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Generalized precompactness and mixed topologies

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Abstract

The equicontinuous sets of locally convex generalized inductive limit (or mixed) topologies are characterized as generalized precompact sets. Uniformly pre-Lebesgue and Lebesgue topologies in normed Riesz spaces are investigated and it is shown that order precompactness and mixed topologies can be used to great advantage in the study of these topologies.

Notions of smallness play an important role in analysis, and one of the best known and most useful of these is precompactness. Many smallness properties appearing in the literature can in fact be thought of as "precompactness-like" conditions. A generalized form of precompactness was introduced in [4] in order to prove an extension of Grothendieck's precompactness lemma. The aim of this paper is to show that in the context of locally convex spaces there is an intimate relationship between generalized precompactness and generalized inductive limit (or mixed) topologies. Generalized precompactness is defined in the first section, and examples are given. Mixed topologies are introduced in the second section and it is shown that the equicontinuous sets of locally convex mixed topologies can often be characterized as generalized precompact sets. In the third section the special case of order precompact sets in normed Riesz spaces and the related mixed topologies are explored in more detail. These topologies turn out to be closely related to the uniformly Lebesgue topologies introduced by Nowak in [11].

1. Generalized precompactness

Let E be a vector space (real or complex). A *bornology* $\mathcal{B}$ on E is a collection of subsets of E which covers E, is closed under finite unions and has the property that if $B \in \mathcal{B}$ and $A \subset B$, then $A \in \mathcal{B}$. Now let E be a topological vector space. Following [4], we call a subset A of E *$\mathcal{B}$-precompact* if for every neighborhood U of 0 in E, there is a $B \in \mathcal{B}$ such that $A \subset B + U$. The reader is referred to [4] for the elementary properties of $\mathcal{B}$-precompact sets. Some examples can also be found there; we give some more.

EXAMPLE 1.1: Let E be a Banach space and $\mathcal{B}$ the collection of relatively weakly compact subset of E. It follows from a result of Grothendieck ([5], Chapter 13, Lemma 2) that the $\mathcal{B}$-precompact sets are exactly the relatively weakly compact sets in E.

EXAMPLE 1.2: Let E be a normed space and $\mathcal{B}$ be the collection of bounded weakly metrizable subsets of E. It follows from [14], Lemma 1.2 that the $\mathcal{B}$-precompact sets are exactly the bounded weakly metrizable sets.

EXAMPLE 1.3: Let E be a locally convex space. If U is a closed absolutely convex neighborhood of 0 in E, we shall denote by E_U the locally convex space obtained by equipping E with the gauge of U as seminorm. Let $\mathcal{B}$ be the collection of bounded subsets of E. Then E is quasinormable if for every closed absolutely convex neighborhood U of 0 in E, there is another such neighborhood V such that V is $\mathcal{B}$-precompact in E_U (cf. [9], 10.5.2).

EXAMPLE 1.4: Let $\mathcal{A}$ be an ideal of operators between Banach spaces in the sense of Pietsch ([13]). For a Banach space F let $\mathcal{B}$ be the collection of all subsets of B of F such that there is a Banach space G and an $S \in \mathcal{A}(G, F)$ such that $B \subset S(B_G)$, where B_G denotes the closed unit ball of G. A bounded linear operator $T: E \to F$ belongs to the surjective hull of the closure of $\mathcal{A}$ if and only if T maps the unit ball of the Banach space E into a $\mathcal{B}$-precompact set. (cf. [10]).

A bornology $\mathcal{B}$ on a vector space E is a *vector bornology* if it is closed under sums, scalar multiples and balanced hulls. A subset $\mathcal{B}_0$ of a bornology $\mathcal{B}$ is a *basis* for $\mathcal{B}$ if for every $B \subset \mathcal{B}$, there is a $B_0 \in \mathcal{B}_0$ such that $B \subset B_0$. A vector bornology will be called *convex* if it has a basis consisting of absolutely convex sets. It is easy to check that if $\mathcal{B}$ is a bornology on a topological vector space E, the collection $\mathcal{B}_p$ of all $\mathcal{B}$-precompact sets is again a bornology on E. If $\mathcal{B}$ is a vector bornology, so is $\mathcal{B}_p$; if $\mathcal{B}$ is convex, $\mathcal{B}_p$ is closed under the formation of convex hulls.

