CONNECTIONS ON FIBER BUNDLES , BIANCHI IDENTITIES , AND MAXWELL AND YANG-MILLS EQUATIONS

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ABSTRACT

The deep relations wich exist between the fundamental notions in Mathematics of principal fiber bundle , associated vector bundle , connection , curvature and Bianchi identities , and the basic concepts in Physics of gauge , potencial , field , Maxwell equations , conservation laws and Yang-Mills equations , are worked out . In detriment of mathematical rigor , intrinsicness or generality , special emphasis is put on the detailed derivation of the explicit expressions wich relate both categories of concepts , degrees of freedom are pointed out .

RESUM

En aquest treball es posen de manifest les profundes correspondències que existeixen entre les nocions fonamentals en Matemàtiques de fibrat , fibrat vectorial associat , connexió , curvatura i identitats de Bianchi , i els conceptes bàsics en la Física de gauge , potencial , camp , equacions de Maxwell , lleis de conservació i equacions de Yang-Mills . Sacrificant generalitat, concisió i rigor matemàtic , en l'exposició es posa un èmfasi especial en la deducció acurada de les expressions explícites que relacionen ambdues categories de conceptes .

1 Introduction.

Laws are written in the Book of Nature in mathematical language. This is not, of course, the only role of Mathematics, but it is the one that really counts for a scientist. Historically, there has always been a traditional reluctance among the bulk of the comunity of physicists against the incorporation of new mathematical concepts. For instance, some decades ago against those of (multi-)linear algebra, as general vector spaces, tensors, spinors, of non-euclidean geometry, of Lie algebras, etc., and now against those of exterior calculus, twistors and fiber bundles. For the grown up physicist mathematical background is as difficult to change as religion or political creed is for the plain man. Making reference to past succesful experiences is commonly no use. Not either to realize the tremendous importance that the new mathematical concepts and methods have had on the impressive advances ocurred in the physical comprehension of our world during the present century.

In spite of the great ammount of books and research articles which have appeared in the last 10 or 15 years dealing with the concepts of fiber bundles, both from the mathematical [1–3] and from the physical [4–8] point of view, this subject continues to be *tabu* —which means no use or highly irrelevant— to the majority of our colleages. Very deep, general and rigorous treatises are now on hand for everyone having enough time to spend in such important questions, and it is not he purpose of this paper to try to compete with them, not even to try to substitute any in some way or other. The intention of this elementary paper is very modest, but maybe not so easy to accomplish. Namely, to explain in very few words and formulas the physical and mathematical concepts involved in this (already not so) new discipline, performing carefully the (rather trivial) proofs that are not to be found in those more advanced texts, and with particular attention being put in small details, as minus signs or factors which differ from one notation to another. The following pages are addressed, in particular, to these physicists who are worried or feel some interest about fiber bundles, want to now the essentials of them, and do not have very much time to spend on the subject.

2 Fibration. Fiber bundle. Principal fiber bundle. Associated vector bundle.

2.1 Fibration.

It is a triplet (P, M, π), with P (called **fiber space**) and M (called **base space**) differentiable manifolds, and $\pi : P \to M$ **projection**, satisfying the **local triviality** condition: $\forall\ x \in M, \exists\ U_x$ neighborhood, F_{U_x} differentiable manifold, and $\varphi : \pi^{-1}(U_x) \to U_x \times F_{U_x}$ diffeomorphism, such that $\pi[\varphi^{-1}(y,z)] = y, \quad \forall\ y \in U_x,\ z \in F_{U_x}$. $\pi^{-1}(x) \equiv P_x \subset P$ is called the **fiber** over $x \in M$. $\quad s : M \to P$ such that $\pi \circ s = I_M$ is a **section**.

2.2 Fiber bundle.

It is a fibration, (P, M, π), such that $P_x \simeq F,\ \forall x \in M$. If F is a vector space, it is called a **vector bundle**, and if F is a Lie group, a **principal fibration**.

A **trivial** fiber bundle is

$$(M \times F, M, pr_1). \tag{2.1}$$

Given (P, M, π) and (P', M', π'), a **morphism** is a couple (f, g) such that $\pi' \circ g = f \circ \pi$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & M \\ g\downarrow & /// & \downarrow f \\ P' & \xrightarrow{\pi'} & M'. \end{array} \tag{2.2}$$

A fiber bundle, (P, M, π), is **trivializable** if there exists a morphism, (f, g), from it into (2.1), being $f = I_M$.

It is immediate that every vector bundle admits a section, namely the zero section. This is not true in general for a principal fibration.

2.3 Principal fiber bundle.

(P, M, G, π) with P, M and π as before, and G a Lie group which acts on P freely on the right (i.e. $p \cdot g = p \Rightarrow g = e$ neutral element). Moreover,

$\pi : P \to M$ must satisfy that $\forall x \in M$, $\exists U_x$ neighborhood and $\varphi : \pi^{-1}(U_x) \to U_x \times G$ diffeomorphism such that

$$\pi(\varphi^{-1}(y,g)) = y, \qquad \varphi^{-1}(y, gg') = \varphi^{-1}(y,g) \cdot g',$$
$$\forall\, y \in U_x, \quad g, g' \in G. \tag{2.3}$$

Proposition 1.

$\pi(p_1) = \pi(p_2)$ iff it exists $g \in G$ such that $p_1 = p_2 \cdot g$.

Proof. $\Rightarrow$) Let $\pi(p_1) = \pi(p_2) \equiv x$. Locally, in U_x, $\pi = \varphi^{-1}(x, g_i)$, $i = 1,2$; therefore, $p_1 = \varphi^{-1}(x, g_1) = \varphi^{-1}(x, g_2) \cdot g_2^{-1} g_1$, so that $p_1 = p_2 \cdot g$, with $g = g_2^{-1} g_1$.

