Al-Samaw'al's Curious Approach to Trigonometry

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Abstract
Al-Samaw'al's work *Exposure of the Errors of the Astronomers* contains a single numerical table, a highly unusual table of sines. In order to avoid the problem of approximating sin(1 degree), al-Samaw'al redefines the number of degrees in a circle from 360 to 480. In the accompanying chapter, al-Samaw'al criticizes his predecessors for deviating from geometrical precision and explains how his table is to be calculated. However, our analysis demonstrates that al-Samaw'al did not follow his own instructions, simply interpolating his entries from a traditional sine table.

Trigonometry was the mathematical heart of ancient Greek and medieval Islamic astronomy. The need to convert geometric models of planetary motion into quantifiable predictions of positions led to the birth of trigonometry and its associated tables, first by the hand of 2nd century BC scientist Hipparchus of Rhodes (whose table is now lost), and later by Claudius Ptolemy (fl. AD 140) in the *Almagest* [Toomer, 1984]. The sine function (or, in Greece, the chord) was therefore the mathematical foundation of astronomy. Methods of table computation and levels of precision improved gradually throughout the medieval period. However, the subject of this paper, the 12th-century scientist Ibn Yaḥyā al-Samaw'al al-Maghribī, brought a voice of critique to the standard techniques, claiming that a mistake was made in allowing inherently approximative methods to enter the geometric realm of trigonometry. His curious resolution of the matter, to redefine the circle to

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contain 240 or 480 parts rather than the standard 360º, appears to have been carried through to a completely new trigonometric table. However, we shall demonstrate that the way he constructed his table violated his own principle.

The sine function, an Indian invention, simplified the geometric procedures needed in mathematical astronomy. Although the sine dominated in Islam as well, the chord continued to be used occasionally, presumably through its appearance in the *Almagest*. The transition between the two functions is simple. If one needs to compute a chord length for a given arc in a circle of radius R, one takes the sine of half of the arc and doubles the result (see Figure 1). Thus, while a table of sines reaches a maximum value of R[1] at $\theta = 90^\circ$ a table of chords reaches its maximum value of $2R$ at $\theta = 180^\circ$. Other than these differences, tables of sines and chords are identical.

Tables usually were calculated similarly to Ptolemy's method in *Almagest* I.10.[2] For a table of chords, one starts with those that are easily found via ruler and compass methods: for instance, using the standard sexagesimal notation[3] in a circle of radius $R = 60$, Crd 60º = 60, Crd 90º = $60\sqrt{2}$ = 84;51,10 (to three places), and Crd 120º = $\sqrt{120^2 - 60^2}$ = 103;55,23. A geometrical argument related to Euclid's construction of the regular pentagon leads to the discovery that Crd 36º = 37;4,55 and Crd 72º = 70;32,3. Once these chords have been obtained, the use of chord equivalents to the sine sum, difference and half-angle laws in various combinations eventually allow the computation of the chord of each multiple of 3º. It is not possible to determine the chords of other integer-valued arcs: to arrive at a value for Crd 1º would require a procedure equivalent to trisecting the angle, which is impossible using ruler and compass.

To proceed further, Ptolemy appeals to the following theorem:

$$\text{If } 0^\circ < \beta < \alpha < 180^\circ, \text{ then } \frac{\alpha}{\beta} > \frac{\text{Crd}\,\alpha}{\text{Crd}\,\beta}.$$

Since Crd $\frac{3}{2}^\circ$ and Crd $\frac{3}{4}^\circ$ may be found by applying the half-angle formula to the known Crd 3º, we may calculate lower and upper bounds to

[1] Most pre-modern scientists defined the sine as a length in a circle of radius R. Although R = 60 was almost universal in Islam, various different radii were used in India. In this article "Sin" will denote such a sine, which is simply R times the modern function.
[2] Accounts of the construction of the *Almagest* chord table may be found in [Aaboe, 1964, 101-125] and [Pedersen, 2010, 56-63].
[3] For instance, 103;55,23 = 103 + 55/60 + 23/60^2.

Crd 1º by substituting first $\alpha = \frac{3}{2}°, \beta = 1°$, and then $\alpha = 1°, \beta = \frac{3}{4}°$, into the theorem. Ptolemy obtains

$$1;2,50 < \text{Crd } 1° < 1;2,50, \qquad (1)$$

from which he concludes that Crd 1º = 1;2,50.