2. Mixed Topologies

The mixed topologies we consider will be a special case of the generalized inductive limit topologies introduced by Garling [7], and a slight generalization of the mixed topologies of Persson [12].

If E is a vector space, $\mathcal{B}$ a convex bornology on E and τ a vector topology on E such that every $B \in \mathcal{B}$ is τ-bounded, we shall call the triple $(E, \mathcal{B}, \tau)$ a *mixed space*. If in addition $\mathcal{B}$ has a basis consisting of τ-closed sets, the mixed space is called *normal*. The *mixed topology* $\gamma_\tau(\mathcal{B})$ is the finest locally convex topology coinciding with τ on the sets in $\mathcal{B}$. If the bornology $\mathcal{B}$ is clear from the context, we shall abbreviate $\gamma_\tau(\mathcal{B})$ to γ_τ.

We write β for the finest locally convex topology for which every $B \in \mathcal{B}$ is bounded; when equipped with this topology E is a bornological space. The space of all linear functionals on E which are bounded on the sets in $\mathcal{B}$ will be denoted by E^b. This is also the dual $(E, \beta)'$ of E equipped with the topology β. The space E^b will always have the topology τ_b of uniform convergence on the sets of $\mathcal{B}$.

It is easy to check that if τ is locally convex, we have $\tau \leq \gamma_\tau \leq \beta$ and hence $(E, \tau)' \subset (E, \gamma_\tau)' \subset E^b$. Furthermore, $(E, \gamma_\tau)'$ is a complete subspace of E^b ([12], Corollary 1.1, Theorem 2.1). In the case where $(E, \mathcal{B}, \tau)$ is normal, it follows from Grothendieck's completeness theorem ([15], Chapter VI, Theorem 2) that $(E, \gamma_\tau)'$ is the closure of $(E, \tau)'$ in E^b.

It follows from [7], Proposition 1, that if $(E, \mathcal{B}, \tau)$ is a mixed space, $\mathcal{B}_0$ a basis for $\mathcal{B}$ and τ locally convex, then a basis for the γ_τ-neighborhoods of 0 is given by the collection of absolutely convex hulls of the sets $\cup\{(B \cap U_B) \colon B \in \mathcal{B}_0\}$, where $(U_B)_{B \in \mathcal{B}_0}$ ranges over families of absolutely convex τ-neighborhoods of 0 in E. This description enables us to generalize a result of Cooper ([Co], Proposition 1.22) characterizing the γ_τ-equicontinuous sets.

Theorem 2.1

Let $(E, \mathcal{B}, \tau)$ be a normal mixed space, with τ locally convex and $\mathcal{B}_0$ a basis for $\mathcal{B}$ consisting of τ-closed sets. A subset A of $(E, \gamma_\tau)'$ is γ_τ-equicontinuous if and only if for every $\varepsilon > 0$ and every $B \in \mathcal{B}_0$ there is a τ-equicontinuous set $A(\varepsilon, B)$ such that $A \subset A(\varepsilon, B) + \varepsilon B^0$, where the polar B^0 is taken in E^b.

Proof. We first note the every $B \in \mathcal{B}_0$ is $\sigma(E,(E,\tau)')$-closed, and hence also $\sigma(E,(E,\gamma_\tau)')$-closed, by [15], Chapter VI, Theorem 2, Corollary 3. If U is an absolutely convex closed τ-neighborhood of 0 in E, it can then be shown as in the proof of [15], Chapter VI, Theorem 2 that $(U \cap B)^0 \subset U^0 + B^0 \subset 2(U \cap B)^0$.

If A is a γ_τ-equicontinuous set and $\varepsilon > 0$, it follows from the characterization of the γ_τ-neighborhoods of 0 that we can find a family $(U_B)_{B\in\mathcal{B}_0}$ of absolutely convex closed τ-neighborhoods of 0 such that $A \subset \varepsilon[ac \cup \{(B \cap U_B) \colon B \in \mathcal{B}_0\}]^0$, (where "$ac$" denotes the absolutely convex hull) and hence for every $B \in \mathcal{B}_0$,

$$A \subset \varepsilon(B \cap U_B)^0 \subset \varepsilon U_B^0 + \varepsilon B^0.$$

The result then follows from the fact that εU_B^0 is τ-equicontinuous.

