\mathcal{C}(L))$ and $f \in \mathcal{C}(L)$ such that $f_\nu \xrightarrow{\tau_A} f$. We will prove that $f \in T_\varphi(\mathcal{C}(L))$. It follows from the definition of τ_A that f is constant on $\varphi^{-1}(l) \cap A$ for every $l \in L$. Hence there is a function $g : B \to \mathbb{R}$ such that $f|A = g \circ (\varphi|A)$. As f is continuous and $\varphi|A$ is a quotient mapping, we get that g is continuous as well. Now, since $L = \beta B$, there is a continuous extension $\tilde{g}$ of g onto L. It follows that $f = \tilde{g} \circ \varphi$ which completes the argument.

(2) $\Rightarrow$ (1) By Theorem 2.1 we have that A is countably compact and $\mathcal{C}_A(K)$ is primarily Lindelöf. Hence B is countably compact (as a continuous image of A). Moreover, $\mathcal{C}_B(L)$ is primarily Lindelöf by Lemma 1.3 and Lemma 2.9(1). It follows from Theorem 2.1 that B is a dense Σ-subset of L. $\square$

Remark. The implication (2) $\Rightarrow$ (1) of the previous theorem can be proved also using [15, Theorem 2].

3. The case of $\mathcal{C}(K)$ spaces

The aim of this section is to prove Theorem 3.2 on relations between K and the dual unit ball of $\mathcal{C}(K)$. Let us recall that due to Riesz theorem we can identify $\mathcal{C}(K)^*$ with the space of all finite signed Radon measures on K. First let us fix some notation.

Notation 3.1 Let K be a compact Hausdorff space.

(1) By $P(K)$ we denote the space of all Radon probabilities on K, endowed with the weak* topology. We consider $P(K)$ as a subset of $\mathcal{C}(K)^*$, by the identification

$$P(K) = \left\{\mu \in \mathcal{C}(K)^* \mid \|\mu\| \le 1 \ \&\ (\mu, 1_K) = 1\right\}.$$

(2) If $x \in K$, we denote by δ_x the Dirac measure supported by the point x.
(3) If $A \subset K$, we denote by $\delta(A)$ the homeomorphic image of A by the mapping $\delta : x \mapsto \delta_x$.

Theorem 3.2

Let K be a compact Hausdorff space. Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4), *where*

(1) K is Valdivia.
(2) There is a dense convex symmetric Σ-subset of $(B_{\mathcal{C}(K)^*}, w^*)$.
(3) There is a dense convex Σ-subset of $(B_{\mathcal{C}(K)^*}, w^*)$.
(4) There is a dense convex Σ-subset of $P(K)$.

Proof. (1) $\Rightarrow$ (2) Let $h : K \to \mathbb{R}^\Gamma$ be a homeomorphic injection with $h(K) \cap \Sigma(\Gamma)$ dense in $h(K)$. Put $A = h^{-1}(h(K) \cap \Sigma(\Gamma))$. For $\gamma \in \Gamma$ let $f_\gamma = \pi_\gamma \circ h$, where π_γ denotes the projection of $\mathbb{R}^\Gamma$ onto the γ-th coordinate. It is clear that the family $(f_\gamma \mid \gamma \in \Gamma)$ separates points of K and that

$$A = \left\{x \in k \mid \{\gamma \in \Gamma \mid f_\gamma(x) \neq 0\} \text{ is countable}\right\}.$$

Let $\tilde{\Gamma}$ be the set of all (possibly empty) finite sequences of elements of Γ. For $\tilde{\gamma} \in \tilde{\Gamma}$ let us define

$$g_{\tilde{\gamma}} = \begin{cases} 1 & \text{if } \tilde{\gamma} = \emptyset\,, \\ f_{\gamma_1} \cdot \ldots \cdot f_{\gamma_n} & \text{if } \tilde{\gamma} = (\gamma_1, \ldots, \gamma_n). \end{cases}$$

It follows from Stone-Weierstrass theorem, that $\overline{\operatorname{span}\{g_{\tilde{\gamma}} \mid \tilde{\gamma} \in \tilde{\Gamma}\}} = \mathcal{C}(K)$, hence the family $\left(g_{\tilde{\gamma}} \mid \tilde{\gamma} \in \tilde{\Gamma}\right)$ separates points of $\mathcal{C}(K)^*$. Therefore, if we define the mapping $\tilde{h} : B_{\mathcal{C}(K)^*} \to \mathbb{R}^{\tilde{\Gamma}}$ by the formula

$$\tilde{h}(\mu)(\tilde{\gamma}) = (\mu, g_{\tilde{\gamma}})\,,$$

it is a homeomorphic injection ($B_{\mathcal{C}(K)^*}$ is considered with the weak* topology). Put

$$\tilde{A} = \tilde{h}^{-1}\left(\tilde{h}(B_{\mathcal{C}(K)^*}) \cap \Sigma(\tilde{\Gamma})\right).$$

