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Painlevé's problem and analytic capacity

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Abstract

In this paper we survey some recent results in connection with the so called Painlevé's problem and the semiadditivity of analytic capacity γ. In particular, we give the detailed proof of the semiadditivity of the capacity γ_+, and we show almost completely all the arguments for the proof of the comparability between γ and γ_+.

This paper arose from a series of three lectures given at the 7^{th} International Conference on Harmonic Analysis and Partial Differential Equations, at El Escorial (Madrid), in June 2004.

In this article we will discuss and review some recent results in connection with the so called Painlevé's problem and the semiadditivity of analytic capacity, as well as other related questions. Some parts of this work, specially those sections which are purely expository, follow quite closely our previous (but very recent) survey paper [53]. However, the present article contains much more detailed information. For instance, unlike [53], it includes the detailed arguments for the proofs of the semiadditivity of γ_+ and of the comparability between γ and γ_+.

The plan of the paper is the following. In the first section, which is introductory and purely expository, the notions of analytic capacity

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and Painlevé's problem are explained. Also, some basic properties of analytic capacity are shown and the theorem of Guy David which solves Vitushkin's conjecture is stated. The final part of this section deals with the Cauchy transform and the capacity γ_+.

In the second section we introduce the notion of curvature of a measure, and we show its close relationship with the L^2 norm of the Cauchy transform on the one hand, and with rectifiability on the other hand. This section is also mainly expository.

In Section 3 we survey several results on Calderón-Zygmund theory with non doubling measures. This theory plays a key role in the study of analytic capacity. We discuss some of the results more useful in connection with analytic capacity, such as the weak $(1, 1)$ boundedness of Calderón-Zygmund operators, Cotlar's inequality, and the $T(1)$ and $T(b)$ theorems.

Section 4 deals with the semiadditivity of the capacity γ_+. We give the detailed proof of this result. Further, we show a new (as far as we know) proof which avoids the use of the $T(1)$ theorem (although we also explain the alternative arguments based on the use of the $T(1)$ theorem).

In Section 5 we state the theorem about the comparability between γ and γ_+, which in particular implies the semiadditivity of analytic capacity. We describe the main ideas and difficulties in the proof of the comparability theorem. We give the "almost complete" proof of this result in Section 6. With the exception of some few technical and lengthy details, we describe all the arguments.

The final Section 7 is again purely expository. It contains a quick overview of some recent results related to analytic capacity. For example, it deals with questions concerning the continuous analytic capacity α, the behavior of analytic capacity under bilipschitz mappings, Lipschitz harmonic capacity, the capacities γ_s associated to signed s-dimensional Riesz kernels with s non integer... Of course, there is no attempt at completeness in this survey.

Some comments about the notation used in the paper: by a cube Q we mean a closed cube with sides parallel to the axes. We denote its side length by $\ell(Q)$. As usual, the letter 'C' stands for an absolute constant which may change its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

I would like to thank the organizers of the conference of El Escorial for inviting me to give the lectures from which this paper arose.

1. Analytic capacity

1.1 Analytic capacity and Painlevé's problem

A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. In order to study removability, in the 1940's Ahlfors [1] introduced the notion of analytic capacity. The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) = \sup |f'(\infty)|, \tag{1}$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

In [1], Ahlfors showed that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$.

Painlevé's problem consists in characterizing removable singularities for bounded analytic functions in a metric/geometric way. By Ahlfors' result this is equivalent to describing compact sets with positive analytic capacity in metric/geometric terms.

Vitushkin in the 1950's and 1960's showed that analytic capacity and the so called continuous analytic capacity play a central role in problems of uniform rational approximation on compact sets of the complex plane. The continuous analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as

$$\alpha(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all *continuous* functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ which are analytic on $\mathbb{C} \setminus E$ and uniformly bounded by 1 on $\mathbb{C}$. Many results obtained by Vitushkin in connection with uniform rational approximation are stated in terms of α and γ. See [58] and [59], for example. See also [55] for a more modern approach. Because of its applications to this type of problems he raised the question of the semiadditivity of γ and α. Namely, does there exist an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)) \ ?$$

And analogously for α.

1.2 Basic properties of analytic capacity

One should keep in mind that, in a sense, analytic capacity measures the size of a set as a non removable singularity for bounded analytic functions. A direct consequence of the definition is that

$$E \subset F \ \Rightarrow \ \gamma(E) \leq \gamma(F).$$

Moreover, it is also easy to check that analytic capacity is translation invariant:

$$\gamma(z + E) = \gamma(E) \qquad \text{for all } z \in \mathbb{C}.$$

Concerning dilation, we have

$$\gamma(\lambda E) = |\lambda|\gamma(E) \qquad \text{for all } \lambda \in \mathbb{C}.$$

Further, if E is connected, then

$$\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E).$$

The second inequality (which holds for any compact set E) follows from the fact that the analytic capacity of a closed disk coincides with its radius, and the first one is a consequence of Koebe's 1/4 theorem (see [9, Chapter VIII] or [15, Chapter I] for the details, for example). Thus if E is connected and different from a point, then it is non removable. This implies that any removable compact set must be totally disconnected.

1.3 Relationship with Hausdorff measure

The relationship between Hausdorff measure and analytic capacity is the following:

- If $\dim_H(E) > 1$ (here $\dim_H$ stands for the Hausdorff dimension), then $\gamma(E) > 0$. This result follows easily from Frostman's Lemma.
- $\gamma(E) \leq \mathcal{H}^1(E)$, where $\mathcal{H}^1$ is the one dimensional Hausdorff measure, or length. This follows from Cauchy's integral formula, and it was proved by Painlevé about one hundred years ago. Observe that, in particular, it implies that if $\dim_H(E) < 1$, then $\gamma(E) = 0$.

By the statements above, it turns out that dimension 1 is the critical dimension in connection with analytic capacity. Moreover, a natural question arises: is it true that $\gamma(E) > 0$ if and only if $\mathcal{H}^1(E) > 0$?

Vitushkin showed that the answer is *no*. He showed that there are sets with positive length and vanishing analytic capacity. A typical example of such a set is the so called corner quarters Cantor set. This set is constructed in the following way: consider a square Q^0 with side length 1. Now replace Q^0 by 4 squares Q_i^1, $i = 1, \ldots, 4$, with side length 1/4 contained in Q^0, so that each Q_i^1 contains a different vertex of Q^0. Analogously, in the next stage each Q_i^1 is replaced by 4 squares with side length 1/16 contained in Q_i^1 so that each one contains a different vertex of Q_i^1. So we will have 16 squares Q_k^2 of side length 1/16. We proceed inductively (see Figure 1), and we set $E_n = \bigcup_{i=1}^{4^n} Q_i^n$ and $E = \bigcap_{n=1}^{\infty} E_n$. This is the corner quarters Cantor set. Taking into account that

$$\sum_{i=1}^{4^n} \ell(Q_i^n) = 1$$

for each n, it is not difficult to see that $0 < \mathcal{H}^1(E) < \infty$. The proof of the fact that $\gamma(E) = 0$ is more difficult, and it is due independently to Garnett [14] and Ivanov[1] [18].

[1] Vitushkin constructed a different example previously.

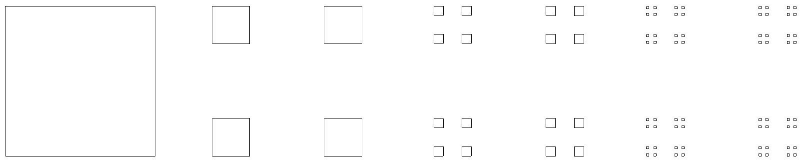

Fig. 1. The square Q^0 and the sets E_1, E_2 and E_3, which appear in the first stages of the construction of the corner quarters Cantor set.

Recall that a set is called rectifiable if it is $\mathcal{H}^1$-almost all contained in a countable union of rectifiable curves. On the other hand, it is called purely unrectifiable if it intersects any rectifiable curve at most in a set of zero length.

It turns out that the corner quarters Cantor set, and also Vitushkin's example, are purely unrectifiable. Motivated by this fact Vitushkin conjectured that pure unrectifiability is a necessary and sufficient condition for vanishing analytic capacity for sets with finite length.

Guy David [4] showed in 1998 that Vitushkin's conjecture is true:

Theorem 1

Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then, $\gamma(E) = 0$ if and only if E is purely unrectifiable.

To be precise, let us remark that the "if" part of the theorem is not due to David. In fact, it follows from Calderón's theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs with small Lipschitz constant. The "only if" part of the theorem, which is more difficult, is the one proved by David. See also [31, 6] and [22] for some previous important preliminary contributions to the proof.

Theorem 1 is the solution of Painlevé's problem for sets with finite length. The analogous result is false for sets with infinite length (see [28] and [21]). For this type of sets there is no such a nice geometric solution of Painlevé's problem, and we have to content ourselves with a characterization such as the one in Corollary 16 below (at least, for the moment).

1.4 The capacity γ_+ and the Cauchy transform

Given a finite complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is defined by

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z} d\nu(\xi).$$

Although the integral above is absolutely convergent a. e. with respect to Lebesgue measure, it does not make sense, in general, for $z \in \text{supp}(\nu)$.

This is the reason why one considers the *truncated Cauchy transform* of ν, which is defined as

$$\mathcal{C}_\varepsilon \nu(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z} d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$.

Given a positive Radon measure μ on the complex plane and a μ-measurable function f on $\mathbb{C}$, we also denote

$$\mathcal{C}_\mu f(z) := \mathcal{C}(f\, d\mu)(z)$$

for $z \notin \operatorname{supp}(f)$, and

$$\mathcal{C}_{\mu,\varepsilon} f(z) := \mathcal{C}_\varepsilon(f\, d\mu)(z)$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. We say that $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ if the operators $\mathcal{C}_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The capacity γ_+ of a compact set $E \subset \mathbb{C}$ is

$$\gamma_+(E) := \sup \big\{ \mu(E) : \operatorname{supp}(\mu) \subset E,\ \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1 \big\}. \tag{2}$$

That is, γ_+ is defined as γ in (1) with the additional constraint that f should coincide with $\mathcal{C}\mu$, where μ is some positive Radon measure supported on E (observe that $(\mathcal{C}\mu)'(\infty) = -\mu(\mathbb{C})$ for any Radon measure μ). To be precise, there is another slight difference: in (1) we asked $\|f\|_{L^\infty(\mathbb{C}\setminus E)} \leq 1$, while in (2), $\|f\|_{L^\infty(\mathbb{C})} \leq 1$ (for $f = \mathcal{C}\mu$). Trivially, we have $\gamma_+(E) \leq \gamma(E)$.