Conversely, suppose $A \subset (E,\gamma_\tau)'$ satisfies the given condition. It follows easily from the definition of γ_τ that A will be γ_τ-equicontinuous if (and only if) the restrictions of the functionals in A to B is τ-equicontinuous for every $B \in \mathcal{B}_0$. If $\varepsilon > 0$ and $B \in \mathcal{B}_0$, then it follows from the assumption that we can find a closed absolutely convex τ-neighborhood U_B of 0 such that $A \subset \frac{1}{2}\varepsilon[U_B^0 + B^0] \subset \varepsilon[U_B \cap B]^0$. Hence $|f(x)| \le \varepsilon$ for every $f \in A, x \in U_B \cap B$, as required. □

As was pointed out in [4], the fact that for a bornology $\mathcal{B}$ on E, $\cup\mathcal{B} = E$, is needed to show that every precompact set is $\mathcal{B}$-precompact. This still holds even if we only assume that $\cup\mathcal{B}$ is dense in E. This slightly generalized version of $\mathcal{B}$-precompactness allows us to restate the above theorem.

Corollary 2.2

Let $(E,\mathcal{B},\tau)$ be a normal mixed space, $\mathcal{E}$ the collection of τ-equicontinuous subsets of $(E,\tau)'$ and let $(E,\gamma_\tau)'$ have the topology τ_b. Then a subset of $(E,\gamma_\tau)'$ is γ_τ-equicontinuous if and only if it is $\mathcal{E}$-precompact.

It follows from the corollary that $\mathcal{B}$-precompact sets in a duality-setting may well signify the presence of a normal mixed space. We illustrate this using Example 1.1. Let E' denote the dual of the Banach space E, $\mathcal{B}'$ the bornology of norm bounded sets in E' and τ the Mackey topology $\tau(E',E)$. Then it is easy to see that $(E',\mathcal{B}',\tau)$ is a mixed space, and it follows from the fact that the closed unit ball in E' is weak* compact that it is in fact a normal mixed space. It now follows from Example 1.1 and Theorem 2.1 that the associated mixed topology is in fact the Mackey topology $\tau(E',E)$.

A further example will be explored in more depth in the next section.

3. Order Precompact Sets and Uniformly Lebesgue Topologies

In this section we look at an example of the duality discussed in the previous section in the setting of Riesz spaces. We briefly summarize some of the notions to be used; more information may be found in [1].

All Riesz spaces will be assumed to be Archimedean. If E is a Riesz space, its set of positive elements will be denoted by E^+, the space of all order bounded linear functionals on it by $E^\sim$, and the space of order continuous linear functionals by $E^\times$. A subset A of E is *solid* if $x \in E, y \in A$ and $|x| \leq |y|$ implies $x \in A$. An *ideal* in E is a solid linear subspace; the ideal of E generated by a subset of A of E is denoted by I_A. If $x, y \in E$, we write $[x, y] = \{z \in E\colon x \leq z \leq y\}$ and call such a set an *order interval*. A set is *order bounded* if it is contained in an order interval.

A linear space topology τ on a Riesz space E is *locally solid* if it has a basis for the neighborhoods of 0 consisting of solid sets; if it is in addition locally convex, it will be called *locally convex-solid*. The space of all τ-continuous linear functionals is denoted by $(E,\tau)'$ (or E' for short); if τ is locally solid, $(E,\tau)'$ is an ideal in $E^\sim$. If E is a Banach lattice with dual E', $E' = E^\sim$. A locally solid topology τ is *pre-Lebesgue* if every disjoint order-bounded sequence is τ-convergent to 0, *Lebesgue* if every decreasing net with infimum 0 is τ-convergent to 0, and *Fatou* if it has a basis for the neighborhoods of 0 consisting of solid order-closed sets. A pre-Lebesgue topology is Lebesgue if and only if it is Fatou. If τ is a Hausdorff Fatou topology, every solid order-closed set is τ-closed. If E is a normed lattice, we shall write E'_a (respectively $E^\times_a$) for the largest ideal of E' (respectively $E' \cap E^\times$) on which the topology induced by the norm of E' is Lebesgue.

If E is a locally solid Riesz space and $\mathcal{B}$ the bornology of order bounded subsets of E, the $\mathcal{B}$-precompact sets were called *Riesz precompact* in [2]. These sets are closely related to the order precompact and quasi-order precompact sets of [6]. We recall that a subset A of E is *order precompact* if for every solid neighborhood U of 0 in E, there is a positive x in the ideal of E generated by A such that $A \subset [-x, x] + U$. We refer to [2] for more detailed information on these notions.