Several Muslim scientists through the 11th century AD improved on Ptolemy's efforts when constructing their own sine tables. In the late 10th-century *Ḥākimī Zīj*, Ibn Yūnus applied essentially Ptolemy's method but used $\text{Sin}\,\frac{9}{8}°$ and $\text{Sin}\,\frac{15}{16}°$ rather than $\text{Sin}\,\frac{3}{2}°$ and $\text{Sin}\,\frac{3}{4}°$ (along with an additional approximative improvement technique) to achieve tighter bounds on Sin 1º [King, 1972, 78-80]. In his determination of $\text{Sin}\,\frac{1}{2}°$, Ibn Yūnus's contemporary Abū'l-Wafā' modified the method slightly to consider the magnitudes of differences between sines rather than the sines themselves, improving the bounds significantly [Woepcke, 1860]:

$$\text{Sin}\,\tfrac{15}{32}° + \tfrac{1}{3}\left(\text{Sin}\,\tfrac{18}{32}° - \text{Sin}\,\tfrac{15}{32}°\right) < \text{Sin}\,\tfrac{1}{2}° < \text{Sin}\,\tfrac{15}{32}° + \tfrac{1}{3}\left(\text{Sin}\,\tfrac{15}{32}° - \text{Sin}\,\tfrac{12}{32}°\right). \qquad (2)$$

In the early 11th-century *Qānūn al-Mas'ūdī* al-Bīrūnī determined Crd 40º (the side length of a regular nonagon) using an iterative procedure, from which Crd 1º may be found by conventional geometric means [Schoy 1926; Schoy 1927, 18-30]. Finally, four centuries later Jamshīd al-Kāshī would provide a numerical method that generates a value for Sin 1º of arbitrary accuracy [see Aaboe 1954 and Rosenfeld/Hogendijk 2002/03].

Not all scientists were satisfied with approaches of this kind. We have on record a criticism of Ptolemy's presentation of his determination of Crd 1º by another leading early 11th-century mathematician/astronomer, Ibn al-Haytham, in the fourth essay of his *Doubts Concerning Ptolemy*:[4]

> [Ptolemy] states in the ninth[5] chapter of the first book concerning the extraction of a chord of one degree that the chord of one [degree] is smaller than 1;2,50 and greater than 1;2,50, so that it is 1;2,50. Thus he makes the same line which is the chord of one degree, which is smaller than the same amount and greater than that same amount to be equal to that same amount. This is a contradictory statement. Besides, it is ugly, and jars the ears and no one is able to hear

[4] Altered slightly from [Voss, 1985, 28-29], which is a translation of the edition [Sabra/Shehaby, 1971, 9-10].
[5] Actually the tenth chapter.

> it...When [the sexagesimal places that follow 1;2,50] are explained, then the chord of one degree is not 1;2,50 but greater than that value...[6] So long as he does not mention these small fractions, his statement on the chord of one degree being smaller than one amount precisely and greater than that same amount precisely is ugly and contradictory and nothing like it is permitted in the mathematics books.

Ibn al-Haytham's objection to Ptolemy's statement is his transition from the statement 1;2;50 < Crd 1° < 1;2,50 to the conclusion that Crd 1° = 1;2,50. This apparently trivial complaint will be echoed more strongly in al-Samaw'al's text.

Al-Samaw'al's Life and Work

Ibn Yaḥya al-Samaw'al al-Maghribī grew up Jewish in Baghdad early in the 12th century, but converted to Islam and spent the latter part of his life near Marāgha, Iran. His best known work is a criticism of the Jewish faith written after his conversion [Perlmann, 1964; Marazka *et al.*, 2006]; he also wrote a medical treatise on sexual diseases and ailments. He studied mathematics and science from a young age, and wrote his most prominent mathematical book –*Al-Bāhir*, "The Dazzling"– before his twentieth birthday [Ahmad/Rashed 1972]. This algebraic work contains an early foreshadowing of negative numbers, uses mathematical induction to establish several relations [Rashed, 1972], and solidifies the reliance of algebra on arithmetic. Many of his other writings are now lost.