2. The curvature of a measure

A Radon measure μ on $\mathbb{R}^d$ has growth of degree n (or is of degree n) if there exists some constant C_0 such that

$$\mu(B(x,r)) \leq C_0 r^n \quad \text{for all } x \in \mathbb{R}^d,\ r > 0. \tag{3}$$

When $n = 1$, we say that μ has linear growth.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x,y,z) = 0$. For a positive Radon measure μ, we set

$$c_\mu^2(x) = \iint c(x,y,z)^2\, d\mu(y) d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c_\mu^2(x)\, d\mu(x) = \iiint c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z). \tag{4}$$

The notion of curvature of a measure was introduced by Melnikov [33] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the big recent advances in connection with analytic capacity. The notion of curvature is connected to the Cauchy transform by the following result, proved by Melnikov and Verdera [34].

Proposition 2

Let μ be a Radon measure on $\mathbb{C}$ with linear growth. We have

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6}\, c_\varepsilon^2(\mu) + O(\mu(\mathbb{C})), \tag{5}$$

where $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$.

In this proposition, $c_\varepsilon^2(\mu)$ stands for the ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (4), but with the triple integral over $\{x,y,z \in \mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$).

The identity (5) is remarkable because it relates an analytic notion (the Cauchy transform of a measure) with a metric-geometric one (curvature). We give a sketch of the proof.

Sketch of the proof of Proposition 2. If we don't worry about truncations and the absolute convergence of the integrals, we can write

$$\begin{aligned}\|\mathcal{C}\mu\|^2_{L^2(\mu)} &= \int \left| \int \frac{1}{y-x}\, d\mu(y) \right|^2 d\mu(x) \\ &= \iiint \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x).\end{aligned}$$

By Fubini (assuming that it can be applied correctly), permuting x, y, z, we get,

$$\|\mathcal{C}\mu\|^2_{L^2(\mu)} = \frac{1}{6} \iiint \sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})\overline{(z_{s_3} - z_{s_1})}}\, d\mu(z_1)d\mu(z_2)d\mu(z_3),$$

where S_3 is the group of permutations of three elements. An elementary calculation shows that

$$\sum_{s \in S_3} \frac{1}{(z_{s_2} - z_{s_1})\overline{(z_{s_3} - z_{s_1})}} = c(z_1, z_2, z_3)^2.$$

So we get

$$\|\mathcal{C}\mu\|_{L^2(\mu)}^2 = \frac{1}{6}\, c^2(\mu).$$

To argue rigorously, above we should use the truncated Cauchy transform $\mathcal{C}_\varepsilon\mu$ instead of $\mathcal{C}\mu$. Then we would obtain

$$\begin{aligned}\|\mathcal{C}_\varepsilon\mu\|_{L^2(\mu)}^2 &= \iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x)\\ &= \iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon\\|y-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\, d\mu(y)d\mu(z)d\mu(x)\\ &\quad + O(\mu(\mathbb{C})). \end{aligned} \tag{6}$$

By the linear growth of μ, it is easy to check that $|O(\mu(\mathbb{C}))| \leq \mu(\mathbb{C})$. As above, using Fubini and permuting x, y, z, one shows that the triple integral in (6) equals $c_\varepsilon^2(\mu)/6$. □

The notion of curvature is related to rectifiability. Indeed, it is strongly connected with the coefficients β which appear in the travelling salesman theorem of P. Jones [20]. In spite of the importance of Jones' ideas (which first appeared in the paper [19], from El Escorial conference in 1987), in our paper we will skip most of the details which have to do with β's and rectifiability. We refer the reader to [41] for additional information.

The following nice result of David and Léger [22], which plays a key role in the proof of Vitushkin's conjecture (Theorem 1), is an example of the relationship between curvature and rectifiability.

Theorem 3

Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. If $c^2(\mathcal{H}^1_{|E}) < \infty$, then E is rectifiable.

Observe that from the preceding result and Proposition 2 one infers that if $\mathcal{H}^1(E) < \infty$ and the Cauchy transform is bounded on $L^2(\mathcal{H}^1_{|E})$, then E must be rectifiable. A more quantitative version of this result proved previously by Mattila, Melnikov and Verdera [31] asserts that if E is such that

$$\mathcal{H}^1(E \cap B(x,r)) \approx r \qquad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam}(E)$$

and the Cauchy transform is bounded on $L^2(\mathcal{H}^1_{|E})$, then E is contained in a regular curve Γ (i.e. a curve which also satisfies the preceding estimates, with Γ instead of E).

3. Calderón-Zygmund theory with non doubling measures

The study of analytic capacity has led to the extension of Calderón-Zygmund (CZ) theory to the situation where the underlying measure μ on $\mathbb{C}$ is non doubling. Recall that μ is said to be doubling if there exists some constant C such that

$$\mu(B(z,2r)) \leq C\mu(B(z,r)) \qquad \text{for } z \in \operatorname{supp}(\mu)) \text{ and } r > 0.$$

Let us remark that in the classical CZ theory this doubling assumption plays an essential role in almost all results. When one deals with analytic capacity one is forced to deal with measures which may be non doubling, and which are only assumed to have linear growth.

The use of CZ theory has been fundamental in most of the recent developments in connection with analytic capacity. For instance, the so called "$T(b)$ type theorems" are essential tools in the proofs of Vitushkin's conjecture by G. David and of the semiadditivity of analytic capacity in [49]. In this section we will describe briefly some results of CZ theory without doubling assumptions that are useful in connection with analytic capacity.

Let us introduce some terminology. We say that

$$k(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\} \to \mathbb{C}$$

is an n-dimensional Calderón-Zygmund kernel if there exist constants $C > 0$ and η, with $0 < \eta \leq 1$, such that the following inequalities hold for all $x, y \in \mathbb{R}^d$, $x \neq y$:

$$\begin{aligned} &|k(x,y)| \leq \frac{C}{|x-y|^n}, \qquad \text{and} \\ &|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leq \frac{C|x-x'|^\eta}{|x-y|^{n+\eta}} \quad \text{if} \\ &|x-x'| \leq |x-y|/2. \end{aligned} \tag{7}$$

For example, the Cauchy kernel $1/(y-x)$, with $x, y \in \mathbb{C}$, is a 1-dimensional CZ kernel.

Given a real or complex Radon measure μ on $\mathbb{R}^d$, we define

$$T\mu(x) := \int k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu). \tag{8}$$

We say that T is an n-dimensional Calderón-Zygmund operator (CZO) with kernel $k(\cdot,\cdot)$. We also consider the following ε-truncated operators T_ε, $\varepsilon > 0$:

$$T_\varepsilon\mu(x) := \int_{|x-y|>\varepsilon} k(x,y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

If μ is non negative and $f \in L^1_{\mathrm{loc}}(\mu)$, we denote

$$T_\mu f(x) := T(f\,d\mu)(x) \qquad x \in \mathbb{R}^d \setminus \operatorname{supp}(f\,d\mu),$$

and

$$T_{\mu,\varepsilon} f(x) := T_\varepsilon(f\,d\mu)(x).$$

We say that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$. Analogously, with respect to the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. We also say that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, where $M(\mathbb{R}^d)$ stands for the space of real (or complex, depending on the context) Radon measures, if there exists some constant C such that for all $\nu \in M(\mathbb{R}^d)$ and all $\lambda > 0$,

$$\mu\{x \in \mathbb{R}^d : |T_\varepsilon \nu| > \lambda\} \leq \frac{C\|\nu\|}{\lambda}$$

uniformly on $\varepsilon > 0$. We have denoted $\|\nu\| = |\nu|(\mathbb{R}^d)$.

As in the doubling case, if a CZO is bounded on $L^2(\mu)$, it turns out that it is also of weak type $(1,1)$, assuming the appropriate growth condition on μ:

Theorem 4

Let μ be a Radon measure on $\mathbb{R}^d$ of degree n. If T_μ is an n-dimensional Calderón-Zygmund operator which is bounded in $L^2(\mu)$, then T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In particular, T_μ is of weak type $(1,1)$.

This result was first obtained in [36], although a previous proof valid only for the Cauchy transform appeared in [44].

Now we turn our attention to Cotlar's inequality. We need to introduce some additional terminology. The *centered maximal Hardy-Littlewood operator* applied to $\nu \in M(\mathbb{R}^d)$ is, as usual,

$$M_\mu \nu(x) = \sup_{r>0} \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} d|\nu|.$$

A useful variant of this operator is the following:

$$\widetilde{M}_\mu \nu(x) = \sup \left\{ \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} d|\nu| : r > 0,\ \mu(\bar{B}(x,5r)) \leq 5^{d+1} \mu(\bar{B}(x,r)) \right\}.$$

For $f \in L^1_{\mathrm{loc}}(\mu)$ we set $M_\mu f := M_\mu(f d\mu)$ and $\widetilde{M}_\mu f := \widetilde{M}_\mu(f d\mu)$. The operators M_μ and $\widetilde{M}_\mu$ are bounded in $L^p(\mu)$, and from $M(\mathbb{R}^d)$ into

$L^{1,\infty}(\mu)$. This fact follows from Besicovitch's covering theorem. Let us remark that the non centered version of $\widetilde{M}_\mu$, given by

$$N_\mu\nu(x) = \sup\left\{\frac{1}{\mu(\bar{B})}\int_{\bar{B}} d|\nu| : \ \bar{B} \text{ closed ball, } x\in\bar{B},\ \mu(5\bar{B})\le 5^{d+1}\mu(\bar{B})\right\},$$

is also bounded in $L^p(\mu)$ and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. This fact can be proved using Vitali's covering theorem with balls $B(x,5r)$.

If T is a CZO, the *maximal operator* T_* is

$$T_*\nu(x) = \sup_{\varepsilon>0}|T_\varepsilon\nu(x)| \qquad \text{for } \nu\in M(\mathbb{R}^d),\ x\in\mathbb{R}^d,$$

and the *δ-truncated maximal operator* $T_{*,\delta}$ is

$$T_{*,\delta}\nu(x) = \sup_{\varepsilon>\delta}|T_\varepsilon\nu(x)| \qquad \text{for } \nu\in M(\mathbb{R}^d),\ x\in\mathbb{R}^d.$$

We also set $T_{\mu,*}\,f := T_*\,(f\,d\mu)$ and $T_{\mu,*,\delta}\,f := T_{*,\delta}\,(f\,d\mu)$ for $f\in L^1_{\mathrm{loc}}(\mu)$.