As $\tilde{h}$ is clearly affine, it follows that $\tilde{A}$ is a convex symmetric set. Moreover, $\tilde{A}$ contains $\delta(A)$. Indeed, if $x \in A$ and $\tilde{\gamma} \in \tilde{\Gamma}$ with $g_{\tilde{\gamma}}(x) \neq 0$, then either $\tilde{\gamma} = \emptyset$ or $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i(x) \neq 0$, $i = 1, \ldots, n$. So clearly $\{\tilde{\gamma} \in \tilde{\Gamma} \mid g_{\tilde{\gamma}}(x) \neq 0\}$ is countable, and therefore $\delta_x \in \tilde{A}$. So $\operatorname{conv}(\delta(A) \cup (-\delta(A))) \subset \tilde{A}$. Now it is a standard fact that $\operatorname{conv}(\delta(A) \cup (-\delta(A)))$ is weak* dense in $B_{\mathcal{C}(K)^*}$ whenever A is dense in K.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (4) This follows easily from Lemma 1.7(3) as $P(K)$ is weak*-G_δ closed convex subset of $B_{\mathcal{C}(K)^*}$.

(4) $\Rightarrow$ (2) Let A be a convex dense Σ-subset of $P(K)$. Then $\mathcal{C}_A(P(K))$ is primarily Lindelöf by Theorem 2.1. Let us consider the injection $T : \mathcal{C}(K) \to \mathcal{C}(P(K))$ defined by the formula $T(f)(\mu) = (\mu, f)$, $\mu \in P(K)$, $f \in \mathcal{C}(K)$. We claim that $F \in \mathcal{C}(P(K))$ belongs to $T(\mathcal{C}(K))$ if and only if F is affine.

The 'only if' part is obvious, so let us prove the 'if' part. Let $F \in \mathcal{C}(P(K))$ be affine. Put $f = (F|\delta(K)) \circ \delta$. Then $f \in \mathcal{C}(K)$. And it easily follows from the facts that F is continuous and affine that $F = T(f)$.

Further let us notice that $T(\mathcal{C}(K))$ is τ_A-closed in $\mathcal{C}(P(K))$. Indeed, let $F_\nu \xrightarrow{\tau_A} F$, where $F_\nu \in T(\mathcal{C}(K))$ and $F \in \mathcal{C}(P(K))$. Each F_ν is affine and A is convex, so $F|A$ is affine. As A is dense and F continuous, we get that F is affine, hence $F \in T(\mathcal{C}(K))$.

So $T(\mathcal{C}(K))$ is τ_A-primarily Lindelöf, and clearly $(\mathcal{C}(K), w_A)$ is homeomorphic to $(T(\mathcal{C}(K)), \tau_A)$, therefore $(\mathcal{C}(K), w_A)$ is primarily Lindelöf. Put $\tilde{A} = \operatorname{conv}(A \cup (-A))$. Then clearly $w_A = w_{\tilde{A}}$, so $(\mathcal{C}(K), w_{\tilde{A}})$ is primarily Lindelöf. In view of Theorem 2.3 it is enough to show that $\tilde{A}$ is weak* countably compact and weak* dense in $B_{\mathcal{C}(K)^*}$.

By Lemma 1.7(4) we have that $A \times A \times [0,1]$ is a dense Σ-subset of $P(K) \times P(K) \times [0,1]$, and hence it is countably compact by Lemma 1.7(1). Consider the mapping

$$\psi : P(K) \times P(K) \times [0,1] \to B_{\mathcal{C}(K)^*}$$

defined by the formula

$$\psi(\mu, \nu, t) = t\mu - (1-t)\nu.$$

Then ψ is continuous and onto, and $\tilde{A} = \psi(A \times A \times [0,1])$, so clearly $\tilde{A}$ is weak* countably compact and weak* dense. This completes the proof. $\square$

Remarks 3.3. (1) The implication (1) $\Rightarrow$ (2) follows also from [15, Corollary 2.2] together with a simple observation [9, Lemma 3]. But our proof is more simple and direct, we use only Stone-Weierstrass theorem.

(2) We do not know whether the condition (2) of Theorem 3.2 is equivalent to $B_{\mathcal{C}(K)^*}$ being Valdivia compact. Neither do we know whether the implication (2) $\Rightarrow$ (1) holds. However, both of these questions have positive answer when K has a dense set of G_δ points, as proved in Theorem 4.10.