The work under consideration here, *Kashf ʻuwar al-munajjimīn* (Exposure of the Errors of the Astronomers, written AD 1165/6), is a 25-chapter collection of what al-Samaw'al perceived as inadequacies and shortcomings in the works of many of his astronomical predecessors. Few scientists escape al-Samaw'al's critical eye in this work of the same tradition as Ibn al-Haytham's *Doubts Concerning Ptolemy*. The book deserves more attention than it has received. Three modern studies have examined particular chapters: [Rosenthal, 1950] discusses and translates part of the book's introduction relating to scientific progress. [Rashed, 1991] is a study and translation into French of the seventh chapter on al-Samaw'al's suggested improvements to second-order interpolation methods. [Berggren/Van Brummelen 2003] describes and translates into English part of the fifteenth

[6] This is odd, because the chord of 1° is actually less than 1;2,50.

chapter, where al-Samaw'al criticizes Abū Sahl al-Kūhī for situating the observer (rather than the Earth's center) at the center of the universe while he was determining the angle of depression to the horizon for an observer on a raised platform. We concern ourselves here with the fourth chapter, "On showing [the astronomers'] incapacity to find the true sine of one degree, on which the sine table is constructed", as well as the associated trigonometric table, the only table in the book. An edition and translation of the text of Chapter 4 may be found at the end of this article; we also translate and analyze the table.

Commentary on the text

Al-Samaw'al begins by referring to the four theorems that astronomers use to calculate chords:

- the chord difference formula;
- the chord sum formula;
- the chord half-angle formula;
- if $0^\circ < \beta < \alpha < 180^\circ$, then $\text{Crd}\,\alpha / \text{Crd}\,\beta < \alpha / \beta$.

These are the four theorems that Ptolemy uses in the *Almagest*. Next al-Samaw'al describes the standard procedure for generating chords of smaller and smaller arcs: begin with the chords of 72º (the length of a side of a regular pentagon inscribed in the circle) and 60º (a hexagon); use the chord difference formula to determine Crd 12º; and then successively apply the half-angle formula to get Crd 6º, Crd 3º, $\text{Crd}\,1\frac{1}{2}^\circ$, and $\text{Crd}\,\frac{3}{4}^\circ$. Finally, applying the fourth theorem as Ptolemy does, al-Samaw'al gets

$$\tfrac{2}{3}\text{Crd}\,1\tfrac{1}{2}^\circ < \text{Crd}\,1^\circ < \tfrac{4}{3}\text{Crd}\,\tfrac{3}{4}^\circ, \qquad (2)$$

resulting in the bounds

$$1;2,49,48 < \text{Crd}1^\circ < 1;2,49,52. \qquad (3)$$

The inclusion of a sexagesimal place beyond Ptolemy's calculation allows al-Samaw'al to avoid Ibn al-Haytham's complaint, although the matter is not remarked upon in the text. Al-Samaw'al's lower bound is correct to all places; the upper bound should be 1;2,49,53. These values are more accurate than Ptolemy's, but tables of this accuracy and better (usually for sines rather than chords) were commonplace at this time.

Al-Samaw'al's disagreement with the astronomers is that Crd 1º is only known to lie between the two bounds, and to take a value between the bounds to be the correct Crd1° requires that we lapse into approximation (*taqrīb*) rather than truth (*taḥqīq*). His solution is to redefine the unit of angle measurement in a circle. Rather than dividing the circle into the standard 360 degrees, he adopts two (unnamed) new units of measurement wherein either 240 or 480 parts constitute a circle. We shall refrain from referring to these new units as degrees, since al-Samaw'al uses the word "degree" (*daraja*) only in the context of a 360-degree circle.

The Ptolemaic sequence of chord calculations with 480-part units proceeds as follows: the pentagon is the chord of 96 parts (72°), while the hexagon is the chord of 80 parts (60°). The chord difference formula gives the chord of 16 parts (12°), and successive application of the half-angle formula generates the chord of eight, four, two, and finally one part ($\frac{3}{4}^{\circ}$). Thus in the new units, the approximation methods of Ptolemy and his successors are unnecessary. Al-Samaw'al does not provide any values for the chords of these arcs; given our analysis below, he may never have computed them.

Al-Samaw'al's chord table

The table of chords is reproduced in Table 1, with errors displayed in units of the last sexagesimal place. It is entitled "Table Constructed by al-Samaw'al on the Correction of the Altitudes of the Stars", referring to its first astronomical use early in the treatise. The table is used later in *Exposure of the Errors of the Astronomers* in a variety of contexts as a source of chords and sines. Obviously it differs from Ptolemy's chord table in how arcs are measured; also, the chords are given for a circle of *diameter* 60, not of *radius* 60. We shall see why al-Samaw'al made this choice in an application below.