We have:

Theorem 5 (Cotlar's inequality)

Let μ be a positive Radon measure on $\mathbb{R}^d$ with growth of degree n. If the T is an n-dimensional CZO bounded in $L^2(\mu)$, then for $0<s\le 1$ we have

$$T_{*,\delta}\,\nu(x) \le C_s\left(\widetilde{M}_\mu(|T_\delta\nu|^s)(x)^{1/s} + M_\mu\nu(x)\right), \quad \text{for } \nu\in M(\mathbb{R}^d),\ x\in\mathbb{R}^d, \tag{9}$$

where C_s depends only on the constant C_0 in (3), s, n, d, and the norm of the T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. As a consequence, $T_{\mu,*}$ is bounded in $L^p(\mu)$, $p\in(1,\infty)$ and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$.

Cotlar's inequality with non doubling measures was proved by Nazarov, Treil and Volberg [36], although not exactly in the form stated above, which is from [45].

Given $\rho>1$, we say that $f\in L^1_{\mathrm{loc}}(\mu)$ belongs to the space $BMO_\rho(\mu)$ if

$$\sup_Q \frac{1}{\mu(\rho Q)}\int_Q |f-m_Q(f)|\,d\mu < \infty,$$

where the supremum is taken over all the cubes in $\mathbb{R}^d$ and $m_Q(f)$ is the μ-mean of f over Q.

Following [35], a Calderón-Zygmund operator T_μ is said to be weakly bounded if

$$\big|\langle T_{\mu,\varepsilon}\chi_Q,\chi_Q\rangle\big| \le C\mu(Q) \qquad \text{for all the cubes } Q\subset\mathbb{R}^d, \text{ uniformly on } \varepsilon>0.$$

Notice that if T_μ is antisymmetric, then the left hand side above vanishes and so T_μ is weakly bounded.

Now we are ready to state the $T(1)$ theorem:

Theorem 6 ($T(1)$ theorem)

Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. The following conditions are equivalent:

(a) *T_μ is bounded on $L^2(\mu)$.*
(b) *T_μ is weakly bounded and, for some $\rho > 1$, we have that $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \in BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$.*
(c) *There exists some constant C such that for all $\varepsilon > 0$ and all the cubes $Q \subset \mathbb{R}^d$,*
$$\|T_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \le C\mu(Q)^{1/2} \quad \textit{and} \quad \|T^*_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \le C\mu(Q)^{1/2}.$$

The classical way of stating the $T(1)$ theorem is the equivalence (a) $\Leftrightarrow$ (b). However, for some applications it is sometimes more practical to state the result in terms of the L^2 boundedness of T_μ and T^*_μ over characteristic functions of cubes, i.e. (a) $\Leftrightarrow$ (c).

Theorem 6 is the extension of the classical $T(1)$ theorem of David and Journé to measures of degree n which may be non doubling. The result was proved by Nazarov, Treil and Volberg in [35], although not exactly in the form stated above. An independent proof for the particular case of the Cauchy transform was obtained almost simultaneously in [44]. Another more recent proof of the $T(1)$ theorem for non doubling measures for the particular case of the Cauchy transform is in [56].

By Proposition 2, the $T(1)$ theorem for the Cauchy transform can be rewritten in the following way:

Theorem 7

Let μ be a Radon measure on $\mathbb{C}$ with linear growth. The Cauchy transform is bounded on $L^2(\mu)$ if and only if $c^2(\mu_{|Q}) \le C\mu(Q)$, for all the squares $Q \subset \mathbb{C}$.

Observe that this result is a restatement of the equivalence (a) $\Leftrightarrow$ (c) in Theorem 6, by an application of (5) to the measure $\mu_{|Q}$, for all the squares $Q \subset \mathbb{C}$.

Let us remark that the boundedness of T_μ on $L^2(\mu)$ does not imply the boundedness of T_μ from $L^\infty(\mu)$ into $BMO(\mu)$ (this is the space $BMO_\rho(\mu)$ with parameter $\rho = 1$), and in general $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \notin BMO(\mu)$ uniformly on $\varepsilon > 0$. See [56] and [24]. On the contrary, one can show that if T_μ is bounded on $L^2(\mu)$, then it is also bounded from $L^\infty(\mu)$ into $BMO_\rho(\mu)$, for $\rho > 1$, by arguments similar to the classical ones for homogeneous spaces. However, the space $BMO_\rho(\mu)$ has some drawbacks. For example, it depends on the parameter ρ and it does not satisfy the John-Nirenberg inequality. To solve these problems, in [46] a new space called $RBMO(\mu)$ has been introduced. $RBMO(\mu)$ is a subspace of $BMO_\rho(\mu)$ for all $\rho > 1$, and it coincides with $BMO(\mu)$ when μ is an AD-regular measure, that is, when

$$\mu(B(x,r)) \approx r^n \quad \text{for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

Moreover, $RBMO(\mu)$ satisfies a John-Nirenberg type inequality, and all CZO's which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. For these reasons $RBMO(\mu)$ seems to be a good substitute of the classical space BMO for non doubling measures of degree n.

The precise definition of $RBMO(\mu)$ is the following. Given a function $f \in L^1_{\mathrm{loc}}(\mu)$, we say that f is in $RBMO(\mu)$ if there is a constant C_1 such that for any Q

$$\int_Q |f - m_Q(f)|\, d\mu \leq C_1\, \mu(2Q),$$

and for all cubes Q, R with $Q \subset R$

$$|m_Q(f) - m_R(f)| \leq C_1 \left(1 + \int_{2R\setminus Q} \frac{1}{|x - c_Q|^n}\, d\mu(x)\right) \left[\frac{\mu(2Q)}{\mu(Q)} + \frac{\mu(2R)}{\mu(R)}\right],$$

where c_Q is the center of Q. Let us remark that the definition of $RBMO(\mu)$ does not depend on the choice of the parameter 2 in $\mu(2Q)$ and $\mu(2R)$ in the estimates above. That is to say, $\mu(2Q)$ and $\mu(2R)$ can be replaced by $\mu(\rho Q)$ and $\mu(\rho R)$, for any $\rho > 1$. See Lemma 2.10 in [46] for several other equivalent definitions of $RBMO(\mu)$.

It can be shown that condition (b) in Theorem 6 is equivalent to

(b') *T_μ is weakly bounded and $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \in RBMO(\mu)$ uniformly on $\varepsilon > 0$.*

$T(b)$ type theorems are other criteria for the $L^2(\mu)$ boundedness of CZO's. To state one of these theorems in detail we need to introduce the notion of weak accretivity. We say that a function $b \in L^1_{\mathrm{loc}}(\mu)$ is weakly accretive if there exists some positive constant C such that

$$\left|\int_Q b\, d\mu\right| \geq C^{-1}\mu(Q) \text{ for all cubes } Q \subset \mathbb{R}^d.$$

Then we have:

Theorem 8 ($T(b)$ theorem)

Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. Let b_1, b_2 be two weakly accretive functions belonging to $L^\infty(\mu)$. The following conditions are equivalent:

(a) *T_μ is bounded on $L^2(\mu)$.*

(b) *$b_2 T_\mu b_1$ is weakly bounded and $T_{\mu,\varepsilon} b_1$, $T^*_{\mu,\varepsilon} b_2$ belong to $BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$, for some $\rho > 1$.*

(c) *$b_2 T_\mu b_1$ is weakly bounded and $T_{\mu,\varepsilon} b_1$, $T^*_{\mu,\varepsilon} b_2$ belong to $RBMO(\mu)$ uniformly on $\varepsilon > 0$.*

The condition that $b_2 T_\mu b_1$ is weakly bounded means that

$$\langle b_2 T_{\mu,\varepsilon}(\chi_Q b_1),\, \chi_Q \rangle \leq C\mu(Q)$$

uniformly on $\varepsilon > 0$, for all cubes $Q \subset \mathbb{R}^d$. Notice that if T_μ is antisymmetric and $b_1 = b_2 = b$, then $bT_\mu b$ is always weakly bounded.

The preceding theorem has been proved in [38], and it is a generalization of a classical theorem of David, Journé and Semmes to the case of non doubling measures (and so it requires new ideas and techniques). Other variants of this result (i.e. other $T(b)$ type theorems) can be found in [37] and [39].

For the particular case of the Cauchy transform, Theorem 8 yields the following result.

Theorem 9

Let μ be a Radon measure on $\mathbb{C}$ with linear growth. Suppose that there exists a function b such that:

(a) *$b \in L^\infty(\mu)$,*

(b) *b is weakly accretive,*

(c) *$\mathcal{C}_{\mu,\varepsilon} b \in BMO_\rho(\mu)$ uniformly in $\varepsilon > 0$, for some $\rho > 1$.*

Then $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$.

Many more results on Calderón-Zygmund theory with non doubling measures have been proved recently. For example, there are results concerning L^p and weak $(1,1)$ estimates [36]; Hardy spaces [47]; weights [12, 23, 40]; commutators [2, 17, 46]; fractional integrals [13, 10]; Lipschitz spaces [11]; etc. See also the survey paper [57].

4. Semiadditivity of γ_+ and its characterization in terms of curvature and L^2 boundedness of the Cauchy transform

Recall that the capacity γ_+ of a compact set $E \subset \mathbb{C}$ is

$$\gamma_+(E) = \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E,\ \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\}.$$

In order to relate the L^2 boundedness of the Cauchy transform with γ and γ_+, the following lemma is a key result.

Lemma 10

Let μ be a Radon measure with linear growth on $\mathbb{C}$. The following statements are equivalent:

(a) *The Cauchy transform is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$.*
(b) *For any set $A \subset \mathbb{C}$ there exists some function h supported on A, with $0 \leq h \leq 1$, such that $\int h\, d\mu \geq C^{-1}\mu(A)$ and $\|\mathcal{C}_\varepsilon(h\, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ for all $\varepsilon > 0$.*

The constant C in (b) *depends only on the norm of the Cauchy transform from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, and conversely.*

This lemma is a particular case of a result which applies to more general linear operators. The statement (b) should be understood as a weak substitute of the $L^\infty(\mu)$ boundedness of the Cauchy integral operator, which does not hold in general.

We will prove the easy implication of the lemma, that is, (b) $\Rightarrow$ (a). For the other implication, which is due to Davie and Øksendal [7] the reader is referred to [3, Chapter VII].