$\Leftarrow$) Let $p_1 = p_2 \cdot g$. Then $p_2 \cdot g = \varphi^{-1}(x, g_2) \cdot g = \varphi^{-1}(x, g_2 g) = p_1$. We see that $\pi(p_1) = x$ and $p_1 = \varphi^{-1}(x, g_1)$, with $g_1 = g_2 g$, that is $g = g_2^{-1} g_1$. In particular, $\pi(p_1) = \pi(p_2)$, and g *is unique.*

We notice that locally the principal fiber bundle is obtained from the zero-section by action of G: for any $p \in \pi^{-1}(U_x)$ there exists $(y, g) \in U_x \times G$ with $p = \varphi^{-1}(y, e) \cdot g$. This can be generalized as follows.

Proposition 2.

A principal fiber bundle is trivializable iff it admits a section ($s : M \to P$ global, everywhere defined and smooth).

Proof. Define $\varphi_s : (P, M.G, \pi) \to (M \times G, M, G, pr_1)$, $\varphi_s^{-1}(x, g) = s(x) \cdot g$. It is a trivialization (C^∞ and $1-1$). In fact, for every $p \in P$, let $\pi(p) = x$. Then $\pi(p) = \pi(s(x))$, and $\exists!\ g \in G$ such that $p = s(x) \cdot g$.

2.4 Vector bundle associated to a principal fiber bundle.

Given (P, M, G, π) and a representation $r : G \to GL(V)$, V vector space, consider $P \times V$, the action $(p, v) \cdot g = (p \cdot g, r(g^{-1})v)$, the orbit space $E = (P \times V)_{/G}$, and the projection $\bar{\pi}(p, v) = \pi(p)$. The associated vector bundle is, by definition, $(E, M, \bar{\pi})$.

It can be proven that $(E, M, \bar{\pi})$ is locally trivializable with neighborhoods of type $U \times V$ and fiber isomorphic to V. Moreover, the associated vector bundle is trivializable iff there exists a maximal set of linearly independent sections (i.e., as much as the dimension of the fiber). Then the principal fiber bundle is also trivializable. In fact,

1. The *projection* $\bar{\pi}$, $\bar{\pi}(\overline{(p,v)}) = \pi(p)$, verifies:

   (a) $\bar{\pi}$ is a well defined mapping: $(p_1, v_1) \in \overline{(p,v)}$ means that there exists $g \in G$ such that $(p_1, v_1) = (p, v) \cdot g = (p \cdot g, r(g^{-1})v)$. Then $\pi(p_1) = \pi(p \cdot g) = \pi(p)$;

   (b) $\bar{\pi}$ is a projection: immediate, because π is, i.e., for any $x \in M$ there exists $p \in P$ such that $\pi(p) = x$, that is $\bar{\pi}(\overline{(p,0)}) = x$.

2. The *local trivialization* of $(E, M, \bar{\pi})$ will be called ψ and it is constructed, starting from φ, (2.3), as follows

$$\begin{aligned} \psi : \bar{\pi}^{-1}(U_x) &\rightarrow U_x \times V, \\ \psi(\overline{(p,v)}) &= (\pi(p), r(g)v). \end{aligned} \tag{2.4}$$

   (a) ψ *is well defined*, because $(p_1, v_1) = (p, v) \cdot g' = (p \cdot g', r(g'^{-1})(v))$ implies $\pi(p_1) = \pi(p \cdot g') = \pi(p)$, and $r(g_1)(v_1) = r(g_1)r(g'^{-1})(v) = r(g_1 g'^{-1})(v) = f(g)(v)$, because $p_1 = p \cdot g'$, i.e., $\varphi^{-1}(\pi(p_1), g_1) = \varphi^{-1}(\pi(p), g) \cdot g' = \varphi^{-1}(\pi(p), gg')$, and therefore $g_1 = gg'$.

   (b) ψ *is injective*: $(\pi(p), r(g)v) = (\pi(p_1), r(g_1)v_1)$ implies $\pi(p) = \pi(p_1)$ and $r(g)v = r(g_1)v_1$, that is $p_1 = p \cdot g'$ and $v_1 = r(g_1^{-1}g)v$. But g and g_1 are not arbitrary: $p = \varphi^{-1}(\pi(p), g)$ and $p_1 = \varphi^{-1}(\pi(p), g_1) = p \cdot g' = \varphi^{-1}(\pi(p), gg')$. That is, $g' = g^{-1}g_1$, thus $(p_1, v_1) = (p, v)g'$, or $\overline{(p_1, v_1)} = \overline{(p, v)}$.

   (c) ψ *is exhaustive*: for any $(y, v) \in U_x \times V$, let $p = \varphi^{-1}(y, e)$; then $\psi(\overline{(p,v)}) = (\pi(p), r(e)v) = (y, v)$.

3. The *fibers are isomorphic to* V, that is $\bar{\pi}^{-1}(x) \simeq V$, for any $x \in M$. In fact, we only need to consider $\psi : \pi^{-1}(U_x) \rightarrow U_x \times V$ restricted to x, that is $\sigma : \pi^{-1}(\{x\}) \rightarrow V$, $\sigma(\overline{(p,v)}) = r(g)v$. σ is well defined and it is an isomorphism, because ψ is.

2.5 Example.

Gauge principal fibration $(M_4 \times SU(N), M_4, SU(N), pr_1)$. V is the N-dimensional representation space of isospin.

The mathematical structure in which a gauge theory is formulated in a natural way is the associated vector bundle (E, M_4, pr_1), with $E = (M_4 \times$

$SU(N) \times V)_{/SU(N)}$. In fact, the elements of this space of orbits correspond to physical vectors, in the sense that those of them which differ by a gauge transformation are equivalent, belong to the same class (orbit), and define a unique vector of E. The physical process of fixing the gauge corresponds in this language in selecting a canonical representative for each of these equivalence classes (elements of E), i.e. in coordinating the vector bundle.