Several of the table's entries are corrupted by scribal errors, but nevertheless it is clear from the pattern in the errors that an interpolation grid exists with nodes at every fourth entry in the table: these entries are close to correct, while the others diverge from the correct values. These are the entries in al-Samaw'al's table that would correspond to every third entry in an ordinary sine table. The errors in the remaining entries are very closely reproduced by interpolating linearly between multiples of 3° of a hypothetical sine table, calculated accurately to three sexagesimal places for every degree of arc.[7] So, for example, Samaw'al calculated the entries between 108 and 112 parts as follows:

[7] Thanks to Clemency Montelle for this suggestion.

- The entries for 108 and 112 parts were taken from the corresponding entries of a conventional sine table[8] for 81º and 84º.
- The entry for 109 parts was computed by interpolating between the entries in a sine table for 81º and 82º: $\text{Crd}\,109 = \frac{1}{4}\text{Sin}\,81° + \frac{3}{4}\text{Sin}\,82°$ (since 109 parts is three times as close to 82º as it is to 81º).
- The entry for 110 parts was computed as $\text{Crd}\,110 = \frac{1}{2}\text{Sin}\,82° + \frac{1}{2}\text{Sin}\,83°$.
- The entry for 111 parts was computed as $\text{Crd}\,111 = \frac{3}{4}\text{Sin}\,83° + \frac{1}{4}\text{Sin}\,84°$.

The difference between the errors produced by this method and the errors in the table is displayed in the column entitled "Difference" in Table 1. Our recomputation accounts for most of the error; the small remaining differences are caused partly by errors of 1º in the last sexagesimal place in some of the entries of the underlying 360º sine table, as well as corrupted entries, rounding, and small computational deviations.

We must therefore conclude that al-Samaw'al did not in fact pursue the project that he advocates in chapter 4. Indeed, his table was computed by interpolating from each entry in a conventional sine table, including the entries that rely on the procedures that he condemns.

The sine table from which al-Samaw'al's table was generated appears to be mostly accurate to three sexagesimal places, with about 1/4 of the entries in error by 1 in the last place. Such tables were commonplace by al-Samaw'al's time, although none of the sine tables we checked provides a close enough fit to the table that al-Samaw'al used for us to be confident of identifying the source.[9] It is thus possible that the underlying table was computed in the usual way by al-Samaw'al himself. The function is symmetric, reaching its peak at 120 parts; most, but not all, of the symmetric pairs of entries on opposite sides of the peak match. A patch of interpolated entries between 156 and 160 parts contains unusually large errors, suggesting that a mistake was made in the interpolation process.

[8] It is also possible that the underlying table was of chords, although sine tables were much more common in al-Samaw'al's time.

[9] Among the sine tables checked are those found in the zījes (astronomical handbooks) of al-Khwārizmī, al-Battānī, Ibn Yūnus, Kushyār ibn Labbān (the *Jāmiʿ Zīj*), the *Mumtaḥan Zīj*, al-Bīrūnī's *al-Qānūn al-Masʿūdī*, and the *Shāmil Zīj*. (The latter postdates al-Samaw'al, but it was checked due to the possibility that the two tables share a common source.)

In addition to his chord table al-Samaw'al includes a "Table of Corrections", which simply tabulates the fraction of the circle that comprises a given angle. This allows al-Samaw'al to work occasionally with fractions of a circle, rather than 360º , 240-part, or 480-part units. The values of the entries increase uniformly by 0;0,15 for each part. The use of this table is illustrated in the following section.

How the tables were used: an application

Al-Samaw'al uses his chord table several times in *Exposure of the Errors of the Astronomers*; we sample one episode here,[10] the application for which the table receives its name: "On the Correction of the Altitude of the Stars". Chapter 2 is a criticism of the use of the astrolabe to measure the altitudes of stars above the horizon. Al-Samaw'al's complaint is that the true altitude of a star should be measured from the center of the Earth rather than from its surface. Al-Samaw'al's universe is finite in extent and he has a measure of its radius, so he is able to propose a correction to all altitude measurements. The only other way to be accurate, he says, is to build an astrolabe with its center at the center of the Earth, and it would need to be so large that its edge extends beyond the Earth's surface.