Proof of (b) $\Rightarrow$ (a). It is enough to show that for any complex measure $\nu \in M(\mathbb{C})$ and any $\lambda > 0$,

$$\mu\left\{x \in \mathbb{C} : \operatorname{Re}(\mathcal{C}_\varepsilon \nu(x)) > \lambda\right\} \leq \frac{C\|\nu\|}{\lambda}.$$

To this end, let us denote by A the set on the left side above, and let h be a function supported on A fulfilling the properties in the statement (b) of the lemma. Then we have

$$\begin{aligned}\mu(A) &\leq C\int h\, d\mu \leq \frac{C}{\lambda}\operatorname{Re}\left(\int (\mathcal{C}_\varepsilon \nu)\, h\, d\mu\right)\\ &= \frac{-C}{\lambda}\operatorname{Re}\left(\int \mathcal{C}_\varepsilon(h\, d\mu)\, d\nu\right) \leq \frac{C\|\nu\|}{\lambda}.\end{aligned}$$ □

Remark 11 Notice that if E supports a non zero Radon measure μ with linear growth such that the Cauchy integral operator $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$, then there exists some nonzero function h with $0 \leq h \leq \chi_E$ such that $\|\mathcal{C}_\varepsilon(h\, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ uniformly on ε, by Theorem 4 and the preceding lemma. Letting $\varepsilon \to 0$, we infer that $|\mathcal{C}(h\, d\mu)(z)| \leq C$ for all $z \notin E$, and so $\gamma(E) > 0$.

A more precise result will be proved in Theorem 12 below.

We denote by $\Sigma(E)$ the set of Radon measures supported on E such that $\mu(B(x,r)) \leq r$ for all $x \in \mathbb{C}$, $r > 0$.

The following theorem characterizes γ_+ in terms of curvature of measures and in terms of the L^2 norm of the Cauchy transform.

Theorem 12

For any compact set $E \subset \mathbb{C}$ we have

$$\begin{aligned}\gamma_+(E) &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \le 1\,\forall\varepsilon > 0\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \le \mu(E)\,\forall\varepsilon > 0\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \le \mu(E)\big\}\\ &\approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \le 1\big\}.\end{aligned}$$

In the statement above, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)}$ stands for the operator norm of $\mathcal{C}_\mu$ on $L^2(\mu)$. That is, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} = \sup_{\varepsilon>0} \|\mathcal{C}_{\mu,\varepsilon}\|_{L^2(\mu),L^2(\mu)}$.

Proof. We denote

$$\begin{aligned}S_1 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \le 1\,\forall\varepsilon > 0\big\},\\ S_2 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \le \mu(E)\,\forall\varepsilon > 0\big\},\\ S_3 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, c^2(\mu) \le \mu(E)\big\},\\ S_4 &:= \sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \le 1\big\}.\end{aligned}$$

We will show that $\gamma_+(E) \lesssim S_1 \lesssim S_2 \lesssim S_3 \lesssim S_4 \lesssim \gamma_+(E)$. The inequality $S_3 \lesssim S_4$ requires more work than the others. We will give two proofs of it. One uses the $T(1)$ theorem and the other not and so it is more elementary.

Proof of $\gamma_+(E) \lesssim S_1$. Let μ be supported on E such that $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \le 1$ with $\gamma_+(E) \le 2\mu(E)$. It is enough to show that μ has linear growth and $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \le C$ uniformly on $\varepsilon > 0$.

First we will prove the linear growth of μ. For any fixed $x \in \mathbb{C}$, by Fubini it turns out that for almost all $r > 0$,

$$\int_{|z-x|=r} \frac{1}{|z-x|}\, d\mu(z) < \infty.$$

For this r we have

$$\mu(B(x,r)) = -\int_{|z-x|=r} \mathcal{C}\mu(z)\, \frac{dz}{2\pi i} \le r.$$

Now the linear growth of μ follows easily.

To deal with the $L^\infty(\mu)$ norm of $\mathcal{C}_\varepsilon$ we use a standard technique: we replace $\mathcal{C}_\varepsilon$ by the regularized operator $\widetilde{\mathcal{C}}_\varepsilon$, defined as

$$\widetilde{\mathcal{C}}_\varepsilon\mu(x) = \int r_\varepsilon(y-x)\, d\mu(y),$$

where r_ε is the kernel

$$r_\varepsilon(z) = \begin{cases} \dfrac{1}{z} & \text{if } |z| > \varepsilon, \\ \dfrac{\overline{z}}{\varepsilon^2} & \text{if } |z| \le \varepsilon. \end{cases}$$

Then, $\widetilde{\mathcal{C}}_\varepsilon\mu$ is the convolution of the complex measure μ with the uniformly continuous kernel r_ε and so $\widetilde{\mathcal{C}}_\varepsilon\mu$ is a continuous function. Also, we have

$$r_\varepsilon(z) = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2},$$

where χ_ε is the characteristic function of $B(0,\varepsilon)$. Since μ is compactly supported, we have the following identity:

$$\widetilde{\mathcal{C}}_\varepsilon\mu = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \mu = \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \mathcal{C}\mu.$$

This equality must be understood in the sense of distributions, with $\mathcal{C}\mu$ being a function of $L^1_{\mathrm{loc}}(\mathbb{C})$ with respect to Lebesgue planar measure. As a consequence, if $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \le 1$, we infer that $\|\widetilde{\mathcal{C}}_\varepsilon\mu\|_{L^\infty(\mu)} \le 1$ for all $\varepsilon > 0$.

Since μ has linear growth, we have

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x) - \mathcal{C}_\varepsilon\mu(x)| = \frac{1}{\varepsilon^2}\left|\int_{|y-x|<\varepsilon} (\overline{y-x})\,d\mu(y)\right| \le C, \tag{10}$$

and so $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \le C$ uniformly on $\varepsilon > 0$.

Proof of $S_1 \lesssim S_2$. Trivial.

Proof of $S_2 \lesssim S_3$. This is a direct consequence of Proposition 2.

Proof of $S_3 \lesssim S_4$ using the $T(1)$ theorem. Let μ supported on E with linear growth such that $c^2(\mu) \le \mu(E)$ and $S_3 \le 2\mu(E)$. We set

$$A := \bigl\{x \in E : c^2_\mu(x) \le 2\bigr\}.$$

By Tchebychev $\mu(A) \ge \mu(E)/2$. Moreover, for any set $B \subset \mathbb{C}$,

$$\begin{aligned} c^2(\mu_{|B\cap A}) &\le \iiint_{x\in B\cap A} c(x,y,z)^2\,d\mu(x)d\mu(y)d\mu(z) \\ &= \int_{x\in B\cap A} c^2_\mu(x)\,d\mu(x) \le 2\mu(B). \end{aligned}$$

In particular, this estimate holds when B is any square in $\mathbb{C}$, and so $\mathcal{C}_{\mu_{|A}}$ is bounded on $L^2(\mu_{|A})$, by Theorem 7. Thus $S_4 \gtrsim \mu(A) \approx S_3$.

Proof of $S_3 \lesssim S_4$ without using the $T(1)$ theorem. Take μ supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_3 \leq 2\mu(E)$. To prove $S_3 \lesssim S_4$ we will show that there exists a measure ν supported on E with linear growth such that $\nu(E) \geq \mu(E)/4$ and $\|\mathcal{C}_\nu\|_{L^2(\nu),L^2(\nu)} \leq C$.

Given $C_2 > 0$, let

$$A_\varepsilon := \big\{x \in E : |\mathcal{C}_\varepsilon\mu(x)| \leq C_2 \text{ and } c_\mu^2(x) \leq C_2^2\big\}.$$

Since $\int c_\mu^2(x)\, d\mu(x) = c^2(\mu) \leq \mu(E)$ and, by Proposition 2, $\int |\mathcal{C}_\varepsilon\mu|^2\, d\mu \leq C\mu(E)$, we infer that $\mu(A_\varepsilon) \geq \mu(E)/2$ if C_2 is chosen big enough, by Tchebychev.

We want to show that the Cauchy integral operator $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{|A_\varepsilon})$. To this end we introduce an auxiliary "curvature operator": for $x, y \in A_\varepsilon$, consider the kernel $k(x,y) := \int c(x,y,z)^2\, d\mu(z)$, and let T be the operator

$$Tf(x) = \int k(x,y) f(y)\, d\mu(y).$$

By Schur's Lemma, T is bounded on $L^p(\mu_{|A_\varepsilon})$ for all $p \in [1,\infty]$, because for all $x \in A_\varepsilon$,

$$\begin{aligned}\int k(x,y)\, d\mu_{|A_\varepsilon}(y) &= \int k(y,x)\, d\mu_{|A_\varepsilon}(y)\\ &= \int_{y\in A_\varepsilon} c(x,y,z)^2\, d\mu(y) d\mu(z) \leq c_\mu^2(x) \leq C_2^2.\end{aligned}$$

Given a non negative (real) function f supported on A_ε, by arguments similar to the ones in the proof of Proposition 2, we have

$$\begin{aligned}4\int |\mathcal{C}_\varepsilon(f\, d\mu)|^2\, d\mu \;=&\; \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon\\ |y-z|>\varepsilon}} c(x,y,z)^2 f(x) f(y)\, d\mu(x) d\mu(y) d\mu(z)\\ &- 2\mathrm{Re}\int (\mathcal{C}_\varepsilon\mu)\, \overline{\mathcal{C}_\varepsilon(f\, d\mu)}\, f\, d\mu + O(\|f\|_{L^2(\mu)}^2).\end{aligned}$$

See [56, Lemma 1] for the details, for example. Thus,

$$\begin{aligned}\int |\mathcal{C}_\varepsilon(f\, d\mu)|^2\, d\mu \leq \frac{1}{4}\big|\langle Tf,\, f\rangle\big| + \frac{1}{2}\int \big|(\mathcal{C}_\varepsilon\mu)\, \mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu&\\ + C\|f\|_{L^2(\mu)}^2.&\qquad (11)\end{aligned}$$

To estimate the first term on the right side we use the $L^2(\mu_{|A_\varepsilon})$ boundedness of T (recall that $\mathrm{supp}(f) \subset A_\varepsilon$):

$$\big|\langle Tf,\, f\rangle\big| \leq \|Tf\|_{L^2(\mu)}\, \|f\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}^2.$$

To deal with the second integral on the right side of (11), notice that $|\mathcal{C}_\varepsilon\mu| \leq C_2$ on the support of f, and so

$$\int \big|(\mathcal{C}_\varepsilon\mu)\, \mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu \leq C_2 \int \big|\mathcal{C}_\varepsilon(f\, d\mu)\, f\big|\, d\mu \\ \leq C_2 \|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \|f\|_{L^2(\mu)}.$$

By (11) we get

$$\|\mathcal{C}_\varepsilon(f\, d\mu)\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)} + \frac{C_2}{2}\, \|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \|f\|_{L^2(\mu)},$$

which implies that $\|\mathcal{C}_\varepsilon(f\, d\mu)\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}$.