2.6 Definition of a fiber bundle by means of local charts.

Let $\{U_i\}_{i\in I}$ be an atlas of the manifold M, such that on each neighborhood U_i there is a local trivialization of P. Let $U_i \cap U_j \neq \emptyset$ and the local trivializations

$$\begin{aligned} \varphi_i &: \pi^{-1}(U_i) \to U_i \times F_i, \\ \varphi_j &: \pi^{-1}(U_j) \to U_j \times F_j. \end{aligned} \tag{2.5}$$

Consider

$$\varphi_{ji} : (U_i \cap U_j) \times F_i \to \pi^{-1}(U_i \cap U_j), \tag{2.6}$$

and

$$\begin{aligned} \psi_{ji} \equiv \varphi_{ij} \circ \varphi_{ji}^{-1} : (U_i \cap U_j) \times F_i &\to (U_i \cap U_j) \times F_j \\ (x, z) &\to (x, f_{ji}(x, z)), \end{aligned} \tag{2.7}$$

where $x \in M$, $z \in F_i$, $f_{ji}(x) : F_i \to F_j$. The ψ_{ji} are called **transition functions**. They satisfy the **patching** or **cocycle condition:**

$$\psi_{ji} = \psi_{jk} \circ \psi_{ki}. \tag{2.8}$$

In the particular case of a principal bundle ($F_i = F_j = G$) we have the local sections

$$\begin{aligned} s_i : U_i &\to \pi^{-1}(U_i), \\ s_i(x) &= \varphi_i^{-1}(x, e), \\ s_j : U_j &\to \pi^{-1}(U_j), \\ s_j(x) &= \varphi_j^{-1}(x, e). \end{aligned} \tag{2.9}$$

Then, **transition functions** g_{ij} are defined such that

$$g_{ij} : U_i \cap U_j \to G, \quad s_j(x) = s_i(x) g_{ij}(x). \tag{2.10}$$

3 Connections on vector bundles.

3.1 The connection 1-form.

Let (E, M, π) be an arbitrary vector bundle, with M a n-dimensional manifold and fibers isomorphic to $\mathbf{R}^m$, and let $S(M, E)$ the vector space of all C^∞ sections of E. Consider $\Lambda^p(TM)$ the bundle of p-forms on the tangent space of M.

A connection or covariant differentiation ∇ is a mapping (for each $p \in \mathbf{N}$)

$$\nabla : S(M, E \otimes \Lambda^p(TM)) \to S(M, E \otimes \Lambda^{p+1}(TM)), \tag{3.1}$$

with the following properties:

1. *linearity*

$$\nabla (a_1 s_1 + a_2 s_2) = a_1 \nabla s_1 + a_2 \nabla s_2, \quad \forall\, a_1, a_2 \in \mathbf{R}, \tag{3.2}$$

2. *Leibniz rule*

$$\nabla (s \otimes \omega) = s \otimes d\omega + \nabla s \wedge \omega, \tag{3.3}$$

with d exterior differential and $\wedge$ exterior product. In particular, for $p = 0$ and f a smooth function

$$\nabla (sf) = s \otimes df + (\nabla s) f. \tag{3.4}$$

Given a local trivialization in $U \times \mathbf{R}^m$ with x^k, $k = 1, 2, ..., n$, local coordinates on $U \subset M$, and s_i, $i = 1, 2, ..., m$, linearly independent sections on U, then

$$\nabla s_i = \sum_{j=1}^{m} s_j \otimes \omega_i^j, \quad \omega_i^j \in T^* M, \tag{3.5}$$

where the matrix ω_i^j is called the connection 1- form (in U). In matrix form (Chern)

$$\nabla s = s \otimes \omega, \quad s = (s_1, ..., s_m), \quad \omega = (\omega_i^j). \tag{3.6}$$

The connection 1-form ω is a matrix of Lie algebra valued 1-forms.

Let us see how it changes under a linear transformation of sections, which locally can be written as

$$s'(x) = s(x)g(x), \quad g(x) \in GL(n, \mathbf{R}), \tag{3.7}$$

where the components of $g(x)$ are smooth functions of $x \in M$. If E is a vector bundle associated to a principal fiber bundle by a certain representation $r : G \to GL(m, \mathbf{R})$, then $g(x)$ is a local section of the principal fiber bundle. In the new reference

$$\nabla s' = s' \otimes \omega', \tag{3.8}$$

that is,

$$\begin{aligned} \nabla(sg) &= s \otimes dg + (\nabla s)g = s \otimes (dg + \omega g) \\ &= s \otimes g\omega', \quad g\omega' = dg + \omega g. \end{aligned} \tag{3.9}$$

Therefore,

$$\omega' = g^{-1}dg + g^{-1}\omega g. \tag{3.10}$$

3.2 The curvature 2-form.

Consider the iterated application ot the covariant differential operator ∇ on sections:

$$\nabla(\nabla s) = \nabla(s \otimes \omega) = s \otimes d\omega + \nabla s \wedge \omega = s \otimes (d\omega + \omega \wedge \omega). \tag{3.11}$$

The **curvature 2-form matrix** Ω is defined by $\nabla(\nabla s) = s \otimes \Omega$, i.e.