Uniformly Lebesgue topologies were introduced by Nowak in [11] in the setting of normed function spaces. In order to generalize this notion to a large class of normed Riesz spaces, we need to generalize the topology of convergence in measure on sets of finite measure. This is done in [3]; we give the facts pertinent to this paper here.

Let E be a Riesz space which contains an order-dense Riesz subspace F which admits a Hausdorff Lebesgue topology τ. The topology τ has a set $\mathcal{P}$ of defining

Riesz pseudonorms. If $p \in \mathcal{P}$ and $0 \leq u \in F$, we can define a Riesz pseudonorm p_u by

$$p_u(x) = p(|x| \wedge u) \qquad (x \in E).$$

The topology τ_m defined by the pseudonorms p_u $(p \in \mathcal{P}, 0 \leq u \in F)$ is independent of F and τ. It is a Hausdorff Lebesgue topology, and is in fact the coarsest such topology on E. If (x_n) is a disjoint sequence in E^+, (x_n) is τ_m-convergent to 0. In the case where E is a Riesz space of measurable functions on a semi-finite measure space, τ_m is the topology of convergence in measure on sets of finite measure. In the rest of this section whenever the existence of the topology τ_m on a Riesz space E is assumed, it will be assumed that E has an order-dense Riesz subspace which admits a Hausdorff Lebesgue topology. It is known that all Hausdorff Lebesgue topologies induce the same topology on the order bounded subsets of a Riesz space ([1], Theorem 12.9). Using this result it is easy to show that a Hausdorff locally solid topology τ on E is Lebesgue if and only if every order bounded net (x_α) which is τ_m-convergent to 0 is also τ-convergent to 0. This motivates the following definition (see also [11], Definition 1.1).

Let E be a normed Riesz space with unit ball B_E. A locally solid topology τ is *uniformly Lebesgue* if every net in B_E which is τ_m-convergent to 0 is also τ-convergent to 0; and τ is *uniformly pre-Lebesgue* if every disjoint sequence in B_E is τ-convergent to 0.

It follows at once that every uniformly (pre-) Lebesgue topology is (pre-) Lebesgue. The converse holds in L_∞-spaces. Clearly τ_m is a uniformly Lebesgue topology. Since disjoint sequences are τ_m-convergent to 0, every uniformly Lebesgue topology is uniformly pre-Lebesgue.

The following duality result will play a crucial role in the rest of this section.

Theorem 3.1

Let E be a normed Riesz space with closed unit ball B and A a solid $\sigma(E^\sim, E)$-bounded subset of $E^\sim$. Define the seminorm p_A on E by $p_A(x) = \sup\{|f(x)| \colon f \in A\}$. Consider the following statements:

(1) B is order p_A-precompact and A is order $|\sigma|(E^\sim, E)$-precompact.
(2) A is order precompact for the norm on E' and B is order $|\sigma|(E, I_A)$-precompact.
(3) $\|f_n\| \to 0$ for every disjoint sequence (f_n) in A.
(4) $p_A(x_n) \to 0$ for every disjoint sequence (x_n) in B.

Then (1) $\Rightarrow$ (4), and if E is a Banach lattice, or if the topology τ_m can be defined in E, (4) $\Rightarrow$ (1). If A is a subset of the norm dual E' of E,
(1) $\iff$ (2) $\iff$ (3).

Proof. This is a special case of [2], Theorem 3.3. If E is a Banach lattice, (4) $\Rightarrow$ (1) follows from the fact that the norm topology on E is a complete topology for which B is bounded. In the case where the topology τ_m can be defined on E, it is a Hausdorff Lebesgue topology on E. To see that B is τ_m-bounded, it suffices to observe that it can be shown (cf. [3], Theorem 5.6) that τ_m is coarser than the norm topology on E. □

Corollary 3.2

Let E be a normed Riesz space and F an ideal in E'. Then the topology induced by the norm of E' on F is Lebesgue if and only if the closed unit ball B_E of E is order $|\sigma|(E,F)$-precompact.