So far we have proved the $L^2(\mu_{|A_\varepsilon})$ boundedness of $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$. If A_ε were independent of ε, we would set $\nu := \mu_{|A_\varepsilon}$ and we would be done. Unfortunately this is not the case and we have to work a little more. We set

$$B_\varepsilon := \big\{x \in E : |\mathcal{C}_{\varepsilon,*}\mu(x)| \leq C_3 \text{ and } c^2_\mu(x) \leq C_3^2\big\},$$

where C_3 is some constant big enough (with $C_3 > C_2$) to be chosen below. By Theorem 5 and the discussion above, we know that $\mathcal{C}_{\varepsilon,*}$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu_{|A_\varepsilon})$ (with constants independent of ε). Thus,

$$\mu\big\{x \in A_\varepsilon : |\mathcal{C}_{\varepsilon,*}\mu(x)| > C_3\big\} \leq \frac{C\mu(E)}{C_3}.$$

If C_3 is big enough, the right hand side of the preceding inequality is $\leq \mu(E)/4 \leq \mu(A_\varepsilon)/2$. Thus, $\mu(B_\varepsilon) \geq \mu(E)/4$.

We set

$$B := \bigcap_{\varepsilon>0} B_\varepsilon.$$

Notice that, by definition, $B_\varepsilon \subset B_\delta$ if $\varepsilon > \delta$ and so we have

$$\mu(B) = \lim_{\varepsilon\to 0} \mu(B_\varepsilon) \geq \frac{1}{4}\, \mu(E).$$

By the same argument used for A_ε, it follows that $\mathcal{C}_{\mu_{|B_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{B_\varepsilon})$ (with constant independent of ε), and thus $\mathcal{C}_{\mu_{|B}}$ is bounded on $L^2(\mu_{|B})$. If we take $\nu := \mu_{|B}$, we are done.

Proof of $S_4 \lesssim \gamma_+(E)$. This is a direct consequence of Lemma 10 and the fact that the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ implies its boundedness from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, as shown by Theorem 4. □

From Theorem 12, since the term

$$\sup\big\{\mu(E) : \mu \in \Sigma(E),\, \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\big\}$$

is countably semiadditive, we infer that γ_+ is also countably semiadditive.

Corollary 13

The capacity γ_+ *is countably semiadditive. That is, if* E_i, $i = 1, 2, \ldots$, *is a countable (or finite) family of compact sets, we have*

$$\gamma_+\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma_+(E_i).$$

Another consequence of Theorem 12 is that the capacity γ_+ can be characterized in terms of the following potential, introduced by Verdera [56]:

$$U_\mu(x) = M_R\mu(x) + c_\mu(x), \tag{12}$$

where $M_R\mu(x) := \sup_{r>0} \frac{\mu(B(x,r))}{r}$ and $c_\mu(x)$ is the pointwise version of curvature defined in (4). The precise result is the following.

Corollary 14

For any compact set $E \subset \mathbb{C}$ *we have*

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\, U_\mu(x) \leq 1\, \forall x \in \mathbb{C}\big\}.$$

The proof of this corollary follows easily from the comparability between γ_+ and the third sup in Theorem 12, and the fact that U_μ satisfies a kind of maximum principle: if $U_\mu(x) \leq C$ for $x \in \operatorname{supp}(\mu)$, then $U_\mu(x) \leq C'$ for any $x \in \mathbb{C}$ (see [44] for the proof).

Let us remark that the preceding characterization of γ_+ in terms of U_μ is interesting because it suggests that some techniques of potential theory can be useful to study γ_+. See [48] and [56] for more detailed information.

5. The comparability between γ and γ_+

In [49] the following result has been proved.

Theorem 15

There exists an absolute constant C *such that for any compact set* $E \subset \mathbb{C}$ *we have*

$$\gamma(E) \leq C\gamma_+(E).$$

As a consequence, $\gamma(E) \approx \gamma_+(E)$.

Let us remark that the comparability between γ and γ_+ had been previously proved by P. Jones for compact connected sets by geometric arguments (see [41, Chapter 3]), very different from the ones in [49]. Also, in [27] it had already been shown that $\gamma \approx \gamma_+$ holds for a big class of Cantor sets. In particular, for the corner quarters Cantor set E (see Fig. 1) it was proved in [27] that $\gamma(E_n) \approx \gamma_+(E_n)$. Recall that E_n is the n-th generation appearing in the construction of E. By results due to Mattila [29] and Eiderman [8] (see also [48]) it was already known that $\gamma_+(E_n) \approx 1/n^{1/2}$. Thus, one has $\gamma(E_n) \approx 1/n^{1/2}$.

An obvious corollary of Theorem 15 and the characterization of γ_+ in terms of curvature obtained in Theorem 12 is the following.

Corollary 16

Let $E \subset \mathbb{C}$ be compact. Then, $\gamma(E) > 0$ if and only if E supports a non zero Radon measure with linear growth and finite curvature.

Since we know that γ_+ is countably semiadditive, the same happens with γ:

Corollary 17

Analytic capacity is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have

$$\gamma\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq C \sum_{i=1}^{\infty} \gamma(E_i).$$

5.1 Some ideas about the proof of Theorem 15

Notice that, by Theorem 12, to prove Theorem 15 it is enough to show that there exists some measure μ supported on E with linear growth, satisfying $\mu(E) \approx \gamma(E)$, and such that the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ with absolute constants. To implement this argument, the main tool used in [49] is the $T(b)$ theorem of Nazarov, Treil and Volberg in [37], which is similar in spirit to the $T(b)$ type theorem stated in Theorem 9 but more appropriate for the present situation. To apply this $T(b)$ theorem, one has to construct a suitable measure μ and a function $b \in L^\infty(\mu)$ fulfilling some precise conditions, analogous to the conditions (a), (b) and (c) in Theorem 9.

Because of the definition of analytic capacity, there exists some function $f(z)$ which is analytic and bounded in $\mathbb{C} \setminus E$ with $f'(\infty) = \gamma(E)$ (this is the so called Ahlfors function). By a standard approximation argument, it is not difficult to see that one can assume that E is a finite union of disjoint segments, so that in particular $\mathcal{H}^1(E) < \infty$. Then one has to construct μ and b and to prove the comparability $\gamma(E) \approx \gamma_+(E)$ with estimates independent of $\mathcal{H}^1(E)$. Since E is a finite union of disjoint segments, there exists some complex measure ν_0 (obtained from the boundary values of $f(z)$) supported on E such that $f = \mathcal{C}\nu_0$. This measure satisfies the following properties:

$$\|\mathcal{C}\nu_0\|_\infty \le 1, \tag{13}$$

$$|\nu_0(E)| = \gamma(E), \tag{14}$$

$$d\nu_0 = b_0\, d\mathcal{H}^1|E, \quad \text{where } b_0 \text{ satisfies } \|b_0\|_\infty \le 1. \tag{15}$$

Given this information, by a more or less direct application of a $T(b)$ type theorem we cannot expect to prove that the Cauchy transform is bounded with respect to a measure μ such us the one described above with absolute constants. Let us explain the reason in some detail. Suppose for example that there exists some function b such that $d\nu_0 = b\, d\mu$ and we use the information about ν_0 given by (13), (14) and (15) (notice the difference between b and b_0). From (13) and (14) we derive

$$\|\mathcal{C}(b\, d\mu)\|_\infty \le 1 \tag{16}$$

and

$$\left|\int b\, d\mu\right| \approx \mu(E). \tag{17}$$

The estimate (16) is very good for our purposes. In fact, most classical $T(b)$ type theorems (like Theorem 9) require only the $BMO_\rho(\mu)$ norm of b to be bounded, which is a weaker assumption. The estimate (17) is likewise good; it is a global accretivity condition, and with some technical difficulties (which may involve some kind of stopping time argument, like in [4] or [37]), one can hope to be able to prove that the accretivity condition

$$\left|\int_Q b\, d\mu\right| \approx \mu(Q \cap E)$$

holds for many squares Q.

Our problems arise from (15). Notice that this implies that

$$|\nu_0|(E) \le \mathcal{H}^1(E), \tag{18}$$

where $|\nu_0|$ stands for the variation of ν_0. This is a very bad estimate since we don't have any control on $\mathcal{H}^1(E)$ (we only know $\mathcal{H}^1(E) < \infty$ because our assumption on E). However, as far as we know, all $T(b)$ type theorems require the estimate $\|b\|_{L^\infty(\mu)} \le C$, or variants of it, which in particular imply that

$$|\nu_0|(E) \le C\mu(E) \approx \gamma(E). \tag{19}$$

That is to say, the estimate that we get from (15) is (18), but the one we need is (19). So by a direct application of a $T(b)$ type theorem we will obtain bad results when $\gamma(E) \ll \mathcal{H}^1(E)$.

To prove Theorem 15, we need to work with a measure "better behaved" than ν_0, which we call ν. This new measure will be a suitable modification of ν_0 with the required estimate for its total variation. To construct ν, in [49] we consider a set F containing E made up of a finite disjoint union of squares: $F = \bigcup_{i\in I} Q_i$. One should think that the squares Q_i approximate E at some "intermediate scale". For each square Q_i, we take a complex measure ν_i supported on Q_i such that $\nu_i(Q_i) = \nu_0(Q_i)$ and $|\nu_i|(Q_i) = |\nu_i(Q_i)|$ (that is, ν_i is a constant multiple of a positive measure). We set $\nu = \sum_i \nu_i$. So ν is some kind of approximation of ν_0, and if the squares Q_i are big enough, the variation $|\nu|$ becomes sufficiently small (because there are "cancellations" in the measure ν_0 in each Q_i). On the other hand, the squares Q_i cannot be too big, because we need

$$\gamma_+(F) \leq C\gamma_+(E). \tag{20}$$

In this way, we will have constructed a complex measure ν supported on F satisfying

$$|\nu|(F) \approx |\nu(F)| = \gamma(E). \tag{21}$$

Taking a suitable measure μ such that $\mathrm{supp}(\mu) \supset \mathrm{supp}(\nu)$ and $\mu(F) \approx \gamma(E)$, we will be ready for the application of an appropriate $T(b)$ type theorem, such as the one in [37], which is a very powerful tool. Indeed, notice that (21) implies that ν satisfies a global accretivity condition and that also the variation $|\nu|$ is controlled. On the other hand, if we have been careful enough, we will have also some useful estimates on $|\mathcal{C}\nu|$, since ν is an approximation of ν_0. Then, using the $T(b)$ theorem in [37], we will deduce $\gamma_+(F) \geq C^{-1}\mu(E)$, and so $\gamma_+(E) \geq C^{-1}\gamma(E)$, by (20), and we will be done. Nevertheless, in order to obtain the right estimates on the measures ν and μ it will be necessary to use an induction argument involving the sizes of the squares Q_i, which will allow to assume that $\gamma(E \cap Q_i) \approx \gamma_+(E \cap Q_i)$ for each square Q_i.