Clearly there would be a few hurdles to overcome in building such an astrolabe, so al-Samaw'al proceeds with the mathematical correction. In Figure 2 (simplified from the treatise) AUB is the celestial sphere, GXD is the Earth's surface, and ZS is an astrolabe being used to measure the altitude of star Q. Suppose that the altitude is measured to be $\angle QHM = 30°$; then the correct altitude is the unknown $\angle QEU$.

Al-Samaw'al prepares for the problem by scaling the universe so that its radius is 60 units; this leads him to determine the radius of the Earth to be approximately 0;2,44. The rest of the mathematics revolves around ΔQHE, with given $\angle H = 120°$. A chord equivalent to the planar law of sines is applied:

$$\frac{\text{Crd } 2\angle Q}{EH} = \frac{\text{Crd } 2\angle H}{60}, \quad \text{or} \quad \text{Crd } 2\angle Q = \text{Crd } 2\angle H \cdot \left(\frac{EH}{60}\right). \tag{4}$$

To look up Crd $2\angle H$ = Crd $(2 \cdot 120°)$ in his table, al-Samaw'al first converts to units of right angles, so that $\angle H$ = 1;20. He wants to look up the

[10] Another may be found in [Berggren/Van Brummelen 2003, 122-124].

chord of twice this value, which is 2;40 right angles; but in units of fractions of a circle he must divide this by 4, so he finds 2;40 / 4 = 0;40 in the Table of Corrections. From this row in the table, he finds that the angle measure for 0;40 in a 240-part circle is 160 parts, and the corresponding chord is 51;57,41. Substituting this value into (4), he finds that Crd $2\angle Q$ = 0;2,22,1,40. Interpolating backward within his table he gets $2\angle Q$ = 0;3,0,51, in a 240-part circle. But this quantity is just $\angle Q$ itself in a 480-part circle, a felicity in an otherwise cumbersome process. Determining $\angle E$ is now just a matter of subtracting $\angle H$ and $\angle Q$ from 240 parts (= 180º); he gets 79;56,59,9. This value is divided by 120 to get the value 0;39,58,29,34 in right-angle units, and the result is subtracted from one right angle to get $\angle QEU$ = 0;20,1,30,26. At the end, here and elsewhere, al-Samaw'al converts his result back to ordinary degrees, obtaining $\angle QEU$ = 30;2,15,39º. Apparently he felt that readers would not accept a final result unless it was given in units that they would recognize.

Conclusion

By adopting a circle broken into 480 parts rather than the usual 360 degrees, al-Samaw'al found an ingenious solution to his complaint with his predecessors: he was able to compute an entire sine table using only purely geometric methods, without having to rely on the approximations that Ibn al-Haytham earlier had rejected. However, our analysis of the table reveals that al-Samaw'al did not in fact compute his sine table in this way; rather, he simply interpolated his sine values between entries in an unidentified conventional sine table. Our discovery is intriguing and certainly undermines what al-Samaw'al's was trying to accomplish. But it does not change the fact that al-Samaw'al felt strongly enough about the proper methodological boundaries between mathematical disciplines that he wrote a long scientific treatise with this table at its heart. Clearly some mathematicians, at least, were worried enough about the appropriate relationship between geometry and computation that they were willing to make public statements on the matter – Ibn al-Haytham with the complaint, and al-Samaw'al with the solution. At the moment it is unknown whether al-Samaw'al's approach was taken up by other authors (although a similar objection was to be expressed several centuries later, in another culture, by Giordano Bruno), or whether similar tables were constructed. In the meantime, al-Samaw'al's chord table stands as a substantial statement of a concern within medieval trigonometry that the methodological boundary between geometric reasoning and numerical methods should be maintained.

Translation of Chapter 4

Exposure of the Errors of the Astronomers exists in two manuscripts. Leiden Or. 98 (which we call **L**) is well composed; the table is laid out carefully on two adjacent pages with 40 rows of entries each. However, it contains a large lacuna that begins about 2/3 of the way through chapter 4. Thus the last part of our translation is based only on Bodleian Hunt 539 (called **B**). In the latter manuscript the first two pages of the table consist of 26 rows of entries; the third and final page has 28 rows. Various scribal errors in the tables that are shared between the two manuscripts and similarities in their colophons suggest that they have a common source. See the section on the Arabic text, below.