$$\Omega = d\omega + \omega \wedge \omega. \tag{3.12}$$

The components Ω_i^j of Ω are Lie algebra valued 2-forms

$$\Omega_i^j = d\omega_i^j + \omega_k^j \wedge \omega_i^k. \tag{3.13}$$

Under the linear transformation (3.7)

$$\begin{aligned} \Omega' &= d\omega' + \omega' \wedge \omega' = d(g^{-1}dg + g^{-1}\omega g) \\ &+ (g^{-1}dg + g^{-1}\omega g) \wedge (g^{-1}dg + g^{-1}\omega g) \\ &= -g^{-1}dgg^{-1} \wedge dg + g^{-1}d^2g - g^{-1}dgg^{-1} \wedge \omega g \\ &+ g^{-1}d\omega g - g^{-1}\omega \wedge dg + g^{-1}dg \wedge g^{-1}dg \\ &+ g^{-1}dg \wedge g^{-1}\omega g + g^{-1}\omega g \wedge g^{-1}dg \\ &+ g^{-1}\omega g \wedge g^{-1}\omega g = g^{-1}(d\omega + \omega \wedge \omega)g. \end{aligned} \tag{3.14}$$

Thus, the curvature 2-form Ω transforms as a tensor under a local gauge transformation

$$\Omega' = g^{-1}\Omega g. \tag{3.15}$$

3.3 Exterior covariant differentiation.

It can be defined in an intrinsic way [2], but for the moment let us introduce it in coordinate form [5]. Recall the ordinary expressions of the covariant differential of vectors, one-forms and tensors

$$\begin{aligned} \nabla_i v^j &= \partial_i v^j + \Gamma^j_{ik} v^k, & (3.16)\\ \nabla_i v_j &= \partial_i v_j - \Gamma^k_{ij} v_k, & (3.17)\\ \nabla_i t^j_k &= \partial_i t^j_k + \Gamma^j_{ih} t^h_k - \Gamma^h_{ik} t^j_h, & (3.18) \end{aligned}$$

respectively, where Γ^i_{jk} are the *Christoffel symbols* of the manifold's connection. Now, let

$$\omega^j_i = dx^k \Gamma^j_{ki}, \quad d = dx^k \partial_k, \quad \nabla = dx^k \nabla_k . \tag{3.19}$$

These expressions can be generalized, in the sense that the natural coordinate basis $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ can be substituted by other ones: $\{e_i\}$ and $\{\theta^i\}$. For the sake of concreteness, however, we shall restrict ourselves to (3.19). Taking into account the expressions (3.16)–(3.18), one defines the **exterior covariant differential** of 1- and 2-forms θ^j and ξ^j_i, respectively, by

$$\begin{aligned} D\theta^j &= d\theta^j + \omega^j_k \wedge \theta^k, & (3.20)\\ D\xi^j_i &= d\xi^j_i + \omega^j_k \wedge \xi^k_i - \omega^k_i \wedge \xi^j_k \\ &= d\xi^j_i + \omega^j_k \wedge \xi^k_i - \xi^j_k \wedge \omega^k_i, & (3.21) \end{aligned}$$

where d is the exterior differential and $\wedge$ the exterior product. In matrix form

$$D\theta = d\theta + \omega \wedge \theta, \quad D\xi = d\xi + \omega \wedge \xi - \xi \wedge \omega. \tag{3.22}$$

In particular, for the connection 1-form and curvature 2-form we have, respectively,

$$\begin{aligned} D\omega &= d\omega + \omega \wedge \omega = \Omega, & (3.23)\\ D\Omega &= d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = D(D\omega). & (3.24) \end{aligned}$$

3.4 Bianchi identities.

We are now going to prove the following equality, called **Bianchi identity** (BI)

$$D\Omega = 0. \tag{3.25}$$

In fact,

$$\begin{aligned} d\Omega^j_i &= d^2\omega^j_i + d\omega^j_k \wedge \omega^k_i - \omega^j_k \wedge d\omega^k_i, \\ \omega^j_k \wedge \Omega^k_i &= \omega^j_k \wedge d\omega^k_i - \omega^j_k \wedge \omega^k_h \wedge \omega^h_i, \\ \Omega^j_k \wedge \omega^k_i &= d\omega^j_k \wedge \omega^k_i - \omega^j_h \wedge \omega^h_k \wedge \omega^k_i, \end{aligned} \tag{3.26}$$

therefore,

$$\begin{aligned} D\Omega^j_i &= d\Omega^j_i + \omega^j_k \wedge \Omega^k_i - \Omega^j_k \wedge \omega^k_i \\ &= d\omega^j_k \wedge \omega^k_i - \omega^j_k \wedge d\omega^k_i + \omega^j_k \wedge d\omega^k_i \\ &- \omega^j_k \wedge \omega^k_h \wedge \omega^h_i - d\omega^j_k \wedge \omega^k_i \\ &+ \omega^j_h \wedge \omega^h_k \wedge \omega^k_i = 0. \end{aligned} \tag{3.27}$$

4 Bianchi identities in General Relativity, Maxwell equations and Yang-Mills equations.

4.1 BI in General Relativity.

The BI (3.25) reduce indeed to the well-known form for the curvature 4-tensor R^i_{jkh}

$$\nabla_l R^i_{jkh} + \nabla_k R^i_{jhl} + \nabla_h R^i_{jlk} = 0 \tag{4.1}$$

(for a Riemannian manifold in the natural basis). In order to prove this, we start from the relation between the curvature 4-tensor R^i_{jkh} and the curvature 2-form Ω^i_j. As is well known,

$$R^i_{jkh} = \partial_k \Gamma^i_{hj} - \partial_h \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{hj} - \Gamma^i_{hm}\Gamma^m_{kj}, \tag{4.2}$$

which contracted with $\frac{1}{2}dx^k \wedge dx^h$, gives

$$\frac{1}{2}R^i_{jkh}dx^k \wedge dx^h = d\omega^i_j + \omega^i_m \wedge \omega^m_j, \tag{4.3}$$

where we have used eq. (3.19). Comparing with (3.13), it turns out that

$$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h. \tag{4.4}$$

Now, the covariant differential of a tensor of the type of R^i_{jkh} is

$$\begin{aligned} \nabla_l R^i_{jkh} &= \partial_l R^i_{jkh} - \Gamma^{j'}_{lj} R^i_{j'kh} - \Gamma^{k'}_{lk} R^i_{jk'h} \\ &- \Gamma^{h'}_{lh} R^i_{jkh'} + \Gamma^i_{li'} R^{i'}_{jkh}. \end{aligned} \tag{4.5}$$