6. The proof of Theorem 15

By a standard discretization argument we may (and will) assume that $\mathcal{H}^1(E) < \infty$, and even that E is a finite union of disjoint compact segments. Then we have to prove that $\gamma(E) \leq C\gamma_+(E)$, with C being an absolute constant (independent of $\mathcal{H}^1(E)$, in particular). Before explaining the construction of the intermediate set F mentioned in Subsection 5.1, we first need to explain some properties of the potential $U_\mu := M_R\mu + c_\mu$ introduced in (12). Recall the characterization of γ_+ in terms of U_μ in Corollary 14.

6.1 Potential theory for γ_+

Theorem 18

There exists an absolute constant C_4 such that for any positive Radon measure σ on $\mathbb{C}$ and all $\lambda > 0$, we have

$$\gamma_+\{x \in \mathbb{C} : U_\sigma(x) > \lambda\} \leq C_4 \frac{\sigma(\mathbb{C})}{\lambda}.$$

The proof of this lemma follows by Fubini and an appropriate maximum principle for U_σ. See [48] for the details.

By variational arguments, we also obtain:

Theorem 19

Let $A \subset \mathbb{C}$ be compact. There exists a Radon measure σ on $\mathbb{C}$ such that $\sigma(\mathbb{C}) \approx \gamma_+(A)$ and $U_\sigma(x) \gtrsim 1$ for all $x \in A$.

See [48] for the proof again.

Notice that from the preceding two theorems we have the following characterization of γ_+ in terms of U_σ, dual to the one in Corollary 14:

Corollary 20

For $A \subset \mathbb{C}$ compact we have

$$\gamma_+(A) \approx \inf\{\sigma(\mathbb{C}) : \mu \in M_+(\mathbb{C}), U_\sigma(x) \geq 1\, \forall x \in E\},$$

where $M_+(\mathbb{C})$ stands for the class of all positive Radon measures on $\mathbb{C}$.

6.2 Construction of the intermediate set F

Next lemma, which deals with the construction of F, is one of the key points of the proof of Theorem 15. We will show an almost complete proof. For the whole arguments see [49, Lemma 5.1].

Lemma 21

Suppose that $\gamma_+(E) \leq C_5 \mathrm{diam}(E)$, where C_5 is small enough. There exists a compact set $F = \bigcup_{i \in I} Q_i$, where $\{Q_i\}_{i \in I}$ is a finite family of squares such that $\sum_i \chi_{10Q_i} \leq C$ and

(a) *$E \subset F$ and $\gamma_+(F) \approx \gamma_+(E)$,*

(b) *$\sum_i \gamma_+(E \cap 2Q_i) \leq C\gamma_+(E)$,*

(c) *$\mathrm{diam}(Q_i) \leq \frac{1}{10} \mathrm{diam}(E)$ for all i.*

EXAMPLE: Consider the 1/4 planar Cantor set in Fig. 1. Let $E_n =$ be the n-th generation of this set, $n \geq 4$. Recall that $\gamma_+(E_n) \approx 1/\sqrt{n}$. A good choice for F in this case would be $E_{n/2}$ (assuming n even). The squares Q_i, $i \in I$, would be the $4^{n/2}$ squares of generation $n/2$. The reader can check easily that (a), (b), and (c) are satisfied (hint: use that $\gamma_+(E_n) \approx \gamma_+(E_{n/2})$ and $\gamma_+(Q_i \cap E_n) = \ell(Q_i)\gamma_+(E_{n/2})$ by "self similarity").

Proof of Lemma 21. Let σ be a maximal measure for $\gamma_+(E)$ described in Theorem 19. Then, $\sigma(E) \approx \gamma_+(E)$ and $U_\sigma(x) \gtrsim 1$ for all $x \in E$. Consider the open bounded set $\Omega = \{x \in \mathbb{C} : U_\sigma(x) > \lambda_0\}$, where $0 < \lambda_0 \ll 1$. Let Q_i, $i \in I$, be the Whitney squares of Ω such that $2Q_i \cap E \neq \varnothing$. We assume also that Q_i have been constructed so that $\sum_i \chi_{10Q_i} \leq C$. We set $F := \bigcup_{i\in I} Q_i$. By compactness, it is easy to check that I is finite.

The statement (a) follows easily: we have $E \subset F$ by construction, and also

$$\gamma_+(F) \leq \gamma_+(\Omega) \leq \frac{C\sigma(E)}{\lambda_0} \leq C\gamma_+(E),$$

by Theorem 18, since $U_\sigma \gtrsim \lambda_0$ on Ω.

To prove (b), notice that for any square Q_i, $i \in I$, there exists some point $x' \in \partial\Omega$ such that $U_\sigma(x') = M_R\sigma(x') + c_\sigma(x') \leq \lambda_0 \ll 1$, with $\operatorname{dist}(x', 2Q_i) \approx \ell(Q_i)$. Since for all $x \in E \cap 2Q_i$ we have $U_\sigma(x) \gtrsim 1$, it can be shown that, roughly speaking, most of the potential U_σ at x is originated by $\sigma_{|4Q_i}$, and then

$$U_{\sigma|4Q_i}(x) \gtrsim \frac{1}{2} \qquad \text{for all } x \in E \cap 2Q_i$$

(at this point the reader should think about the typical arguments involved in good λ inequalities for CZO's). As a consequence,

$$\gamma_+(E \cap 2Q_i) \leq \gamma_+\Big\{x : U_{\sigma|4Q_i}(x) \gtrsim \frac{1}{2}\Big\} \lesssim \sigma(4Q_i),$$

and so

$$\sum_i \gamma_+(E \cap 2Q_i) \lesssim \sum_i \sigma(4Q_i) \lesssim \sigma(E) \lesssim \gamma_+(E).$$

The proof of (c) follows easily from the fact that for $x \in \mathbb{C}$ one has $U_\sigma(x) \lesssim \sigma(\mathbb{C})/\operatorname{dist}(x, E)$ and the assumption that $\gamma_+(E) \leq C_5 \operatorname{diam}(E)$, with C_5 small enough. The details are left for the reader. □

6.3 Construction of μ and ν

The following lemma is another key step for the proof of Theorem 15. It deals with the construction of μ and ν, which will be good measures for the application of a suitable $T(b)$ type theorem.

Lemma 22

Let $F = \bigcup_i Q_i$ be intermediate set constructed in Lemma 21. Let $A \geq 1$ be fixed. If $\gamma(E \cap 2Q_i) \leq A\,\gamma_+(E \cap 2Q_i)$ for all i, then:

(1) *either $\gamma(E) \leq A\,\gamma_+(E)$,*

(2) *or $\gamma(E) > A\,\gamma_+(E)$, and there exist measures μ, ν supported on F ($\mu \geq 0$ and ν complex) and a subset $H \subset F$, such that, such that*

   (a) *$\gamma(E) \approx \mu(F)$,*

   (b) *$d\nu = b\,d\mu$, with $\|b\|_{L^\infty(\mu)} \leq C$,*

   (c) *$|\nu(F)| = \gamma(E)$,*

   (d) *$|\nu(Q)| \lesssim \ell(Q)$ for any square Q.*

   (e) *$\displaystyle\int_{F\setminus H} \mathcal{C}_*\nu\,d\mu \leq C\mu(F)$.*

   (f) *If $\mu(B(x,r)) > K_0 r$ (for some big constant K_0), then $B(x,r) \subset H$. In particular, $\mu(B(x,r)) \leq K_0 r$ for all $x \in F \setminus H$, $r > 0$.*

   (g) *H is of the form $H = \bigcup_{k\in I_H} B(x_k, r_k)$, with $\sum_{k\in I_H} r_k \lesssim K_0^{-1}\mu(F)$.*

The constants do **not** *depend on A.*

Let us insist on the fact that all the constants different from A which appear in the lemma do not depend on A. This fact will be essential for the proof of Theorem 15.

For the reader's convenience, before going on we will make some comments on the lemma. Assume we are in Case (2). The properties (a), (b), (c) and (e) are basic properties which must satisfy μ and ν in order to apply a $T(b)$ type theorem with absolute constants. In (e) notice that instead of the $L^\infty(\mu)$ or $BMO(\mu)$ norm of $\mathcal{C}\nu$, we estimate the $L^1(\mu)$ norm of $\mathcal{C}_*\nu$ out of H.

Roughly speaking, the *exceptional set* H contains the part of μ without linear growth. The properties (f) and (g) describe H. Observe that (g) means that H is rather small if we choose K_0 big enough. For example, (g) combined with (d), (a) and (b), implies that

$$|\nu(H)| \lesssim \sum_{k\in I_H} \ell(R_k) \lesssim K_0^{-1}\mu(F) \lesssim K_0^{-1}|\nu(F)| \ll |\nu(F)|$$

(assuming K_0 big enough).

Below we will show the construction of μ and ν and the detailed proof of (a), (b) and (c) in Case (2). We think that these are the most interesting points of Lemma 22. The proof of (d)$-$(g) is more technical, and it uses more standard estimates (perhaps with the exception of (e)).

We will give only some hints about the arguments in this case. For the details, see [49, Lemma 5.1].

Construction of μ and ν and proof of (a), (b), (c) in Case (2) of Lemma 22. We assume that $\gamma(E) > A\,\gamma_+(E)$.