Proof. Let B_E be order $|\sigma|(E,F)$-precompact and $f \in F^+$. Then (1) $\Rightarrow$ (3) of 3.1, with $A = [-f, f]$, shows that every disjoint sequence in A is norm convergent to 0, and it follows that the norm of E' induces a pre-Lebesgue topology on F. The norm topology is Fatou on E', and since F is an ideal, also on F. It follows that the norm topology on F is Lebesgue.

Conversely, if the norm topology on F is Lebesgue, hence pre-Lebesgue, it follows from (3) $\Rightarrow$ (1) of 3.1 that B_E is $|\sigma|\,(E,F)$-precompact. □

Corollary 3.3

Let τ be a locally convex pre-Lebesgue topology on a normed Riesz space E. If B_E is order τ-precompact, τ is uniformly pre-Lebesgue. Conversely, if τ is uniformly Lebesgue, or E is a Banach lattice and τ is uniformly pre-Lebesgue, B_E is order τ-precompact.

Proof. Since τ is pre-Lebesgue, every τ-equicontinuous subset A of $F = (E,\tau)'$ is order $|\sigma|(F,E)$-precompact ([2], Theorem 2.7); also F is an ideal in $E^\sim$. If B_E is order τ-precompact, it follows from (1) $\Rightarrow$ (4) of 3.1 that τ is uniformly pre-Lebesgue. Conversely, under the stated conditions it follows as before that (4) $\Rightarrow$ (1) of 3.1 holds, and the result follows. □

Corollary 3.4

If E is a Banach lattice and τ a uniformly pre-Lebesgue locally convex topology on E, then $F = (E,\tau)' \subset E'_a$.

Proof. This follows from 3.3, 3.2 and the fact that τ is finer than $|\sigma|(E,F)$. □

The converse of 3.4 does not hold. As an example, let $E = L_2[0,1]$ and τ be the usual norm topology of E. Then $(E,\tau)' = L_2[0,1] = E'_a$. If τ were uniformly pre-Lebesgue, B_E would be order τ-precompact (by 3.3), and it would then follow from [8], Lemma 4.4 that $L_2[0,1]$ is finite-dimensional.

Theorem 3.5

Let τ be a locally convex-solid topology on a normed Riesz space E. If every τ-equicontinuous set is an order precompact subset of E'_a, τ is uniformly pre-Lebesgue, and the converse holds if E is a Banach lattice.

Proof. If every τ-equicontinuous set A is order precompact in E'_a, then in particular $F = (E,\tau)' \subset E'_a$. It follows from 3.2 that B_E is order $|\sigma|(E,F)$-precompact, and the result then follows from (2) $\Rightarrow$ (4) of 3.1.

Conversely, if E is a Banach lattice, we have $F = (E,\tau)' \subset E^{\sim} = E'$. Also, (4) $\Rightarrow$ (2) of 3.1 shows that every τ-equicontinuous set A in E' is order precompact in E' and that B_E is order $|\sigma|(E,F)$-precompact. It follows from 3.2 that $F \subset E'_a$. □

We immediately obtain a partial converse to 3.4:

Corollary 3.6

Let E be a normed Riesz space and F an ideal in E'_a. Then $|\sigma|(E,F)$ is a uniformly pre-Lebesgue topology.

The following result, reminiscent of the fact that Hausdorff Lebesgue topologies coincide on order bounded sets, will be needed to analyse uniformly Lebesgue topologies:

Theorem 3.7

Let τ be a Lebesgue topology on the Riesz space E. Then τ_m induces a finer topology than τ on the order τ-precompact sets of E. If τ is Hausdorff, the two topologies coincide on the order τ-precompact sets.

Proof. Let A be τ-precompact and suppose (x_α) is a net in A which is τ_m-convergent to $x \in A$. It follows from [1], Theorem 12.8 that the topology induced by τ_m on order intervals of E is finer than that induced by τ. Let p be a τ-continuous pseudonorm and $\varepsilon > 0$. Choose $u \in I_A^+$ such that $p(|x| - |x| \wedge u) < \varepsilon$ for every $x \in A$. Since (x_α) is τ_m-convergent to x, $(|x - x_a| \wedge 2u)$ is τ_m-convergent to 0, and so it follows a priori that we can find an α_0 such that $p(|x - x_a| \wedge 2u) < \varepsilon$ for $\alpha \geq \alpha_0$. Therefore, for $\alpha \geq \alpha_0$

$$\begin{aligned} p(x - x_0) &\leq p(|x - x_\alpha| - |x - x_\alpha| \wedge 2u) + p(|x - x_\alpha| \wedge 2u) \\ &\leq p(|x| - |x| \wedge u) + p(|x_\alpha| \wedge u) + p(|x - x_\alpha| \wedge 2u) \\ &< 3\varepsilon \end{aligned}$$

and so (x_α) is τ-convergent to x. If τ is Hausdorff, $\tau_m \leq \tau$ on E, and the result follows. □

Proposition 3.8

Let τ be a locally solid topology on a normed Riesz space E. If τ is Lebesgue and B_E is order τ-precompact, τ is uniformly Lebesgue. The converse holds if τ is locally convex.