The BI (3.25), in terms of R^i_{jkh} and Γ^i_{jk}, is

$$\begin{aligned} & \frac{1}{2} \partial_l R^i_{jkh} dx^l \wedge dx^k \wedge dx^h \\ + & \frac{1}{2} \Gamma^i_{lm} R^m_{jkh} dx^l \wedge dx^k \wedge dx^h \\ - & \frac{1}{2} \Gamma^m_{lj} R^i_{mkh} dx^k \wedge dx^h \wedge dx^l = 0. \end{aligned} \tag{4.6}$$

The coefficient of the term $dx^l \wedge dx^k \wedge dx^h$ of the basis of 3-forms, is

$$\sum_{perm\ lkh} \left(\partial_l R^i_{jkh} + \Gamma^i_{lm} R^m_{jkh} - \Gamma^m_{lj} R^i_{mkh} \right) = 0 \tag{4.7}$$

(each summand with its corresponding sign), and because of the antisymmetry of R^i_{jkh} in kh, we have

$$\sum_{cyclic\ lkh} \left(\partial_l R^i_{jkh} + \Gamma^i_{lm} R^m_{jkh} - \Gamma^m_{lj} R^i_{mkh} \right) = 0. \tag{4.8}$$

The cyclic sum in lkh of eq. (4.5) is

$$\begin{aligned} & \sum_{cyclic\ lkh} \nabla_l R^i_{jkh} \\ = & \sum_{cyclic\ lkh} \left(\partial_l R^i_{jkh} + \Gamma^i_{lm} R^m_{jkh} - \Gamma^m_{lj} R^i_{mkh} \right) \\ + & \Gamma^m_{lh} R^i_{jmh} - \Gamma^m_{lk} R^i_{jmh} + \Gamma^m_{hk} R^i_{jml} \\ - & \Gamma^m_{kh} R^i_{jml} + \Gamma^m_{kl} R^i_{jmh} - \Gamma^m_{hl} R^i_{jmk} \\ = & -S^m_{lk} R^i_{jmh} - S^m_{kh} R^i_{jml} - S^m_{hl} R^i_{jmk}, \end{aligned} \tag{4.9}$$

where eq. (4.8) has been used. Summing up,

$$\sum_{cyclic\ lkh} \nabla_l R^i_{jkh} = - \sum_{cyclic\ lkh} S^m_{lk} R^i_{jmh}, \tag{4.10}$$

where

$$S^m_{lk} = \Gamma^m_{lk} - \Gamma^m_{kl}. \tag{4.11}$$

Eqs. (4.10) are the well-known BI in General Relativity. For a Riemannian manifold in the natural basis one has $S^m_{lk} = 0$, $\forall m, l, k$, and the BI are given by (4.1) in this case.

4.2 Maxwell equations.

They are [9]

$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}, \tag{4.12}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}. \tag{4.13}$$

In terms of the electromagnetic tensor $F_{\mu\nu}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{4.14}$$

and of the current J^μ

$$J^\mu = (\rho, \vec{J}), \tag{4.15}$$

eqs. (4.12) and (4.13) are (with $F^{\mu\nu} = g^{\mu\lambda} F_{\lambda\rho} g^{\rho\nu}$)

$$\partial_\mu F^{\mu\nu} = J^\nu, \tag{4.16}$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{4.17}$$

respectively [5,9]. In terms of the dual tensor

$$F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{4.18}$$

the homogeneous equations (4.13) or (4.17) are written as

$$\partial_\mu F^{*\mu\nu} = 0. \tag{4.19}$$

Summing up, the Maxwell equations can be either given by (4.16) and (4.19), or by (4.17) and

$$\partial_\lambda F^*_{\mu\nu} + \partial_\mu F^*_{\nu\lambda} + \partial_\nu F^*_{\lambda\mu} = J^*_{\lambda\mu\nu}, \tag{4.20}$$

which is (4.16) in terms of the dual tensors, with

$$J^*_{\lambda\mu\nu} = \epsilon_{\rho\lambda\mu\nu} J^\rho. \tag{4.21}$$

In terms of the potential A_μ, the electromagnetic field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.22}$$

We recognize in the homogeneous Maxwell equations (4.17) the BI (3.25), (3.24), with

$$D = dx^\lambda \partial_\lambda, \quad \Omega = F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.23}$$

In the particular case $J = 0$, the inhomogeneous Maxwell equations (4.20) are recognized as a sort of BI (3.25) for the dual tensor

$$D\Omega^* = 0, \quad \Omega^* = F^*_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.24}$$

In general ($J \neq 0$) the inhomogeneous eqs. (4.20) or (4.16) give rise to a different BI for the current J^μ

$$\partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = \frac{1}{2}[\partial_\nu, \partial_\mu] F^{\mu\nu} = 0. \tag{4.25}$$

This is the continuity equation for the 4-current. Using Stokes' theorem, one gets from (4.25) the conservation law for the electric charge.

4.3 Yang-Mills equations.

We shall follow here the opposite direction. The BI (3.25) for the dual Ω^* of the curvature 2-form will give rise to the **Yang-Mills (YM) equations**, while the ordinary BI (3.25) for the curvature 2-form Ω will continue to be identities, which are normally neglected in the literature. We shall work in Euclidean space.

