Collect. Math. **44** (1993), 261–268

Random rearrangements in functional spaces

E.M. Semenov

Department of Mathematics, Voronezh State University,

Voronezh 394693, Russia

Abstract

We give an operator approach to several inequalities of S. Kwapien and C. Schütt, which allows us to obtain more general results.

Section 0

Let n be an integer, $x = \{x_{ij}\}, 1 \le i, j \le n, \Pi = \Pi_n$ be the set of rearrangements of $\{1, 2, \ldots, n\}$. Denote by $s_1, s_2, \ldots, s_{n^2}$ the rearrangement of $|x_{ij}|$ in the decreasing order. S. Kwapien and C. Schütt proved the following statements.

Theorem A ([3])

The inequalities

$$\frac{1}{2n}\sum_{k=1}^{n} s_k \le \frac{1}{n!}\sum_{\pi\in\Pi} \max_{1\le i\le n} |x_{i\pi(i)}| \le \frac{1}{n}\sum_{k=1}^{n} s_k$$

are valid.

Theorem B ([5])

If $1 \le p \le q < \infty$, *then*

$$\frac{1}{10}\left(\left(\frac{1}{n}\sum_{k=1}^{n}s_k^p\right)^{1/p}+\left(\frac{1}{n}\sum_{k=n+1}^{n^2}s_k^q\right)^{1/q}\right)$$
$$\leq\left(\frac{1}{n!}\sum_{\pi\in\Pi}\left(\sum_{i=1}^{n}|x_{x\pi(i)}|^q\right)^{p/q}\right)^{1/p}$$
$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}s_k^p\right)^{1/p}+\left(\frac{1}{n}\sum_{k=n+1}^{n^2}s_k^q\right)^{1/q}.$$

The operator approach to such problems is presented in this article. It allows to obtain more general results.

Section 1

If $x(t)$ is a measurable function on $[0,1]$, we denote by $x^*(t)$ the decreasing rearrangement of $|x(t)|$. A Banach functional space E on $[0,1]$ with the Lebesgue measure m is said to be rearrangement invariant (r.i.) if it satisfies the following condition: if $y\in E$ and $x^*(t)\leq y^*(t)$ for all $t\in[0,1]$, then $x\in E$ and $\|x\|_E\leq\|y\|_E$. Let $\tau>0$. The compression operators

$$\sigma_\tau x(t)=\begin{cases}x(\frac{t}{\tau}), & 0\leq t\leq\min(\tau,1)\\ 0, & \min(\tau,1)<t\leq 1\end{cases}$$

act in every r.i. space. The numbers

$$\alpha_E=\lim_{\tau\to 0}\frac{\ln\|\sigma_\tau\|_E}{\ln\tau},\quad \beta_E=\lim_{\tau\to\infty}\frac{\ln\|\sigma_\tau\|_E}{\ln\tau}$$

are named Boyd indexes of the r.i. space E. It's known that $0\leq\alpha_E\leq\beta_E\leq 1$. Let $x,y\in L_1$. We denote $x\prec y$ if

$$\int_0^\tau x^*(t)dt\leq\int_0^\tau y^*(t)dt$$

for each $\tau\in[0,1]$. If a r.i. space E is separable or isometric to the conjugate of some separable r.i. space, then $x\prec y$ implies $\|x\|_E\leq\|y\|_E$. For simplicity we shall assume that a r.i. space E satisfies this assumption. The Hardy operator

$$Hx(t)=\int_t^1\frac{x(s)}{s}ds$$

is bounded in a r.i. space E iff $\alpha_E>0$. Without loss of generality $\|1\|_E=1$.

Orlicz, Lorentz, Marcinkiewicz spaces are r.i. ones. If a function $M(u)$ is even, convex, increasing on $[0,\infty)$ and

$$\lim_{u\to\infty}\frac{M(u)}{u}=\infty,\quad M(0)=0$$

then

$$\|x\|_{L_M}=\inf\left\{\lambda\colon \lambda>0,\ \int_0^1 M\Big(\frac{x(t)}{\lambda}\Big)dt\le 1\right\}.$$

Let $\varphi(t)$ be an increasing concave function on $[0,1]$ s.t. $\varphi(0)=0$. Then

$$\|x\|_{\Gamma(\varphi)}=\int_0^1 x^*(t)d\varphi(t).$$

All the above mentioned properties of r.i. spaces can be found in [1,2,4].

