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A characterization of Orlicz spaces isometric to L_p-spaces

Grzegorz Lewicki

Department of Mathematics, Jagiellonian University, 30-059 Kraków, Reymonta 4, Poland

E-Mail: lewicki@im.uj.edu.pl

Abstract

In this note we present an affirmative answer to the problem posed by M. Baronti and C. Franchetti (oral communication) concerning a characterization of L_p-spaces among Orlicz sequence spaces. In fact, we show a more general characterization of Orlicz spaces isometric to L_p-spaces.

0. Introduction

Let (Ω, Σ, μ) be a measure space. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an Orlicz function, i.e. f is continuous and nondecreasing in $\mathbb{R}^+$, $f(0) = 0$ and $\lim_{t\to+\infty} f(t) = +\infty$. Denote by $\mathcal{M}$ the set of all measurable, real or complex valued functions defined on Ω. For $g \in \mathcal{M}$ set

(0.1) $$\rho_f(g) = \int_\Omega f(| g(t) |)d\mu(t).$$

Let us define an Orlicz space L_f by

(0.2) $$L_f = \{g \in \mathcal{M} : \lim_{t\to 0} \rho_f(tg) = 0\}.$$

If f is an s-convex function for some $s \in (0, 1]$ we can equip L_f with a functional $\| \cdot \|_f$ given by

(0.3) $$\|g\|_f = \inf \left\{ c > 0 : \rho_f\left(\frac{g}{c^{1/s}}\right) \le 1 \right\} \text{ for } g \in L_f,$$

called the Luxemburg s-norm (norm if $s = 1$). Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is s-convex for some $s \in (0, 1]$ if and only if

$$f(tx + ry) \leq t^s f(x) + r^s f(y)$$

for any $x, y \in \mathbb{R}^+$, $0 \leq t,\ \ r \leq 1$, $r^s + t^s = 1$.

If $\Omega = \mathbb{N}, \Sigma = 2^{\mathbb{N}}$ and μ is a counting measure, we call L_f a sequence Orlicz space and we will denote it by l_f. For more information about Orlicz spaces the reader is referred to [4].

The aim of this note is to present an affirmative answer (Corollary 1.11) to the following problem posed by M. Baronti and C. Franchetti (oral communication).

Problem 0.1

Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function such that $f(0) = 0$ and $f(x) > 0$ for some $x > 0$. Assume that $\| \cdot \|_f$ (see (0.3)) satisfies the following:

Property (P): *For every $a, b, c, d \in \mathbb{R}$ if $\|(a, b, 0, ...)\|_f = \|(c, d, 0, ...)\|_f$ then $\|(a, b, x)\|_f = \|(c, d, x)\|_f$ for every $x = (x_3, x_4, ...) \in l_f$.*

Is it true that l_f is linearly isometric to l_p-space for some $p \geq 1$?

In fact, we present a more general characterization of L_f-spaces isometric to L_p-spaces (Theorem 1.10). For other results concerning this topic the reader is referred to [1-3], [5], [6].

1. The results

We start with the following proposition.

Proposition 1.1

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, strictly increasing in $f^{-1}((0, +\infty))$ function such that

$$f(0) = 0, f(x) = 1 \quad \text{for some} \quad x > 0. \tag{1.1}$$

Let $s \in (0, 1]$, $r_1 = 1$, $r_2 \geq 1$ and $r_3 > 0$. Assume additionally that $f(y) = 1/r$ for some positive y, where $r = \min\{1, r_3\}$. For every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ define a functional $\| \cdot \|_f$ by

$$\|x\|_f = \inf\Big\{c > 0 : \sum_{n=1}^{3} r_n f(\mid x_n \mid / c^{1/s}) \leq 1\Big\}.$$

Then the following conditions are equivalent:

Property (A). *For every nonnegative real numbers a, b, c, d such that*

$$\|(a, b, 0)\|_f = \|(c, d, 0)\|_f$$

we have:

$$\|(a, b, x)\|_f = \|(c, d, x)\|_f$$

for any $x \in \mathbb{R}$;