(3) In view of the implication (1) $\Rightarrow$ (2) it is natural to ask whether $B_{\mathcal{C}(K)^*}$ is super-Valdivia whenever K is super-Valdivia. This is answered in the negative by Example 4.12.

4. Valdivia compact spaces with a dense set of G_δ points

The class of compact Hausdorff spaces with a dense set of G_δ points contains several important classes of spaces, as summed up in the following proposition. It turns out that within this class some properties of Valdivia compacta are easier to describe.

Proposition 4.1

Let K be a compact Hausdorff space satisfying at least one of the following conditions.

(1) K is scattered.
(2) K is a Radon-Nikodym compact (or even a continuous image of a Radon-Nikodym compact).
(3) K is fragmentable.
(4) K belongs to the Stegall class $\mathcal{S}$.
(5) $\mathcal{C}(K)$ is a weak Asplund space
(6) $\mathcal{C}(K)$ is a Gâteaux differentiability space.
(7) K is Corson.

Then K contains a dense subset of G_δ points.

Proof. The definitions and proofs of cases (1)–(6) can be found e.g. in [6]. For the proof of the case (7) we refer to [13]. □

Remark. Let us notice that the conditions of the above proposition are not mutually exclusive. On the contrary, many inclusions hold. But we would have liked to name important classes of compacta for which the results of this section hold.

Due to the case (7), the results of this section contain as a special case some known results on Corson compact spaces.

Lemma 4.2

Let K be a compact Hausdorff space. with a dense set of G_δ points. Then there is at most one dense Σ-subset of K.

Proof. Let M be the set of all G_δ points of K, A and B be two dense Σ-subsets of K. By Lemma 1.7(3) we have $M \subset A \cap B$, hence $A \cap B$ is dense in K. Now $A = B$ by Lemma 2.15. □

The proof of the following lemma was shown to the author by L. Zajíček.

Lemma 4.3

Let K be a compact Hausdorff space with a dense set of G_δ points. Then every open continuous image of K has the same property.

Proof. Let $\varphi : K \to L$ be a continuous open surjection. It is enough to prove that $\varphi(x)$ is a G_δ point in L whenever x is a G_δ point in K.

Let $\{x\} = \bigcap_{n\in\mathbb{N}} U_n$ where each U_n is open in K. As K is regular, we can suppose without loss of generality that $\overline{U_{n+1}} \subset U_n$ for every n. Then U_n, $n \in \mathbb{N}$, form a neighborhood basis of x. To see it let V be an arbitrary open neighborhood of x. If $U_n \setminus V \neq \emptyset$ for every n, then, by compactness, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \setminus V \neq \emptyset$. This intersection does not contain x (as it is contained in $K \setminus V$). But in the same time it is contained in $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}$, a contradiction.

As φ is open, $V_n = \varphi(U_n)$ is open for every n. Let us show that $\{\varphi(x)\} = \bigcap_{n\in\mathbb{N}} V_n$. Choose an arbitrary $y \in L \setminus \{\varphi(x)\}$. The set $L \setminus \{y\}$ is a neighborhood of $\varphi(x)$, so, by continuity of φ, there is $n \in \mathbb{N}$ with $\varphi(U_n) \subset L \setminus \{y\}$. Therefore $y \notin V_n$, which completes the proof. □

Lemma 4.4

Let $\varphi : K \to L$ be a continuous open surjection between compact Hausdorff spaces. If L contains a dense set of G_δ points and A is a dense Σ-subset of K, then $T_\varphi(\mathcal{C}(L))$ is τ_A-closed in $\mathcal{C}(K)$.

Proof. We will use Lemma 2.9(2). Put $E = \{(x, y) \in K \times K \mid \varphi(x) = \varphi(y)\}$. We will show that $E \cap (A \times A)$ is dense in E.

Choose an arbitrary pair $(u, v) \in E$, and U, V open neighborhoods of u, v, respectively. Put $z = \varphi(u) = \varphi(v)$. Then $\varphi(U)$ and $\varphi(V)$ are open neighborhoods of z, since φ is open. Therefore $W = \varphi(U) \cap \varphi(V)$ is a nonempty open set, so there is $g \in W$, a G_δ point of L. The set $\varphi^{-1}(g)$ is G_δ in K, and hence $\varphi^{-1}(g) \cap A$ is dense in $\varphi^{-1}(g)$ by Lemma 1.7(3). It follows that we can choose $x \in \varphi^{-1}(g) \cap A \cap U$ and $y \in \varphi^{-1}(g) \cap A \cap V$. Then $(x, y) \in (U \times V) \cap E \cap (A \times A)$, which completes the proof. □

Theorem 4.5

Let $\varphi : K \to L$ be a continuous open surjection between compact Hausdorff spaces. Suppose, moreover, that L has a dense set of G_δ points. Then the following hold.