The connection 1-form and the curvature 2-form are now

$$\begin{aligned} \omega &= A_\mu dx^\mu, \\ \Omega = d\omega + \omega \wedge \omega &= (\partial_\mu A_\nu + A_\mu A_\nu) dx^\mu \wedge dx^\nu, \end{aligned} \tag{4.26}$$

with A_μ Lie algebra valued (generally $SU(2)$ or $SU(3)$). The components of ω are the potential, and those of Ω the YM field-strength

$$\Omega = F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{4.27}$$

The BI (3.25) formulated for the dual 2-form Ω^*

$$\Omega^* = F^*_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{4.28}$$

are

$$D\Omega^* = d\Omega^* + \omega \wedge \Omega^* - \Omega^* \wedge \omega = 0, \tag{4.29}$$

and, in terms of A_μ and $F^*_{\mu\nu}$,

$$\begin{aligned} &\sum_{perm\ \lambda\mu\nu} \left(\partial_\lambda F^*_{\mu\nu} + A_\lambda F^*_{\mu\nu} - F^*_{\lambda\mu} A_\nu\right) \\ = \ & 2 \sum_{cyclic\ \lambda\mu\nu} \left(\partial_\lambda F^*_{\mu\nu} + A_\lambda F^*_{\mu\nu} - F^*_{\lambda\mu} A_\nu\right) = 0. \end{aligned} \tag{4.30}$$

In particular, for $\lambda = 1,\ \mu = 2,\ \nu = 3$, we get

$$\begin{aligned} \partial_1 F_{01} &+ \partial_2 F_{02} + \partial_3 F_{03} + A_1 F_{01} + A_2 F_{02} \\ &+ A_3 F_{03} - F_{03} A_3 - F_{02} A_2 - F_{01} A_1 = 0, \end{aligned} \tag{4.31}$$

i.e. (notice that $F_{00} = 0$)

$$\partial_\mu F_{\mu 0} + [A_\mu, F_{\mu 0}] = 0. \tag{4.32}$$

In general, for the rest of the components, we obtain

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0. \tag{4.33}$$

which are the YM equations, ordinarily written as

$$D_\mu F_{\mu\nu} = 0, \quad D_\mu \equiv \partial_\mu + [A_\mu, \quad . \tag{4.34}$$

It is now immediate that starting from the BI (3.25) we get (notice that $F^{**} = F$)

$$D_\mu F^*_{\mu\nu} = 0, \tag{4.35}$$

or, in terms of $F_{\mu\nu}$,

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0, \tag{4.36}$$

which correspond to the homogeneous Maxwell equations.

Solutions of the YM equations (4.33) (or (4.34)) are the celebrated *instantons* and *merons* [10,11], while the BI (4.36) (or (4.35)) are identically satisfied by $F_{\mu\nu}$ given by (4.27). Combining (4.34) and (4.35), we see that imposing

$$F^*_{\mu\nu} = F_{\mu\nu}, \tag{4.37}$$

the YM equations (4.34) are an immediate consequence of the BI (4.35), so that the solutions of (4.37) are also solutions of (4.34). These are the very important **self-dual solutions**, of which the **instantons** constitute the most famous example. In Euclidean space and for $G = SU(2)$ it is given by the gauge potential

$$\begin{aligned} A_\mu(x) &= \frac{x^2}{x^2 + a^2} g^{-1}(x)\partial_\mu g(x), \\ g(x) &= \frac{x_0 + i\vec{\sigma}\cdot\vec{x}}{\sqrt{x^2}}, \quad x^2 = x_\mu x^\mu, \\ \mu &= 0,1,2,3, \end{aligned} \tag{4.38}$$

being $\vec{\sigma}$ the $SU(2)$ Lie algebra generators (Pauli matrices). Then $F_{\mu\nu}$ is given by (4.27). Notice that, when $x^2 \to \infty$, $A_\mu(x)$ is pure gauge

$$A_\mu(x) \sim g^{-1}(x)\partial_\mu g(x), \quad x^2 \to \infty. \tag{4.39}$$

Analogously, the solutions of

$$F_{\mu\nu} = -F^*_{\mu\nu} \tag{4.40}$$

are also solutions of the YM equations (3.4). They are also called **anti-self-dual solutions (anti-instantons)**.

We shall not say anything about the physical importance nor about the classification in Pontryagin classes of the solutions of the YM equations. There is a very extense literature dealing with these subjects [12,13].

5 Connections on principal fiber bundles.

Also called principal connections. There are several equivalent definitions in the literature. Chern, Koszul, Elie Cartan and Ehresmann are names associated with them. In what follows (P, M, G, π) will be a principal fiber bundle.

5.1 First definition of connection.

It is a **smooth family of vector subspaces** H_p of T_pP (called *horizontal*), such that

1. $H : p \mapsto H_p$ is a C^∞ distribution on P.

2. $H_{p \cdot g} = H_p \cdot g = R_{g*} H_p$, $\forall g \in G$, i.e. the family of horizontal subspaces transforms naturally under the right action of G.

3. $\forall p \in P$, $T_pP = H_p \oplus V_p$, where V_p is the tangent space to the fiber over $\pi(p)$. V_p is called *vertical* subspace.

   The third axiom can be replaced by the following one:

4. $\pi_* H_p = T_{\pi(p)}M$, i.e. the H_p project onto the tangent bundle of M.

The star * means here the Jacobian map. If M has dimension n and G has m parameters (its Lie algebra $\mathcal{G}$ has dimension m), then

$$dim\ T_pP = n + m, \quad V_p = Ker\ \pi_* \simeq \mathcal{G}. \tag{5.1}$$

An *equivalent definition of connection* is the following: it is a **smooth family of morphisms** $\Gamma_p : T_xM \to T_pP$, $\forall\, p \in \pi^{-1}(x) = F_x$, such that

1. $\Gamma : p \mapsto \Gamma_p$ is C^∞.

2. $\Gamma_{p \cdot g} = R_{g*} \Gamma_p$.

3. $\pi_* \circ \Gamma_p = I_{T_xM}$.