Property (B). *For every nonnegative real numbers a, b, c, d such that $f(a) + r_2 f(b) = f(c) + r_2 f(d) = 1$ we have: $f(\alpha a) + r_2 f(\alpha b) = f(\alpha c) + r_2 f(\alpha d)$ for any $\alpha \in (0, 1)$.*

Proof. Suppose that (B) does not hold. This means that there exist real nonnegative numbers a, b, c, d such that

$$f(a) + r_2 f(b) = f(c) + r_2 f(d) = 1 \tag{1.2}$$

and $\alpha \in (0, 1)$ such that $f(\alpha a) + r_2 f(\alpha b) < f(\alpha c) + r_2 f(\alpha d) \leq 1$. Hence we can choose a positive number x such that $f(\alpha a) + r_2 f(\alpha b) + r_3 f(x) = 1$. This means that $\|(\alpha a, \alpha b, x)\|_f = 1$. Note that $f(\alpha c) + r_2 f(\alpha d) + r_3 f(x) > 1$. This implies that $\|(\alpha c, \alpha d, x)\|_f > 1$. Consequently $\|(a, b, x/\alpha)\|_f < \|(c, d, x/\alpha)\|_f$ which contradicts (A) (by (1.2) $\|(a, b, 0)\|_f = \|(c, d, 0)\|_f = 1$).

To prove the converse, assume $\|(a, b, 0)\|_f = \|(c, d, 0)\|_f = A$. This means that

$$f\Big(\frac{a}{A^{1/s}}\Big) + r_2 f\Big(\frac{b}{A^{1/s}}\Big) = f\Big(\frac{c}{A^{1/s}}\Big) + r_2 f\Big(\frac{d}{A^{1/s}}\Big) = 1.$$

For $x \in \mathbb{R}$, denote $E = \|(a, b, x)\|_f$. Note that $E \geq A$. Put $F = \|(c, d, x)\|_f$. We show that $F = E$. By the definition of $\|\cdot\|_f$, $f(a/E^{1/s}) + r_2 f(b/E^{1/s}) + r_3 f(\mid x \mid /E^{1/s}) = 1$. Take $\alpha = (A/E)^{1/s}$. Applying (B) to the numbers $a/A^{1/s}, b/A^{1/s}, c/A^{1/s}, d/A^{1/s}$ and $\alpha = (A/E)^{1/s}$ we get

$$f\Big(\frac{c}{E^{1/s}}\Big) + r_2 f\Big(\frac{d}{E^{1/s}}\Big) + r_3 f\Big(\frac{\mid x \mid}{E^{1/s}}\Big) = 1.$$

By the definition of $\|\cdot\|_f$, $F = E$, which completes the proof. □

Theorem 1.2

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, nondecreasing function satisfying (1.1) *and* (B). *Then $f(t) = C \cdot t^p$ for some $C, p > 0$ and $t \in [0, x]$, where x is so chosen that $f(x) = 1$.*

First we prove some preliminary results in which we assume additionally that $f(1)=1$.

Lemma 1.3

Let f be as in Theorem 1.2. Assume additionally that f is strictly increasing in $[0,1]$. If $f(a)+r_2f(b)=f(d)$ then $f(a/d)+r_2f(b/d)=1$ for any $a,b\in[0,1]$, $d\in(0,1]$.

Proof. If d=1, the statement is obvious. Suppose $d<1$. If $f(a/d)+r_2f(b/d)\neq 1$ then we can choose $d_1\neq d$ with $f(a/d_1)+r_2f(b/d_1)=1$. By (B)

$$f(d)=f\Big(\frac{d\cdot a}{d_1}\Big)+r_2f\Big(\frac{d\cdot b}{d_1}\Big).$$

Since $f(d)>0$, this gives immediately $d=d_1$, a contradiction. □

Lemma 1.4

Let f be as in Lemma 1.3. Then for every $n\in\mathbb{N}$, $a,b_i,d\in[0,1]$ for $i=1,...,n$, if

$$f(a)+r_2\sum_{i=1}^{n}f(b_i)=f(d)$$

then

$$f(\alpha a)+r_2\sum_{i=1}^{n}f(\alpha b_i)=f(\alpha d)$$

for every $\alpha\in[0,1]$.