(1) If K is Valdivia, then so is L.
(2) If K is super-Valdivia, then L is Corson.

Proof. (1) Let A be a dense Σ-subset of K. Then by Lemma 4.4 and Theorem 2.20 we get that $\varphi(A)$ is a dense Σ-subset of L.

(2) Let $y \in L$ be arbitrary. There is $x \in K$ with $\varphi(x) = y$. As K is super-Valdivia, there is a dense Σ-subset $A \subset K$ which contains x. By Lemma 4.4 and Theorem 2.20 we get that $\varphi(A)$ is a dense Σ-subset of L. Moreover, clearly $y \in \varphi(A)$. So L is super-Valdivia. Now it follows from Lemma 4.2 that L is Corson. □

Remark. It is easy to see that Lemma 4.4 does not hold without assumption that L has a dense set of G_δ points. Indeed, let L be a compact Hausdorff space such that there are $A, B \subset L$ two disjoint dense Σ-subsets. (One can take for example $L = [0,1]^\Gamma$ for Γ uncountable, $A = L \cap \Sigma(\Gamma)$ and $B = \{x \in L \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 1\}$ is countable$\}$.) Then $C = A \times \{0\} \cup B \times \{1\}$ is a dense Σ-subset of $K = L \times \{0,1\}$. The natural projection of K onto L is open and continuous, but the image of C is $A \cup B$ which is not a Σ-subset of L (by Lemma 2.15). However, it seems not to be clear whether the assumption on L is necessary in Theorem 4.5(1).

Corollary 4.6

Let K be a Valdivia compact space with a dense set of G_δ points. Then every continuous open image of K is Valdivia.

Proof. This follows immediately from Theorem 4.5 and Lemma 4.3. □

Proposition 4.7

Let K and L be nonempty compact Hausdorff spaces such that K has a dense set of G_δ points. Then the following hold.

(1) If $K \times L$ is Valdivia, then so are K and L.

(2) If $K \times L$ is super-Valdivia, then K is Corson and L super-Valdivia.

Proof. Let $K \times L$ be Valdivia (super-Valdivia). Then K is Valdivia (Corson) by Theorem 4.5, as the projection of $K \times L$ onto K is open. Further, let $x \in K$ be a G_δ point. Then $\{x\} \times L$ is a G_δ set in $K \times L$ which is homeomorphic to L. It easily follows from Lemma 1.7(3) that L is Valdivia (super-Valdivia, respectively). □

Theorem 4.8

Let K_a, $a \in \Lambda$ be an arbitrary family of nonempty compact Hausdorff spaces such that each K_a has a dense subset of G_δ points. Then the following two conditions are equivalent.

(1) $\prod_{\alpha \in \Lambda} K_\alpha$ is a Valdivia compact.

(2) K_α is a Valdivia compact for every $\alpha \in \Lambda$.

Proof. (1) ⇒ (2) This follows immediately from Proposition 4.7(1).

(2) ⇒ (1) This follows from [10, Theorem 4.1]. □

Theorem 4.9

Let K_a, $a \in \Lambda$ be an arbitrary family of nonempty compact Hausdorff spaces such that each K_a has a dense subset of G_δ points. Then the following two conditions are equivalent.

(1) $\prod_{\alpha\in\Lambda} K_\alpha$ is a super-Valdivia compact.
(2) K_α is a Corson compact for every $\alpha \in \Lambda$.

Proof. (1) ⇒ (2) This follows easily from Proposition 4.7(2).

(2) ⇒ (1) This follows from [10, Theorem 4.1]. □

Next we will prove the following theorem strengthening, within compact spaces with a dense set of G_δ points, our Theorem 3.2.

Theorem 4.10

Let K be a compact Hausdorff space with a dense set of G_δ points. Then the following assertions are equivalent.

(1) K is a Valdivia compact.
(2) The space $P(K)$ of all Radon probabilities on K, endowed with the weak topology, is a Valdivia compact.*
(3) The dual unit ball $B_{\mathcal{C}(K)^*}$, endowed with the weak topology, is a Valdivia compact.*
(4) The dual unit ball $B_{\mathcal{C}(K)^*}$, endowed with the weak topology, has a convex symmetric dense Σ-subset.*

To prove this theorem we need a lemma.