Section 2

Let us fix some one-to-one correspondence ℓ of Π into $\{1,2,\dots,n!\}$. Let $1\le q\le\infty$ and x is an n-square matrix. We define the quasi-linear operator

$$T_q x(t)=\Big(\sum_{i=1}^n |x_{i\pi(i)}|^q\Big)^{1/q},\quad t\in\Big(\frac{\ell(\pi)-1}{n!},\frac{\ell(\pi)}{n!}\Big)$$

with usual modification for $q=\infty$. It's evident that $\|T_q x\|_E$ does not depend on ℓ if E is a r.i. space. Define the operator

$$Sx(t)=s_k,\quad t\in\Big(\frac{k-1}{n},\frac{k}{n}\Big),\quad 1\le k\le n.$$

The following statement generalizes Theorem A.

Theorem 1

Let E be a r.i. space. Then

$$\frac{1}{2}\|Sx\|_E\le\|T_\infty x\|_E\le\|Sx\|_E.$$

Proof. For simplicity we assume that $s_1>s_2>\ldots>s_n>0$. Then $m\{t\colon Sx(t)\ge s_k\}=k/n$ for each $k=1,2,\dots,n$. As

$$m\{t\colon T_\infty x(t)=s_k\}\le\frac{(n-1)!}{n!}=\frac{1}{n},\ k=1,2,\dots,n$$

then

$$m\{t\colon T_\infty x(t) \geq s_k\} \leq \frac{k}{n}.$$

Hence $(T_\infty x)^*(t) \leq Sx(t)$ and $\|T_\infty x\|_E \leq \|Sx\|_E$.

To prove the left side of the inequality, we fix $k \in \{1, 2, \ldots, n\}$ and construct the matrix

$$y_{ij} = \begin{cases} |x_{ij}|, & |x_{ij}| \geq s_k \\ 0, & |x_{ij}| < s_k \end{cases}.$$

Applying Theorem A to matrix $\underline{Y} = \{y_{ij}\}$ we have

$$\frac{1}{2n}\sum_{1}^{k} s_i \leq \frac{1}{n!}\sum_{\pi\in\Pi} \max_{1\leq i\leq n} y_{i\pi(i)}. \tag{1}$$

Denote $e_k = \{t\colon T_\infty y(t) \neq 0\}$. Then $me_k \leq k/n$ and

$$\begin{aligned} \frac{1}{2n}\sum_{\pi\in\Pi} \max_{1\leq i\leq n} y_{i\pi(i)} &= \int_{e_k} T_\infty y(t)dt \\ &\leq \int_{e_k} T_\infty x(t)dt \leq \int_0^{k/n} (T_\infty x)^*(t)dt\,. \end{aligned} \tag{2}$$

As

$$\frac{1}{n}\sum_{i=1}^{n} s_i = \int_0^{k/n} Sx(t)dt$$

then (1) and (2) imply the inequalities

$$\frac{1}{2}\int_0^{k/n} Sx(t)dt \leq \int_0^{k/n} (T_\infty x)^*(t)dt.$$

The function

$$\int_0^{\tau} (T_\infty x)^*(t)dt$$

is concave on $[0, 1]$ and the function

$$\frac{1}{2}\int_0^{\tau} Sx(t)dt$$

is linear on each interval $[\frac{k-1}{n}, \frac{k}{n}]$, $1 \leq k \leq n$. Therefore the inequality

$$\frac{1}{2}\int_0^{\tau} Sx(t)dt \leq \int_0^{\tau} (T_\infty x)^*(t)dt$$

is valid for each $\tau \in [0, 1]$. Hence $\frac{1}{2}Sx \prec T_\infty x$ and

$$\frac{1}{2}\|Sx\|_E \leq \|T_\infty x\|_E. \ \square$$

Usually the Orlicz space L_M where $M(u) = e^{|u|} - 1$ is denoted by $\exp L$.