Proof. First we consider the case $n=1$. If $d=0$, the statement is obvious. Suppose $f(a)+r_2f(b)=f(d)\neq 0$. By Lemma 1.3,

$$f\Big(\frac{a}{d}\Big)+r_2f\Big(\frac{b}{d}\Big)=1.$$

Taking $\beta=\alpha\cdot d$, by (B), we get

$$f(\alpha a)+r_2f(\alpha b)=f\Big(\frac{\beta a}{d}\Big)+r_2\Big(\frac{\beta b}{d}\Big)=f(\beta)=f(\alpha d)$$

as required. The case $n>1$ follows from the previous one by the induction argument. □

DEFINITION 1.5. Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous, nondecreasing function satisfying (1.1) and (B). Put

$$\mathcal{A}=\{(a,b);0<a,b<1,f(a)+r_2f(b)=1\}. \tag{1.3}$$

For $(a,b)\in\mathcal{A}$ define by $g(a,b)$ the unique $p>0$ such that $a^p+r_2b^p=1$.

Lemma 1.6

Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, nondecreasing function satisfying (1.1) *and* (B). *Let $p > 0$ be so chosen that there is $t_o > 0$ such that $f(t) < t^p$ (or $f(t) > t^p$) for every $t \in (0, t_o)$. Then $g(a,b) \neq p$ for every $(a,b) \in \mathcal{A}$.*

Proof. Suppose, on the contrary, that $g(a,b) = p$ for some $(a,b) \in \mathcal{A}$. Then

$$1 = f(a) + r_2 f(b) = a^p + r_2 b^p.$$

This gives that there is $c \in [a,b]$ (we can assume without loss of generality that $a \leq b$) with $f(c) = c^p$. Put $c_o = \inf\{c \geq t_o : f(c) = c^p\}$. By assumptions on f, $c_o > 0$ and $f(c_o) = c_o^p$. Note that by (B)

$$f(c_o a) + r_2 f(c_o b) = f(c_o) = c_o^p = (c_o a)^p + r_2 (c_o b)^p. \tag{1.4}$$

Since $a, b < 1$,

$$f(c_o a) < \ (> \text{resp.}\ )(c_o a)^p$$

and

$$f(c_o b) < \ (> \text{resp.}\ )(c_o b)^p$$

which leads to a contradiction with (1.4). □

Lemma 1.7

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, nondecreasing, strictly increasing $[0,1]$ *function satisfying* (1.1) *and* (B). *If $f(a) = 1/(r_2 q) = a^p$ for some $q \in \mathbb{N} \setminus \{0\}$, $p > 0$ then $f(a^n) = (a^n)^p$ for n=2,3,...*

Proof. Suppose that $f(a) = 1/(r_2 q) = a^p$. Then, by Lemma 1.4, $r_2 q \cdot f(a^2) = f(a)$ and consequently,

$$f(a^2) = f(a) \cdot \frac{1}{(r_2 q)} = \big(f(a)\big)^2 = (a^2)^p.$$

To finish the proof it is necessary to apply the induction argument. □

Proof of Theorem 1.2. Take $a \in (0,1)$ with $f(a) = 1/r_2$ if $r_2 > 1$ or $f(a) = 1/2$ if $r_2 = 1$. Then $f(a) = a^l$ for some $l > 0$. Applying (B) and the induction argument one can easily get that $f(a^n) = (f(a))^n = a^{nl}$ for $n = 1, 2\ldots$ Now take any $c \in (0, a)$. Then there is $n \in \mathbb{N}$ with $a^{n+1} < c \le a^n$. Consequently, since f is nondecreasing,

$$f(a^{n+1}) \le f(c) \le f(a^n)$$

and

$$\frac{1}{(a^n)^l} \le \frac{1}{c^l} < \frac{1}{(a^{n+1})^l}.$$

This gives

$$a^l \le \frac{f(c)}{c^l} < \frac{1}{a^l} \tag{1.5}$$

for every $c \in (0, a)$. Now we show that $g(i, j) = l$ (see Def. 1.5) for every $(i, j) \in \mathcal{A}$ (see (1.3)). Note that, by (1.5), for any $q \ne l$

$$\lim_{t\to 0} \frac{f(t)}{t^q} = \lim_{t\to 0} \frac{f(t)}{t^l} \cdot t^{l-q} = 0 \quad \text{or} \quad +\infty\,.$$