Lemma 4.11

Let K be a compact Hausdorff space, $x_1,\dots,x_n$ be G_δ points of K and $t_1,\dots,t_n \ge 0$ with $t_1+\dots+t_n = 1$. Then $t_1\delta_{x_1}+\dots+t_n\delta_{x_n}$ is a G_δ point of $P(K)$.

Proof. At first let us show that the set of probability measures on K supported by the set $F = \{x_1, \dots, x_n\}$ is G_δ. As F is clearly G_δ, it is enough to show the following assertion.

(∗) $F \subset K$ is closed and $G_\delta \Rightarrow \{\mu \in P(K) \mid \mu(F) = 1\}$ is G_δ in $P(K)$.

Let $f : K \to [0, 1]$ be continuous with $f^{-1}(1) = F$. Then it is clear that $\mu(F) = 1$ if and only if $(\mu, f) = 1$, which proves (*).

Finally observe, that $\{\mu \in P(K) \mid \mu(F) = 1\}$ is homeomorphic to $P(F)$, which is metrizable whenever F is finite. So every point of $P(F)$ is G_δ in $P(F)$, and the assertion of the lemma follows. □

Proof of Theorem 4.10 (1) ⇒ (4) This follows from Theorem 3.2.

(4) ⇒ (3) This is trivial.

(3) ⇒ (2) This follows easily from Lemma 1.7(3) as $P(K)$ is weak*-G_δ closed subset of $B_{\mathcal{C}(K)^*}$.

(2) ⇒ (1) Let A be a dense Σ-subset of $P(K)$. Let x be a G_δ point of K. By Lemma 4.11 we get that δ_x is a G_δ point of $P(K)$, so $\delta_x \in A$ by Lemma 1.7(3). Hence $\delta(K) \cap A$ is dense in $\delta(K)$, and it easily follows that K is Valdivia. □

In view of Theorem 3.2 it is natural to ask whether $B_{\mathcal{C}(K)^*}$ is super-Valdivia whenever K is super-Valdivia. But it easily follows from [2, Theorem 3.12] that it is not the case.

Example 4.12: Under continuum hypothesis there is a Corson compact space K such that $B_{\mathcal{C}(K)^*}$ is not super-Valdivia.

Proof. By [2, Theorem 3.12] there is, under continuum hypothesis, a Corson compact space K such that $B_{\mathcal{C}(K)^*}$ is not Corson. We will prove that $B_{\mathcal{C}(K)^*}$ is not super-Valdivia.

Suppose that $B_{\mathcal{C}(K)^*}$ is super-Valdivia. Then it follows from Lemma 1.7(3) that $P(K)$ is super-Valdivia as well (since $P(K)$ is G_δ in $B_{\mathcal{C}(K)^*}$). But K has a dense set of G_δ points (Lemma 4.1), so $P(K)$ has a dense set of G_δ points, too, by Lemma 4.11. Now it follows from Lemma 4.2 that $P(K)$ is Corson, hence $B_{\mathcal{C}(K)^*}$ is Corson as well (as a continuous image of $P(K) \times P(K) \times [0, 1]$, using [1, Corollary IV.3.15]). This is a contradiction which completes the proof. □

Finally we formulate several questions which are, up to our knowledge, open.

Questions 4.13

(1) Is every open continuous image of a Valdivia (super-Valdivia) compact again Valdivia (super-Valdivia)?
(2) Suppose that $K \times L$ is Valdivia (super-Valdivia). Are both K and L Valdivia (super-Valdivia)?
(3) Suppose that $B_{\mathcal{C}(K)^*}$ is Valdivia. Is K Valdivia as well?

All these questions have positive answers within the class of spaces with a dense set of G_δ points. We do not know whether this assumption is essential.

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Added in proof. The implication (3) $\Rightarrow$ (1) of Theorem 2.20 follows also from a result of S.P. Gul'ko (Properties of sets that lie in Σ-products, *Dokl. Akad. Nauk SSSR* **237**(3) (1977), 505–508 (in Russian)) together with our Lemma 1.7(2).

Another Lindelöf properties of Banach spaces with Valdivia dual unit ball were studied by J. Orihuela (On weakly Lindelöf Banach spaces, *Progress in Funct. Anal.*, Eds: K.D. Bierstedt, J. Bonet, J. Horváth, M. Maestre, Elsevier Sci. Publ. B.V. (1992), 279–291). In the proof of Corollary 5 of the mentioned paper there is contained the proof of the implication (1) $\Rightarrow$ (2) of our Theorem 3.2.

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