Theorem 2

There exists a constant $C > 0$ such that

$$\|T_1 x\|_{\exp L} \le C\Bigg(\max_{1\le i,j\le n} |x_{ij}| + \frac{1}{n}\sum_{i,j=1}^{n} |x_{ij}| \Bigg). \qquad (3)$$

Proof. It is well known that the extremal points of the convex set

$$\max_{1\le i,j\le n} |x_{ij}| \le 1 \quad \text{and} \quad \sum_{i,j=1}^{n} |x_{ij}| \le n$$

are matrices such that some n elements x_{ij} $(1 \le i, j \le n)$ are equal to ± 1 and $n^2 - n$ elements are equal to 0. It is sufficient to prove inequality (3) only for such matrices.

Let matrix z belong to this set and $z \ge 0,\ 1 \le j \le n$. We have

$$m\left\{t\colon T_1\big(z(t)\big) = j\right\} \le C_n^j \frac{(n-j)!}{n!} = \frac{1}{j!}\,. \qquad (4)$$

Therefore

$$\int_0^1 (e^{\frac{T_1 z(t)}{\lambda}} - 1)dt \le \sum_{j=1}^{\infty} \frac{e^{j/\lambda}}{j!} - 1 = e^{e^{1/\lambda}} - 2\,.$$

This means that C in (3) may be chosen as $\frac{1}{\ln\ln 3}$. □

Lemma 3

If x is an $n \times n$ matrix and

$$\big|\{(i,j)\colon x_{ij} \ne 0\}\big| \le n$$

then

$$T_1 x \prec 8HSx\,.$$

Proof. First we consider the case:

$$s_i = \begin{cases} 1, & 1 \le i \le k \\ 0, & k < i \le n \end{cases}$$

for some $k \le n$. Given $1 \le j \le k$ we denote

$$R_j = \Big\{\pi\colon \pi \in \Pi,\ \sum_{i=1}^{n} x_{i\pi(i)} = j\Big\}, \quad Q_j = \bigcup_{m=j}^{k} R_m$$

and $\tau_j = \frac{|Q_j|}{n!}$. It is clear that

$$\tau_j \le \frac{2C_k^j (n-j)!}{n!} = \frac{2k!(n-j)!}{j!(k-j)!n!} = \frac{2(k-j+1)\dots k}{j!(n-j+1)\dots n} \le \frac{2k}{j!n}\,.$$

As

$$HSx(t) = \begin{cases} \ln \frac{k}{nt}, & 0 < t \le \frac{k}{n} \\ 0, & \frac{k}{n} \le t \le 1 \end{cases}$$

then

$$m\{t\colon HSx(t) \ge j\} = \frac{k}{n} e^{-j}.$$

Therefore

$$m\{t\colon T_1x(t) \ge j\} = \tau_j \le \frac{2e^2 k}{2n} e^{-j} \le 8m\{t\colon HSx(t) \ge j\}.$$

So

$$T_1 x \prec 8HSx.$$

Let us consider the general case. There exist $a_k \ge 0$, n-square matrices z_k $(1 \le k \le n)$ such that some k elements of z_k are equal to 1 and the other $n^2 - k$ elements are equal to 0,

$$\{(i,j)\colon (z_k)_{ij} = 1\} \subset \{(i,j)\colon (z_{k+1})_{ij} = 1\}$$

for each $k = 1, 2, \dots, n-1$ and

$$x = \sum_{k=1}^{n} a_k z_k.$$

Then

$$\begin{aligned} \int_0^\tau (T_1 x)^*(t)dt &\le \sum_{k=1}^{n} a_k \int_0^\tau (T_1 z_k)^*(t)dt \\ &\le 8 \sum_{k=1}^{n} a_k \int_0^\tau HSz_k(t)dt = 8 \int_0^\tau HSx(t)dt. \ \square \end{aligned}$$

Theorem 4

Let $1 \le q < \infty$, E be a r.i. space, $\alpha_E > 0$. Then

$$\|T_q x\|_E \le C\left(\|Sx\|_E + \Big(\frac{1}{n} \sum_{k=n+1}^{n^2} s_k^q \Big)^{1/q} \right) \tag{5}$$

where C depends only on E.