Chapter 4: On showing their [i.e. astronomers'] incapacity to find the true sine of one degree, on which the table of sine is constructed

And one of those that keep them [i.e. astronomers] from truth in their computational operations is their incapacity to determine the sine of one degree, and from it to construct the sine table. Now, we will cite the method which brings them to the sine of one degree by approximation, not by truth; that is, the path that they have all taken when deriving the sine of one degree. They have no method other than giving the following four lemmata: 1) if the chords of two arcs are known, then the chord of their difference is known; 2) if the chords of two arcs are known, then the chord of their sum is known; 3) if the chord of an arc is known, then the chord of its half is known; 4) the ratio of a larger chord to a smaller chord is smaller than the ratio of the arc of the larger chord to the arc of the smaller chord.

After they establish these lemmata, they derive the chord of a fifth [of a circle] and the chord of a sixth [of a circle] by geometrical methods that lead to truth, so that the chord of 60 [degrees] is known and the chord of 72 [degrees] is known. By the first lemma, the chord of 12 [degrees] is known; by the third lemma, the chord of 6 [degrees] is known, and the chord of 3 [degrees] is known, and the chord of one and a half [degrees] is known, and the chord of three fourths [of a degree] is known.

It is implied by the fourth lemma that the ratio of the chord of one and a half degrees to the chord of one degree is smaller than the ratio of the arc that is one and a half degrees to the arc whose amount is one degree. Thus, the ratio of the chord <of one degree> to the chord of one and a half degrees is larger than the ratio of one degree to one and a half degrees. But the ratio of one degree to one and a half degrees is two thirds, so the chord of one degree is larger than two thirds of the chord of one and a half degrees. And two thirds of the chord of one and a half degrees is 1;2,49,48. Hence, the chord of one degree is larger than 1;2,49,48.

Next, the ratio of one degree to three fourths of a degree is larger than the ratio of the chord of one degree to the chord of three fourths of a degree. But the ratio of one degree to three fourths of a degree is one and a third, so the chord of one degree is smaller than one and a third times the chord of three fourths [of a degree]. One and a third times the chord of three fourths [of a degree] is 1;2,49,52. Hence, the chord of one degree is smaller than 1;2,49,52. It was already shown that it is larger than 1;2,49,48. Therefore, they rely on the amount whose distance from the two limits is the same [that is, the mean of the two limits, i.e. 1;2,49,50], neglecting the effect of the inequality, since it [i.e. the distance] belongs to the third [sexagesimal] unit. Enclosing the chord of one degree between two different amounts, each one of whose distance to it [i.e. the chord of one degree] is not known, is how it stands for them [i.e. astronomers] at this approximation.

The truth is that there is no great diligence behind what they rely on. But if they divide the circumference [of a circle] not into their division, i.e. 360 [degrees], and they make [the division] equal to 240 or 480 [parts], it is possible for them to derive the chord of one part with truth; because when a circle is 240 [parts], the chord of the fifth [i.e., pentagon] and sixth [i.e., hexagon] are known, and [from the first lemma] the chord of their difference is known, so the chord of 16 [parts] is known, and [from the third lemma] the chord of 8 [parts] is known, and the chord of 4 [parts] is known, and the chord of 2 [parts] is known, and the chord of one [part] is known in truth. If they do that, it is possible for them to derive these chords by this division and derive the chord of each of the 360 degrees. Q.E.D.

Arabic Text

List of the manuscripts we use

L: Leiden, University Library MS or. 98; ff. 1a–100a; copied on Saturday, 19 Ramaḍān 682 AH (i.e. 11 December 1283 AD) by Nūr al-Dīn b. Maḥmūd b. ʻIzzat al-Taflīsī.
Colophon:

والتاريخ النسخة المنقولة منه : وفرغ عن ارتجاله واضعه السموول يحيى عبّاس في سنة ٥٦١ . وتاريخ النقل كانت في سنة اثني وثمانين وستّمائة بيوم السبت التاسع عشر شهر المبارك رمضان ، كتب نور الدين بن محمود بن عزّت محمّد التفليسي المغربي المعروف .

The date of the transcription of it [i.e. the book]: it was completed by its orator and composer al-Samaw'al Yaḥyā ʻAbbās in the year 561 [AH, i.e. 1165/6 AD]. The date of the copying is on Saturday, the 19th of blessed Ramaḍān in 682 [AH, i.e. 11 December 1283 AD], written by Nūr al-Dīn b. Maḥmūd b. ʻIzzat al-Taflīsī al-Maghribī, a well-known [scholar].