It is easily seen that there exists a family of $\mathcal{C}^\infty$ functions $\{g_i\}_{i\in J}$ such that, for each $i \in J$, $\mathrm{supp}(g_i) \subset 2Q_i$, $0 \leq g_i \leq 1$, and $\|\nabla g_i\|_\infty \leq C/\ell(Q_i)$, so that $\sum_{i\in J} g_i = 1$ on Ω. Notice that by the definition of I we also have $\sum_{i\in I} g_i = 1$ on E.

Let $f(z)$ be the Ahlfors function of E (i.e. f which maximizes the supremum in the definition of $\gamma(E)$ in (1)), and consider the complex measure ν_0 such that $f(z) = \mathcal{C}\nu_0(z)$ for $z \notin E$, with $|\nu_0(B(z,r))| \leq r$ for all $z \in \mathbb{C}$, $r > 0$ (see [29, Theorem 19.9], for example). So we have

$$|\mathcal{C}\nu_0(z)| \leq 1 \quad \text{for all } z \notin E,$$

and

$$\nu_0(E) = \gamma(E).$$

The measure ν will be a suitable modification of ν_0. As we explained in Subsection 5.1, the main drawback of ν_0 is that the only information that we have about its variation $|\nu_0|$ is that $|\nu_0| = b_0\, d\mathcal{H}^1_E$, with $\|b_0\|_\infty \leq 1$. This is a very bad estimate if we try to apply some kind of $T(b)$ theorem in order to show that the Cauchy transform is bounded (with absolute constants). The main advantage of ν over ν_0 is that we will have a much better estimate for the variation $|\nu|$.

First we define the measure μ. For each $i \in I$, let Γ_i be a circumference concentric with Q_i and radius $\gamma(E \cap 2Q_i)/10$. Observe that $\Gamma_i \subset \frac{1}{2}Q_i$ for each i. We set

$$\mu = \sum_{i\in I} \mathcal{H}^1|\Gamma_i.$$

Let us define ν now:

$$\nu = \sum_{i\in I} \frac{1}{\mathcal{H}^1(\Gamma_i)} \int g_i\, d\nu_0 \cdot \mathcal{H}^1|\Gamma_i.$$

Notice that $\mathrm{supp}(\nu) \subset \mathrm{supp}(\mu) \subset F$. Moreover, we have $\nu(Q_i) = \int g_i\, d\nu_0$, and since $\sum_{i\in I} g_i = 1$ on E, we also have

$$\nu(F) = \sum_{i\in I} \nu(Q_i) = \nu_0(E) = \gamma(E)$$

(which yields (c)).

We have $d\nu = b\, d\mu$, with $b = \frac{\int g_i\, d\nu_0}{\mathcal{H}^1(\Gamma_i)}$ on Γ_i. To estimate $\|b\|_{L^\infty(\mu)}$, notice that

$$|\mathcal{C}(g_i\nu_0)(z)| \leq C \quad \text{for all } z \notin E \cap 2Q_i. \tag{22}$$

This follows easily from the formula

$$\mathcal{C}(g_i\nu_0)(\xi) = \mathcal{C}\nu_0(\xi)\, g_i(\xi) + \frac{1}{\pi}\int \frac{\mathcal{C}\nu_0(z)}{z-\xi}\,\bar\partial g_i(z)\, d\mathcal{L}^2(z), \tag{23}$$

where $\mathcal{L}^2$ stands for the planar Lebesgue measure on $\mathbb{C}$. Let us remark that this identity is used often to split singularities in Vitushkin's way. Inequality (22) implies that

$$\left|\int g_i\, d\nu_0\right| = \left|(\mathcal{C}(g_i\,\nu_0))'(\infty)\right| \leq C\gamma(E\cap 2Q_i) = C\mathcal{H}^1(\Gamma_i). \tag{24}$$

As a consequence, $\|b\|_{L^\infty(\mu)} \leq C$, and (b) is proved.

It remains to check that (a) also holds. Using (24), the assumption $\gamma(E\cap 2Q_i) \leq A\gamma_+(E\cap 2Q_i)$, (b), and the hypothesis $A\gamma_+(E) \leq \gamma(E)$, we obtain the following inequalities:

$$\begin{aligned}
\gamma(E) = |\nu_0(E)| = \left|\sum_{i\in I}\int g_i\, d\nu_0\right| &\leq \sum_{i\in I}\left|\int g_i\, d\nu_0\right| \\
&\leq C\sum_{i\in I}\gamma(E\cap 2Q_i) = C\mu(F) \\
&\leq C\,A\sum_{i\in I}\gamma_+(E\cap 2Q_i) \\
&\leq C\,A\gamma_+(E) \leq C\gamma(E),
\end{aligned}$$

which gives (a) (with constants independent of A). □

Sketch of the proof of (d)–(g) in Case (2) of Lemma 22

(d) Since $\|\mathcal{C}\nu_0\|_{L^\infty(\mathbb{C})} \leq 1$, by Fubini it follows easily that $|\nu_0(Q)| \lesssim \ell(Q)$ for any square Q. Since ν is a kind of approximation of ν_0, then it can also be shown (rather easily) that $|\nu(Q)| \lesssim \ell(Q)$ also holds, by comparison with ν_0. See [49, Lemma 5.1] for the details.

(e) The estimate $\int_{F\setminus H}\mathcal{C}_*\nu\, d\mu \lesssim \mu(F)$ is more delicate than the one in (d). However, it is also a direct consequence of the fact that $\|\mathcal{C}\nu_0\|_{L^\infty(\mathbb{C})} \leq 1$, and the fact that ν is an approximation of ν_0. See [49, Lemma 5.1] for the details again.

(f) Given $x \in F$, $r > 0$, we say that $B(x,r)$ is a *non Ahlfors disk* if $\mu(B(x,r)) > K_0 r$. For a fixed $x\in F$, if there exists some $r>0$ such that $B(x,r)$ is a non Ahlfors disk, then we say that x is a *non Ahlfors point.* For any $x\in F$, we denote

$$\mathcal{R}(x) = \sup\{r>0 :\, B(x,r) \text{ is a non Ahlfors disk}\}.$$

If $x\in F$ is an Ahlfors point, we set $\mathcal{R}(x) = 0$.

Observe (a) implies that $\mu(F) \lesssim \gamma(E) \leq \gamma(F) \leq \text{diam}(F)$. Therefore, for $r \geq \text{diam}(F)/100$

$$\mu(B(x,r)) \leq \mu(F) \leq C\text{diam}(F) \leq K_0 r,$$

assuming K_0 big enough. Thus $\mathcal{R}(x) \leq \text{diam}(F)/100$ for all $x \in F$.

We denote

$$H_0 = \bigcup_{x\in F,\, \mathcal{R}(x)>0} B(x, \mathcal{R}(x)).$$

By Vitali's $5r$-Covering Theorem there is a disjoint family $\{B(x_k, \mathcal{R}(x_k))\}_k$, $k \in I_H$, such that $H_0 \subset \bigcup_{k\in I_H} B(x_k, 5\mathcal{R}(x_k))$. We set

$$H := \bigcup_{k\in I_H} B(x_k, 5\mathcal{R}(x_k)).$$

Observe that by construction $H_0 \subset H$, and so all non Ahlfors disks are contained in H.

(g) Since the balls $B(x_k, \mathcal{R}(x_k))$ are disjoint, we have

$$\sum_{k\in I_H} \mathcal{R}(x_k) \lesssim K_0^{-1} \sum_{k\in I_H} \mu(B(x_k, \mathcal{R}(x_k)) \leq K_0^{-1}\mu(F). \qquad \square$$

6.4 Application of the $T(b)$ theorem of Nazarov, Treil and Volberg

The next step of the proof of Theorem 15 consists of proving the following result. We use the same notation as above (i.e. F, μ, and ν are as in Lemma 22).

Lemma 23

Assume $\gamma_+(E) \leq C_5\text{diam}(E)$, $\gamma(E) \geq A\gamma_+(E)$, and $\gamma(E \cap 2Q_i) \leq A\gamma_+(E \cap 2Q_i)$ for all $i \in I$. Then there exists some subset $G \subset F$, with $\mu(F) \leq C_5\mu(G)$, such that $\mu(G \cap B(x,r)) \leq C_6 r$ for all $x \in G$, $r > 0$, and the Cauchy transform is bounded on $L^2(\mu_{|G})$ with $\|\mathcal{C}\|_{L^2(\mu|G),L^2(\mu|G)} \leq C_7$. The constants C_6, C_5, C_5, C_7 are absolute constants, and do not depend on A.

This lemma follows from a suitable version of a $T(b)$ type theorem of Nazarov, Treil and Volberg [37] to the measure μ and the function b constructed in Lemma 22. To state this $T(b)$ theorem we need to introduce the dyadic lattices $\mathcal{D}(w)$, $w \in \Omega$, called *random dyadic lattices* by Nazarov, Treil and Volberg. Suppose that $F \subset B(0, 2^{N-3})$, where N is a big enough integer. If we denote by $\mathcal{D}$ the collection of all the usual dyadic squares in $\mathcal{D}$, given $w \in \left[-2^{N-1}, 2^{N-1}\right)^2 =: \Omega$ we set $\mathcal{D}(w) := w + \mathcal{D}$. That is to say, a square from $\mathcal{D}(w)$ is the translation by w of a dyadic square from $\mathcal{D}$.

We are ready to state the $T(b)$ theorem of [37] (see also [60]).

Theorem 24 ($T(b)$ theorem)

Let μ be a measure supported on $F \subset \mathbb{C}$. Suppose that there exist a complex measure ν and, for each $w \in \Omega$, two exceptional sets $H_{\mathcal{D}}(w)$ and $T_{\mathcal{D}}(w)$ made up of dyadic squares from $\mathcal{D}(w)$ such that:

(a) *Every ball B_r of radius r such that $\mu(B_r) > C_6 r$ is contained in $H_{\mathcal{D}}(w)$, for all $w \in \Omega$.*
(b) *$d\nu = b\, d\mu$, with $\|b\|_{L^\infty(\mu)} \leq C$.*
(c) *$\int_{\mathbb{C}\setminus H_{\mathcal{D}}(w)} \mathcal{C}_*\nu\, d\mu \leq C\, \mu(F)$, for all $w \in \Omega$.*
(d) *If $Q \in \mathcal{D}(w)$ is such that $Q \not\subset T_{\mathcal{D}(w)}$, then $\mu(Q) \leq K_1|\nu(Q)|$ (i.e. Q is an accretive square).*
(e) *$\mu\big(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)\big) \leq \delta_0\, \mu(F)$, for some $\delta_0 < 1$ and all $w \in \Omega$.*

Then there exists a subset $G \subset F \setminus \bigcap_{w\in\Omega}\big(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)\big)$ such that the Cauchy transform is bounded on $L^2(\mu|G)$, with the L^2 norm bounded above by some constant depending on the constants above.