Proof. Let

$$\|Sx\|_E \le 1, \quad \frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q \le 1. \tag{6}$$

We find n-square matrices y and z such that their supports are disjoint, $x = y + z$, $|\operatorname{supp} y| \le n$ and $Sx = Sy$. By Lemma 3, we have

$$\begin{aligned}\|T_q y\|_E &\le 8\|HSy\|_E \le 8\|H\|_E\|Sy\|_E \\ &= 8\|H\|_E\|Sx\|_E.\end{aligned}$$

Denote $|z_{ij}|^q = u_{ij},\ 1 \le i, j \le n$. Then

$$\|T_q z\|_E = \|(T_1 u)^{1/q}\|_E \le \|T_1 u\|_E^{1/q}.$$

Assumptions (6) imply that

$$0 \le u_{ij} \le 1,\ 1 \le i,\ j \le n,\ \sum_{i,j=1}^{n} u_{ij} \le n.$$

Applying Theorem 2 we have

$$\|T_1 u\|_{\exp L} \le \frac{1}{\ln\ln 3}.$$

It is well known that the assumption $\alpha_E > 0$ implies $E \supset L_\tau$ for some $\tau < \infty$. So $E \supset \exp L$ and

$$\|x\|_E \le C_1\|x\|_{\exp L}$$

for some $C_1 > 0$ and every $x \in \exp L$. Therefore

$$\|T_q z\|_E \le \Big(\frac{C_1}{\ln\ln 3}\Big)^{1/q}$$

and

$$\begin{aligned}\|T_q x\|_E &\le \|T_q y\|_E + \|T_q z\|_E \\ &\le 8\|H\|_E + \Big(\frac{C_1}{\ln\ln 3}\Big)^{1/q}. \ \square\end{aligned}$$

The assumption $\alpha_E > 0$ in Theorem 4 is essential, however it is not necessary. In fact, the function $T_q I_n(t)$ takes the value $n^{1/q}$ on some interval of length $1/n!$. Hence

$$\lim_{n\to\infty} \|T_q I_n\|_{L_\infty} = \infty\,.$$

On the other hand, $SI_n(t) = 1$ and $s_k = 0$ for $n < k \le n^2$.

The inequality inverse to (5) is true without any restrictions.

Theorem 5

Let E be a r.i. space and $1 \le q < \infty$. Then

$$\frac{1}{12}\left(\|Sx\|_E + \Big(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\Big)^{1/q}\right) \le \|T_q x\|_E.$$

Proof. By Theorem 3,

$$\|Sx\|_E \le 2\|T_\infty x\|_E \le 2\|T_q x\|_E.$$

A space E is embedded into L_1 with constant 1 ([2], II.4.1). Applying Theorem B with $p = 1$ we have

$$\left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q} \le 10\|T_q x\|_{L_1} \le 10\|T_q x\|_E\,.$$

From the above given inequality we obtain the needed one. □

Corollary 6

If $M \in \Delta_2,\ 1 \le q < \infty$, then

$$\|T_q x\|_{L_M} \approx \|Sx\|_{L_M} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}.$$

Corollary 7

$1 \le p,\ q < \infty$ *then*

$$\|T_q x\|_{L_p} \approx \left(\frac{1}{n}\sum_{k=1}^{n} s_k^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}.$$

Corollary 7 states that the restriction $p \le q$ in Theorem B is superfluous.

References

1. C. Bennett, R. Sharpley, *Interpolation of Operators*. Academic Press, London, 1988.
2. S.G. Krein, Ju.I. Petunin, E.M. Semenov, *Interpolation of Linear Operators*. Transl. Math. Monogr., Amer. Math. Soc., Providence, 1982.
3. S. Kwapien, C. Schütt, Some combinatorial and probabilistic inequalities and their applications to Banach space theory, *Studia Math.* **82** (1985), 91–106.
4. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces. II*. Springer Verlag 1979.
5. C. Schütt, Lorentz spaces that are isomorphic to subspace L_1. *Trans Amer. Math. Soc.* **89**, No 2 (1985), 583–595.