Proof. The first part is immediate from Theorem 3.7, and the second follows from Corollary 3.3. □

Corollary 3.9

Let τ be a locally convex uniformly pre-Lebesgue topology on a normed Riesz space E. Then the following are equivalent:
(a) τ is uniformly Lebesgue
(b) τ is Lebesgue
(c) τ is Fatou.

Proof. The implications (a) $\Rightarrow$ (b) $\Longleftrightarrow$ (c) are all easy, and (b) $\Rightarrow$ (a) follows from 3.8 and (4) $\Rightarrow$ (1) of 3.1, recalling that by convention we are assuming that the topology τ_m can be defined on E. □

Corollary 3.10

If E is a normed Riesz space, $|\sigma|(E, E_a^\times)$ is a uniformly Lebesgue topology.

Proof. Immediate from 3.6 and 3.9. □

It is now possible to give a non-trivial example of a uniformly pre-Lebesgue topology which is not uniformly Lebesgue. Let E be a Banach lattice such that $E^\times \subset E' = E'_a$, $(E^\times \neq E')$. Then $\tau = |\sigma|(E, E') = |\sigma|(E, E'_a)$ is uniformly pre-Lebesgue (by 3.6), but since $(E, \tau)' = E'_a \subset E^\times$, $(E'_a \neq E^\times)$, τ is not Lebesgue and therefore not uniformly Lebesgue. An example of such a Banach lattice is the direct sum $L_\infty[0,1] \oplus L_2[0,1]$, equipped with the coordinate-wise ordering and the norm

$$\|f \oplus g\| = \max\{\|f\|_\infty, \|g\|_2\}.$$

The dual space is $L_\infty^*[0,1] \oplus L_2[0,1]$, with norm

$$\|\varphi \oplus \psi\| = \|\varphi\|_* + \|\psi\|_2,$$

where $\|\cdot\|_*$ denotes the norm of $L_\infty^*[0,1]$.

Proposition 3.11

Let τ be a uniformly Lebesgue topology on a normed Riesz space E. Then $(E,\tau)' \subset E_a^\times$.

Proof. Let $f \in (E,\tau)'$ and suppose the sequence (x_n) converges in norm to 0. Then there is no loss in generality in assuming that $x_n \in B_E$ for all $n \in \mathbb{N}$, and the same argument as in the proof of 3.1 shows that (x_n) is τ_m-convergent to 0. It then follows from 3.3 and 3.7 that (x_n) is τ-convergent to 0, and so $f(x_n) \to 0$. Hence $(E,\tau)' \subset E'$ and the result now follows in the same way as in 3.4. □

Theorem 3.12

A locally convex-solid topology τ on a normed Riesz space E is uniformly Lebesgue if and only if every τ-equicontinuous set is an order-precompact subset of $E_a^\times$.

Proof. If ever τ-equicontinuous subset is an order precompact subset of $E_a^\times \subset E_a'$, it follows from 3.5 that τ is uniformly pre-Lebesgue; since $(E,\tau)' \subset E_a^\times \subset E^\times, \tau$ is also Lebesgue and hence by 3.9 uniformly Lebesgue. Conversely, if τ is uniformly Lebesgue, $(E,\tau)' \subset E_a' \cap E^\times = E_a^\times$ (by 3.11) and the result then follows from (4) $\Rightarrow$ (1) of 3.1. □

It is clear from the above result that if the topology τ_m can be defined on a normed Riesz space E, then there is a finest uniformly Lebesgue locally convex topology on E, namely the topology of uniform convergence on the order precompact subsets of $E_a^\times$. Likewise it follows from 3.5 that there is a finest uniformly pre-Lebesgue topology on a Banach lattice E, namely, the topology of uniform convergence on the order precompact subsets of E_a'. We now show that these topologies are often mixed topologies. It is clear from 2.2 that the appropriate mixed spaces to consider are $(E,\mathcal{B},|\sigma|(E,E_a^\times))$ and $(E,\mathcal{B},|\sigma|(E,E_a'))$ respectively, where $\mathcal{B}$ is the bornology of norm-bounded subsets of E.