Proof of Lemma 23 As mentioned above, we intend to apply Theorem 24. Let us define the sets $H_{\mathcal{D}}(w)$ and $T_{\mathcal{D}}(w)$. Recall that $H = \bigcup_{k\in I_H} B(x_k, r_k)$, with $\sum_{k\in I_H} r_k \lesssim K_0^{-1}\mu(F)$. Each ball $B(x_k, r_k)$ is covered by at most 4 dyadic squares from $\mathcal{D}(w)$ with side length $\leq 4r_k$. Let $\mathcal{D}_H(w)$ denote the family of these squares. We set $H_{\mathcal{D}}(w) := \bigcup_{R\in\mathcal{D}_H(w)} R$. Observe that by construction, $H \subset H_{\mathcal{D}}(w)$, and it is also clear that

$$\sum_{R\in\mathcal{D}_H(w)} \ell(R) \lesssim K_0^{-1}\mu(F).$$

Now we deal with $T_{\mathcal{D}}(w)$. We say that a square $R \subset \mathbb{C}$ is accretive if

$$\mu(R) \leq K_1|\nu(R)|, \tag{25}$$

where K_1 is some big constant to be chosen below. We denote by $\mathcal{D}_T(w)$ the collection of **non** accretive squares from $\mathcal{D}(w)$.

It is clear that the assumptions (a), (b), (c) and (d) of Theorem 24 hold. It only remains to check that (e) is satisfied too. Let $\mathcal{D}_{HT}(w)$ be the subfamily of maximal (and thus disjoint) squares from $\mathcal{D}_H(w) \cup$

$\mathcal{D}_T(w)$. Then, using (d) from Lemma 22 and the fact that $\mu(F) \approx |\nu(F)|$, we get

$$\begin{aligned}|\nu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))| &\leq \sum_{R \in \mathcal{D}_{HT}(w)} |\nu(R)| \\ &= \sum_{R \in \mathcal{D}_H(w) \cap \mathcal{D}_{HT}(w)} |\nu(R)| + \sum_{R \in \mathcal{D}_T(w) \cap \mathcal{D}_{HT}(w)} |\nu(R)| \\ &\leq C \sum_{R \in \mathcal{D}_H(w)} \ell(R) + K_1^{-1} \sum_{R \in \mathcal{D}_{HT}(w)} \mu(R) \\ &\leq CK_0^{-1}\mu(F) + K_1^{-1}\mu(F) \lesssim (K_0^{-1} + K_1^{-1})|\nu(F)|.\end{aligned}$$

Thus, if we choose K_0 and K_1 big enough we have

$$|\nu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))| \geq \frac{1}{2}|\nu(F)|.$$

Therefore, taking into account that $\|b\|_{L^\infty(\mu)} \leq C$,

$$\begin{aligned}\mu(F) &\leq C|\nu(F)| \leq 2C|\nu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))| \\ &\leq C_8\mu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)).\end{aligned}$$

Thus $\mu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)) \leq \delta_0\mu(F)$, with $\delta_0 = 1 - C_8^{-1}$.

So from Theorem 24 we infer that there exists some set $G \subset F$ like the one required in Lemma 23. □

6.5 Proof of Theorem 15 by induction on the size of rectangles

If we put together Lemmas 21, 22, and 23, we get:

Lemma 25

There exists some absolute constant B such that if $A \geq 1$ is any fixed constant and

(a) $\gamma_+(E) \leq C_5 \operatorname{diam}(E)$,

(b) $\gamma(E \cap Q) \leq A\gamma_+(E \cap Q)$ *for all squares Q with* $\operatorname{diam}(Q) \leq \operatorname{diam}(E)/5$,

(c) $\gamma(E) \geq A\gamma_+(E)$,

then $\gamma(E) \leq B\gamma_+(E)$.

Proof. By Lemmas 21, 22, and 23, there are sets F, G and a measure μ supported on F such that

(i) $E \subset F$ and $\gamma_+(E) \approx \gamma_+(F)$,

(ii) $\mu(F) \approx \gamma(E)$,

(iii) $G \subset F$ and $\mu(G) \approx \mu(F)$,

(iv) $\mu(G \cap B(x,r)) \leq Cr$ for all $x \in G$, $r > 0$, and $\|\mathcal{C}\|_{L^2(\mu|G),L^2(\mu|G)} \leq C_7$.

From (iv) and (iii), we get

$$\gamma_+(F) \geq C^{-1}\mu(G) \geq C^{-1}\mu(F).$$

Then, by (ii), the preceding inequality, and (i),

$$\gamma(E) \leq C\mu(F) \leq C\gamma_+(F) \leq B\gamma_+(E). \qquad \square$$

As a corollary we deduce:

Lemma 26

There exists some absolute constant A_0 such that if $\gamma(E \cap Q) \leq A_0\gamma_+(E \cap Q)$ for all squares Q with $\operatorname{diam}(Q) \leq \operatorname{diam}(E)/5$, then $\gamma(E) \leq A_0\gamma_+(E)$.

Proof. If $\gamma_+(E) > C_5\operatorname{diam}(E)$, then we get $\gamma_+(E) > C_5\gamma(E)$, and we are done provided that $A_0 \geq C_5^{-1}$. We set $A_0 = \max(1, C_5^{-1}, B)$. If $\gamma_+(E) \leq C_5\operatorname{diam}(E)$, then we also have $\gamma(E) \leq A_0\gamma_+(E)$. Otherwise, from the preceding lemma we infer that $\gamma(E) \leq B\gamma_+(E) \leq A_0\gamma_+(E)$, which is a contradiction. $\square$

Now we are ready to prove Theorem 15.

Proof of Theorem 15 Remember that we are assuming that E is a finite union of disjoint compact segments L_j. We set

$$d := \frac{1}{10}\min_{j\neq k}\operatorname{dist}(L_j, L_k).$$

We will prove by induction on n that if R is a closed rectangle with sides parallel to the axes and diameter $\leq 4^n d$, $n \geq 0$, then

$$\gamma(R \cap E) \leq A_0\gamma_+(R \cap E). \tag{26}$$

Notice that if $\operatorname{diam}(R) \leq d$, then R can intersect at most one segment L_j. So either $R \cap E = \varnothing$ or $R \cap E$ coincides with a segment, and in any case, (26) follows (assuming A_0 sufficiently big).

Let us see now that if (26) holds for all rectangles R with diameter $\leq 4^n d$, then it also holds for a rectangle R_0 with diameter $\leq 4^{n+1}d$. We only have to apply Lemma 26 to the set $R_0 \cap E$, which is itself a finite union of disjoint compact segments. Indeed, take a square Q with diameter $\leq \operatorname{diam}(R_0 \cap E)/5$. By the induction hypothesis we have

$$\gamma(Q \cap R_0 \cap E) \leq A_0 \gamma_+(Q \cap R_0 \cap E),$$

because $Q \cap R_0$ is a rectangle with diameter $\leq 4^n d$. Therefore,

$$\gamma(R_0 \cap E) \leq A_0 \gamma_+(R_0 \cap E)$$

by Lemma 26. □

7. Other results

In [50], some results analogous to Theorems 12 and 15 have been obtained for the continuous analytic capacity α. Recall that this capacity is defined like γ in (1), with the additional requirement that the functions f considered in the sup should extend continuously to the whole complex plane. In particular, in [50] it is shown that α is semiadditive. This result has some nice consequences for the theory of uniform rational approximation on the complex plane. For example, it implies the so called *inner boundary conjecture* (see [7] and [54] for previous contributions).

Corollary 16 yields a characterization of removable sets for bounded analytic functions in terms of curvature of measures. Although this result has a definite geometric flavour, it is not clear if this is a really good geometric characterization. Nevertheless, in [51] it has been shown that the characterization is invariant under bilipschitz mappings, using a corona type decomposition for non doubling measures. See also [16] for an analogous result for some Cantor sets.

Using the corona type decomposition for measures with finite curvature and linear growth obtained in [51], it has been proved in [52] that if μ is a measure without atoms such that the Cauchy transform is bounded on $L^2(\mu)$, then any CZO associated to an odd kernel sufficiently smooth is also bounded in $L^2(\mu)$.

Volberg [60] has proved the natural generalization of Theorem 15 to higher dimensions. In this case, one should consider Lipschitz harmonic capacity instead of analytic capacity (see [32] for the definition and properties of Lipschitz harmonic capacity). The main difficulty arises from the fact that in this case one does not have any good substitute of the notion of curvature of measures, and then one has to argue with a potential very different from the one defined in (12). See also [26] for related results about Cantor sets in $\mathbb{R}^d$ which avoid the use of any notion similar to curvature.

However, the relationship of Lipschitz harmonic capacity with rectifiability is not well understood. That is to say, a result analogous to David's Theorem 1 is missing for this capacity. The reason is that, given a set $E \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(E) < \infty$ (where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure), it is not known if the fact that the Riesz transform, i.e. the CZO associated to the vectorial kernel $(x-y)/\|x-y\|^d$, is bounded on $L^2(\mathcal{H}^{d-1}_{|E})$ implies that E is $(d-1)$-rectifiable.

The techniques for the proof of Theorem 15 have also been used by Prat [43] and Mateu, Prat and Verdera [25] to study the capacities γ_s associated to s-dimensional signed Riesz kernels with s non integer:

$$k(x,y) = \frac{x-y}{|x-y|^{s+1}}.$$

In [43] it is shown that sets with finite s-dimensional Hausdorff measure have vanishing capacity γ_s when $0 < s < 1$. Moreover, for these s's it is proved in [25] that γ_s is comparable to the capacity $C_{\frac{2}{3}(n-s),\frac{3}{2}}$ from nonlinear potential theory. The case of non integer s with $s > 1$ seems much more difficult to study, although in the AD regular situation some results have been obtained [43]. The results in [43] and [25] show that the behavior of γ_s with s non integer is very different from the one with s integer.

For more information, we recommend the interested reader to look at the recent surveys [5] and [42], where the geometric part of the recent developments in connection with Painlevé's problem are treated in more detail than in the present paper. For open questions about the relationship between the length of projections of sets and their analytic capacity, as well as other related problems, see [30].

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