Hence there is $t_q > 0$ such that $f(t) < t^q$ or $f(t) > t^q$ for $t \in (0, t_q)$. By Lemma 1.6, $g(i, j) \ne q$ for any $q \ne l$ and consequently $g(i, j) = l$ for any $(i, j) \in \mathcal{A}$.

Now we show that f is a strictly increasing function in [0,1]. By (1.5), $f(t) > 0$ for $t > 0$. By the continuity of f and (B), $f(t) < 1$ for $0 \le t < 1$. Note that for every $(i, j) \in \mathcal{A}$

$$1 = f(i) + r_2 f(j) = i^l + r_2 j^l.$$

Hence, since the function $t \to t^l$ is strictly increasing, $f(t_1) = f(t_2)$ implies $t_1 = t_2$ for any $t_1, t_2 \in (0,1)$.

To finish the proof (in the case $f(1) = 1$), by the continuity and monotonicity of f, it is sufficient to show that $f(t) = t^l$ for every $t \in f^{-1}((0,1) \cap \mathbb{Q})$. To do this, suppose that $f(a_q) = 1/(r_2 q) = a_q^{p_q}$ for $q = 1, 2, \ldots$ Then by Lemma 1.7 $f(a_q^n) = a_q^{np_q}$ for $n = 1, 2, \ldots$ By (1.5), $p_q = l$ for $q = 1, 2, \ldots$

Now fix $q \in \mathbb{N}$, $q > 1$. Take any rational number $p/q \in (0,1)$ and suppose that $f(t_p) = p/q$. We show by the induction argument that $f(t_p) = t_p^l$. Note that $f(t_1) = r_2 f(a_q)$. By Lemma 1.3, $r_2 f(a_q/t_1) = 1$. Since f is strictly increasing in $[0,1]$ and $f(1) = 1$, $a_q/t_1 = a_1$. Consequently,

$$f(t_1) = \frac{f(a_q)}{f(a_1)} = \frac{a_q^l}{a_1^l} = t_1^l,$$

as required.

Now suppose that $f(t_{p-1}) = t_{p-1}^l$. Note that

$$f(t_{p-1}) + r_2 f(a_q) = f(t_p).$$

By Lemma 1.3,

$$f\Big(\frac{t_{p-1}}{t_p}\Big) + r_2 f\Big(\frac{a_q}{t_p}\Big) = 1.$$

Hence $(t_{p-1}/t_p, a_q/t_p) \in \mathcal{A}$ (see Def. 1.5). Since $g(t_{p-1}/t_p, a_q/t_p) = l$,

$$1 = r_2\Big(\frac{a_q}{t_p}\Big)^l + \Big(\frac{t_{p-1}}{t_p}\Big)^l.$$

Consequently, by the induction argument

$$t_p^l = r_2 a_q^l + t_{p-1}^l = r_2 f(a_q) + f(t_{p-1}) = \frac{p}{q} = f(t_p)$$

as required. This completes the proof in the case $f(1) = 1$. If this assumption is not satisfied, take a positive x such that $f(x) = 1$. Consider a function $g(t) = f(tx)$. It is easy to see that $g(1) = 1$ and g satisfies the assumptions of Theorem 1.2. By the proof given above

$$f(t) = g(t/x) = (t/x)^l = x^{-l} \cdot t^l,$$

where l is the index corresponding to g. The proof of Theorem 1.2 is fully complete. □