Then, $\Gamma_p(T_xM) = H_p$, $V_p = T_p(F_x)$.

It is usual to define the maps

$$\begin{array}{ll} v : T_pP \to T_p(F_x), & h : T_pP \to H_p, \quad T_p(F_x) \stackrel{i}{\simeq} \mathcal{G}. \\ X = X_h + X_v \mapsto X_v & X \mapsto X_H \end{array} \tag{5.2}$$

5.2 Dual definition.

The dual version of the preceding definition of connection is: smooth family of $\mathcal{G}$-valued 1-forms ω_p (called connection 1-forms), $\omega_p : T_pP \to \mathcal{G}, \ X \mapsto i(X_v), \ \omega_p = i \circ v$, such that

1. $\omega : p \mapsto \omega_p$ is C^∞.
2. $\omega(X)$ is vertical, $\forall X \in T_pP$, i.e. $\omega(X) = \omega(X_v) = X_v \in \mathcal{G}$, and $\omega(X) = 0 \Longleftrightarrow X \in H_p$.
3. $(R_g^*\omega)(X) = \omega(R_{g*}X) = (ad\ g^{-1})\omega(X), \quad \forall\ g \in G, \ X \in TP$, where $ad\ g^{-1} \in \mathcal{L}(\mathcal{G},\mathcal{G})$, being $ad : G \to \mathcal{G}$ the adjoint representation.

This definition of connection is equivalent to the former one. In fact, Γ_p (or H_p) gives ω_p satisfying these three conditions, and reciprocally, ω_p satisfying the last three axioms defines Γ_p. ω_p is called an Ehresmann connection. One has

$$H_p = Ker\ \omega_p. \tag{5.3}$$

Under very general conditions, there exists a great number of connections on a principal fiber bundle [2].

5.3 Exterior covariant differential (intrinsic definition).

It is the operator

$$\begin{aligned} D : V \otimes \Lambda^p(TP) &\to V \otimes \Lambda^{p+1}(TP), \\ (D\omega)(X_1, ..., X_{p+1}) &= d\omega(hX_1, ..., hX_{p+1}), \end{aligned} \tag{5.4}$$

i.e. $D\omega = d\omega \circ h, \quad h : T_pP \to H_p$ horizontal projection. Here V is a (finite dimensional) vector space of representation (say r) of G, and ω are V-valued forms such that $R_g^*\omega = r(g^{-1})\omega$ (pseudotensorial).

Given ω, the connection 1-form, the curvature 2-form Ω is defined as

$$\Omega = D\omega = d\omega \circ h. \tag{5.5}$$

It turns out that

$$\Omega = d\omega + \omega \wedge \omega \tag{5.6}$$

and

$$D\Omega = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega. \tag{5.7}$$

It can also be proven that [2]

$$D\Omega = 0, \tag{5.8}$$

which are the BI in this general case.

5.4 Definition of connection on local charts.

It is the usual definition in Physics. Let U_i and U_j local charts on M and, in order to make contact with section 3, consider the particular case $\mathcal{G} = GL(m, \mathbf{R})$. We have already seen that for vector bundles we have

$$\omega_{U_i} = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_{U_j} g_{ij}, \quad g_{ij} \in G, \tag{5.9}$$

where $g_{ij}(x)$ depends differentially on $x \in U_i \cap U_j$. ω is the $\mathcal{G}$-valued connection 1-form. In general, one has [2]

$$\omega_{U_i} = g_{ij}^{-1} dg_{ij} + ad(g_{ij}^{-1}) \omega_{U_j}. \tag{5.10}$$

A connection is a family of $\mathcal{G}$-valued 1-forms ω_{U_i} adapted to the covering $\{U_i\}_{i \in I}$ of M and related on the intersections $U_i \cap U_j \neq \emptyset$ by eq. (5.10).

5.5 A note on flat connections.

Let $P = M \times G$ a trivial principal fiber bundle. Then $M \times \{g\}$ is a submanifold of P, $\forall g \in G$, and $M \times \{e\}$ is a subbundle of P (because $\{e\}$ is a Lie group). The canonical flat connection on P consists in taking, for $p = (x, g)$,

$$H_p = T_p(M \times \{g\}). \tag{5.11}$$

It is reducible to a unique connection on $M \times \{e\}$. A connection on (P, M, G, π) is called **flat** if $\forall x \in M$, $\exists U_x$ such that the induced connection in $\pi^{-1}(U_x)$ is isomorphic with the canonical flat connection in $U_x \times G$. It can be proven that a connection ω is flat iff Ω (curvature 2-form) vanishes [2].

6 Parallel transport.

6.1 Definition. The holonomy group.

Let (P, M, G, π) a principal fiber bundle with connection 1-form ω (matrix of $\mathcal{G}$-valued 1-forms on T^*M). In a local trivialization $U \times G$, $U \subset M$, (x, g) local coordinates, ω is given by

$$A(x) = A^a_\mu(x)\frac{\lambda_a}{2i}dx^\mu, \tag{6.1}$$

with x^μ, $\mu = 1, 2, ..., n$ local coordinates on $U \subset M$, and λ_a, $a = 1, 2, ..., m$ constant matrices which generate $\mathcal{G}$. Let $x(t)$, $0 \leq t \leq 1$, curve on M piecewise C^1. An **horizontal lift** of $x(t)$ is another curve $p(t)$ in P such that $\pi(p(t)) = x(t)$, $0 \leq t \leq 1$, and such that all its tangent vectors are horizontal. It can be proven [2] that given $x(t)$ and $p_0 \in P$ with $\pi(p_0) = x_0 = x(0)$, there exists a unique horizontal lift $p(t)$ with $p(0) = p_0$.