Lemma 3.13

Let E be a Fatou-normed Riesz space and $\mathcal{B}$ its bornology of norm-bounded sets. If τ is a Hausdorff Fatou topology on E such that every $B \in \mathcal{B}$ is τ-bounded, then $(E,\mathcal{B},\tau)$ is a normal mixed space.

Proof. Clearly $\mathcal{B}_0 = \{nB_E : n \in \mathbb{N}\}$ is a basis for $\mathcal{B}$ consisting of solid order-closed sets, and since τ is a Hausdorff Fatou topology, each such set is τ-closed ([1], Theorem 12.7). □

Theorem 3.14

Let E be a Fatou-normed Riesz space, $\mathcal{B}$ the bornology of norm-bounded subsets of E and F an ideal in $E_a^\times$ which separates the points of E. Then $(E, \mathcal{B}, |\sigma|(E,F))$ is a normal mixed space and $\gamma_{|\sigma|(E,F)}$ is the topology of uniform convergence on the order precompact sets of the norm closure $\overline{F}$ of F in E'. In particular, if $E_a^\times$ separates the points of E, the finest uniformly Lebesgue topology is a mixed topology.

Proof. It is enough to note that $|\sigma|(E,F)$ is a Hausdorff Lebesgue, hence Fatou, topology with dual F. □

If the topology τ_m can be defined on a Fatou-normed Riesz space E, 3.13 shows that $(E, \mathcal{B}, \tau_m)$ is also a normal mixed space. Since τ_m is in general not a locally convex topology, Theorem 2.1 cannot be used to identify the mixed topology γ_{τ_m}. However, Theorem 3.7 comes to the rescue in many cases.

Lemma 3.15

Let E be a normed Riesz space such that $E_a^\times$ separates the points of E. Then τ_m and $|\sigma|(E, E_a^\times)$ coincide on the closed unit ball B_E of E.

Proof. The topology $|\sigma|(E, E_a^\times)$ is Hausdorff and hence τ_m can be defined on E. The result now follows from 3.2 and 3.7. □

To formulate the next result, we introduce the notation γ'_τ for the finest vector topology which coincides with a vector topology τ on the norm-bounded subsets of a normed Riesz space E. It follows from [7], Proposition 5 that if τ is locally convex $\gamma'_\tau = \gamma_\tau$.

Theorem 3.16

Let E be a normed Riesz space such that $E_a^\times$ separates the points of E. Then

$$\gamma_{\tau_m} = \gamma_{|\sigma|(E,E_a^\times)} = \gamma'_{|\sigma|(E,E_a^\times)} = \gamma'_{\tau_m}\,.$$

Proof. The result follows at once from 3.15 and the above remark. □

Corollary 3.17 (cf. [11], Theorem 3.3)

Let E be a normed Riesz space. The following are equivalent:

(a) γ'_{τ_m} is locally convex
(b) $E_a^\times$ separates the points of E
(c) τ_m and $|\sigma|(E, E_a^\times)$ coincide on the closed unit ball B_E of E.

Proof. (a) $\Rightarrow$ (b): Since γ'_{τ_m} is clearly a uniformly Lebesgue topology, $(E, \gamma'_{\tau_m})' \subset E_a^\times$ by 3.11. If γ'_{τ_m} is locally convex, it follows that $E_a^\times$ separates the points of E.
(b) $\Rightarrow$ (c) : This is 3.15.
(c) $\Rightarrow$ (a) : This follows as in 3.16. $\square$

Proposition 3.18

Let E be a normed Riesz space. Then γ'_{τ_m} is the finest uniformly Lebesgue topology on E.

Proof. Clearly γ'_{τ_m} is uniformly Lebesgue. If τ is a uniformly Lebesgue topology on E, τ_m is finer than τ on B_E, and so $\tau \leq \gamma'_\tau \leq \gamma'_{\tau_m}$. $\square$

Applications of the material in this section to the characterization of compact sets and compact operators in Banach lattices will be given in a forthcoming paper.

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