DEFINITION 1.8. Let (Ω, Σ, μ) be a measure space such that Σ contains at least three pairwise disjoint sets of positive and finite measure. Let f be as in Proposition 1.1 and let $r > 0$ will be given. We say that f satisfies property (A_r) if and only if

$$f(x) = 1/r \text{ for some positive } x; \tag{1.6}$$

there exist $X_1, X_2, X_3 \in \Sigma$ of positive and finite measure, $0 < \mu(X_1), \mu(X_2) \le r$, such that for every $a, b, c, d \in \mathbb{R}$ if

$$\|a\chi_1 + b\chi_2\|_f = \|c\chi_1 + d\chi_2\|_f$$

then

$$\|a\chi_1 + b\chi_2 + x\chi_3\|_f = \|c\chi_1 + d\chi_2 + x\chi_3\|_f$$

(see (0.3)) for any $x \in \mathbb{R}$. (By (1.6) $\|\cdot\|_f$ can be properly defined). Here χ_i denotes the characteristic function of X_i, $i = 1, 2, 3$.

Theorem 1.9

Let (Ω, Σ, μ) and f be as in Definition 1.8. If f satisfies property (A_r) for some $r > 0$ then there exist $c, p > 0$ such that $f(x) = c \cdot x^p$ for $x \in [0, f^{-1}(1/r)]$.

Proof. We can assume without loss of generality that $\mu(X_1) \leq \mu(X_2)$. Put $f_1 = \mu(X_1) \cdot f$, $r_2 = \mu(X_2)/\mu(X_1)$, $r_3 = \mu(X_3)/\mu(X_1)$. Note that if f satisfies (A_r) then f_1, r_2, r_3 satisfy (A). By Proposition 1.1 and Theorem 1.2, there exist $c, p > 0$ such that $f_1(x) = c \cdot x^p$ for $x \in [0, f_1^{-1}(1)]$. Consequently, $f(x) = c_1 \cdot x^p$ for $x \in [0, f^{-1}(1/r)]$. □

Theorem 1.10

Let (Ω, Σ, μ) be as in Theorem 1.9. Suppose that f is an s-convex, continuous function, $f(0) = 0$ and $f(x) > 0$ for some positive x. Put

$$r_o = \inf\{r > 0; (A_r) \text{ is satisfied}\},$$

$$z_o = \inf\{z > 0; \text{ there exists } X \in \Sigma, 0 < \mu(X) \leq z\}$$

($r_o = +\infty$ if (A_r) is not satisfied for any $r > 0$). If $r_o = z_o$ then the space $L_f(\Omega, \Sigma, \mu)$ is linearly isometric to $L_p(, \Omega, \Sigma, \mu)$ for some $p > 0$.

Proof. By Theorem 1.9, there exist $c, p > 0$ such that $f(x) = c \cdot x^p$ for $x \in [0, f^{-1}(1/r_o)]$. Note that by the definition of z_o the function $\|\cdot\|_f$ is uniquely determined by the values of f in $[0, 1/z_o]$. Since $r_o = z_o$ the space $L_f(\Omega, \Sigma, \mu)$ is linearly isometric to $L_p(, \Omega, \Sigma, \mu)$, as required. □

Corollary 1.11

Let f be a convex, nonnegative function, $f(0) = 0$ and $f(x) > 0$ for some positive x. If the function f satisfies (P) *(see Problem 0.1) then the space l_f is linearly isometric to l_p for some $p \geq 1$.*

Proof. By Proposition 1.1 and Theorem 1.2, there exist $c, p > 0$ such that $f(x) = c \cdot x^p$ for $x \in [0, f^{-1}(1)]$. Note that in our case $r_o = z_o = 1$. By Theorem 1.10, l_f is linearly isometric to the space l_p. By the proof of Theorem 1.2 and the convexity of f, $p \geq 1$, as required. □

Corollary 1.11 gives an affirmative answer to Problem 0.1.

Remark 1.12. During a preparation of this note the author has received a preprint of H. Cuenya and M. Marano [2] in which a similar characterization of L_p-spaces has been proved.

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