The **parallel transport of fibers** can be defined as follows: when p_0 varies over the fiber $\pi^{-1}(x_0)$ one obtains $p_1 = p(1)$ such that $\pi(p_1) = x_1 = x(1)$, i.e. a mapping from

$$\pi^{-1}(x_0) \mapsto \pi^{-1}(x_1). \tag{6.2}$$

This map is a fiber isomorphism.

In the particular case that $x_0 = x_1$ (closed curve), we have an automorphism of the fiber $\pi^{-1}(x_0)$. For each closed curve on $x_0 = x_1$ we have one such automorphism. Under composition they constitute a group, called the **holonomy group** of the connection with reference point x_0. When we restrict ourselves to parallel transport about loops on x_0 which are homotopic to zero, we get the **restricted holonomy group**. It is easy to see that both are subgroups of G in a natural sense [2].

6.2 Expression in coordinates.

In the local trivialization formerly defined, the section $g_{ij}(t)$ is, by definition, parallel-transported along the curve $x(t)$ iff

$$\dot{g}_{ij} + A_{\mu ik}(x)\dot{x}^\mu g_{kj} = 0 \tag{6.3}$$

(this comes from the general expression of parallel transport $\dot{x}^\mu(\partial_\mu z^i + \Gamma^i_{\mu k} z^k) = 0$). Differentiation along the lifted curve is given by

$$\frac{d}{dt} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{g}_{ij} \frac{\partial}{\partial g_{ij}}, \tag{6.4}$$

and since it is a parallel lift, using (6.3) we get

$$\begin{aligned} \frac{d}{dt} &= \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - \frac{1}{2i} A^a_\mu(x)(\lambda_a)_{ik} g_{kj} \frac{\partial}{\partial g_{ij}} \right) \\ &= \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - A^a_\mu(x) R_a \right) \equiv \dot{x}^\mu D_\mu, \end{aligned} \tag{6.5}$$

where the covariant derivative is defined as

$$D_\mu = \frac{\partial}{\partial x^\mu} - A^a_\mu(x) R_a, \quad R_a = tr \left(\frac{1}{2i} \lambda_a g \frac{\partial}{\partial g^T} \right), \tag{6.6}$$

$\{R_a\}$ being a right invarriant basis of the tangent space of G. In fact, if $g' = gh$ then $g_{kj}\frac{\partial}{\partial g_{ij}} = g'_{kj}\frac{\partial}{\partial g'_{ij}}$, because from $g_{kj} = g'_{kl}(h^{-1})_{lj}$ and $\frac{\partial}{\partial g_{ij}} = h_{jm}\frac{\partial}{\partial g'_{im}}$ (chain rule), one has $g_{kj}\frac{\partial}{\partial g_{ij}} = g'_{kl}(h^{-1})_{lj} h_{jm} \frac{\partial}{\partial g'_{im}} = g'_{km}\frac{\partial}{\partial g'_{im}}$. The R_a verify

$$[R_a, R_b] = -f_{abc} R_c, \tag{6.7}$$

with f_{abc} structure constants of $\mathcal{G}$. The spliting of T_pP into horizontal and vertical part, $T_pP = H_p \oplus V_p$, is given by

$$(D_\mu, R_a). \tag{6.8}$$

It is immediate to see that the curvature defined as the commutator of the basis for the horizontal subspace H_p has only vertical components. In fact,

$$[D_\mu, D_\nu] = -F^a_{\mu\nu} R_a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu. \tag{6.9}$$

In the dual approach, the connection ω is a $\mathcal{G}$-valued 1-form in T^*P whose vertical component is the **Maurer-Cartan form** $g^{-1}dg$

$$\omega = g^{-1} A g + g^{-1} dg, \tag{6.10}$$

with $A(x)$ given by (6.1). As it must be, ω annihilates the horizontal vectors and is constant on vertical ones

$$< \omega, D_\mu >= 0, \quad < \omega, R_a >= \frac{1}{2i}\lambda_a. \tag{6.11}$$

The Maurer-Cartan form is invariant under the left action $g \mapsto g_0 g$ for constant $g_0 \in G$

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} dg. \tag{6.12}$$

It can be expanded as

$$g^{-1} dg = \frac{1}{2i}\theta_a \lambda_a, \tag{6.13}$$

with $\frac{1}{2i}\lambda_a \in \mathcal{G}$ constant matrices, and θ_a a basis of left-invariant 1-forms. From $d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg = 0$, it turns out that θ_a obeys the Maurer-Cartan equation

$$\theta_a + \frac{1}{2} f_{abc}\theta_b \wedge \theta_c = 0. \tag{6.14}$$

The dual of the θ_a are

$$L_a = tr\left(\frac{1}{2i} g\lambda_a \frac{\partial}{\partial g^T}\right) = \frac{1}{2i} g_{ik}(\lambda_a)_{kj}\frac{\partial}{\partial g_{ij}}, \tag{6.15}$$

which obey

$$< \theta_a, L_b >= \delta_{ab}, \quad [L_a, L_b] = f_{abc} L_c. \tag{6.16}$$

On the other hand, ω transforms as a tensor under the right action $g \mapsto g g_0$ for constant $g_0 \in G$

$$\omega \mapsto g_0^{-1} \omega g_0. \tag{6.17}$$

In general $g \mapsto g g_1$ (g_1 *not* constant)

$$\omega \mapsto g_1^{-1}\omega g_1 + g_1^{-1} dg_1. \tag{6.18}$$

For the curvature 2-form, we have

$$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega = g^{-1} F g, \\ F &= dA + A \wedge A = \frac{1}{2} F^a_{\mu\nu} \frac{1}{2i}\lambda_a dx^\mu \wedge dx^\nu. \end{aligned} \tag{6.19}$$

It obeys the BI

$$D\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0, \tag{6.20}$$

and transforms as a tensor

$$\Omega \mapsto g_1^{-1} \Omega g_1, \quad \forall\, g_1 \in G, \tag{6.21}$$

as we already know.

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