B: Oxford, Bodleian Library, MS Hunt. 539; ff. 1a–99b; copied on Wednesday, 6th night of Rabīʻ al-Awwal in 988 AH (i.e. 21 April 1580 AD) by Muḥammad b. Yūsuf b. ʻAbd al-Qādī b. Muḥammad al-Dumyāṭī.
Colophon:

قال مصنّفه : وفرغ عن ارتجاله واضعه السموول يحيى عبّاس المغربي في سنة ٥٦١ (اتهي : فوق السطر) . هكذا وجدت في نسخة الأصل . تمّ الكتاب والحمد للّه وحده . وفرغ من كتابته وتصوير أشكاله الفقير محمّد بن يوسف بن عبد القاضي بن محمّد الدمياطي الحنفي الموقّت ، بمدرسة السلطان الملك الأشرف برسباي بالحرير (؟) ، بين يوم الأربعاء لستّ ليال خلود من شهر ربيع الأوّل بالحساب من شهور عامّ ثمان وثمانين وتسعمائة هجرية .

The author of it dictated: "it was completed by its orator and composer al-Samaw'al Yaḥyā ʻAbbās al-Maghribī in the year 561" (end of [quotation]). I found [a sentence] like that in a manuscript of the autograph. The book is finished, thank God alone. The copying of it and drawing of its diagrams were completed by Muḥammad b. Yūsuf b. ʻAbd al-Qādī b. Muḥammad al-Dumyāṭī, a Hanafite muwaqqit, at the Madrasa of the sultan al-Malik al-Ashraf Barsbāy –with silk garments– during Wednesday, 6th night of Rabī ʻ al-Awwal –according to the calculation of general months– in 988 AH [i.e. 21 April 1580 AD].

Their colophons confirm that al-Samaw'al composed this book in 561 AH (1165/6 AD). And the fact that they have almost the same sentence about the autograph implies that these two manuscripts were based on manuscripts connected to the author's autograph.

Signs used in the Arabic text and apparatus

< > added by the editor
(L-15b) denotes the beginning of the verso of 15th folio of the Leiden MS.
– Missing from
+ Added in

الباب الرابع[1] في بيان عجزهم عن تحقيق جيب درجة الذي عليه بني جدول الجيب

ومما يقف بهم دون تحقيق أعمالهم الحسابية (L-15b) عجزُهم عن تحقيق جيب درجة واحدة ومنه يتركّب جدول الجيب · ونحن الآن موردون الطريق الذي ادّاهم[2] إلى جيب درجة بالتقريب لا بالتحقيق ، وهو الطريق الذي اشتركوا جميعاً في سلوكه في استخراج جيب درجة · ولا طريق لهم سواه بعد أن تذكر[3] مقدّمات ذلك وهو أربع : آ أنّ كلّ قوسين يكون وتراهما معلومين فإنّ وتر تفاضلهما معلوم ، بَ أنّ كلّ قوسين معلومي الوترين (B-16b) فإنّ وتر مجموعهما يكون معلوماً ، جَ أنّ كلّ قوس معلومة الوتر فإنّ وتر نصفها معلوم ، دَ أنّ نسبة الوتر الأعظم إلى الوتر الأصغر أقلّ من نسبة قوس الوتر الأعظم إلى قوس الوتر الأصغر ·

فلمّا قرّروا هذه المقدّمات ، استخرجوا وتر الخمس ووتر السدس بالطرق الهندسية المؤدية إلى حقيقتها ، فصار وتر سَ معلوماً ووتر عبَ معلوماً · فبالمقدّمة الأولى يكون وتر ١٢ معلوماً ، وبالمقدّمة الثالثة يكون وتر ٦ معلوماً ووتر ٣ معلوماً ووتر جزء ونصف معلوماً ووتر ثلاثة أرباع جزء معلوماً ·

ويلزم من المقدّمة الرابعة أنّ وتر جزء نصف نسبته إلى وتر جزء واحد أقلّ من نسبة القوس التي هي جزء ونصف إلى القوس التي مقدارها جزء واحد ، فنسبة وتر < جزء > إلى وتر جزء ونصف أعظم من نسبة جزء إلى جزء ونصف · لكنّ نسبة الجزء إلى الجزء والنصف ثلثان ، فوتر جزء واحد أعظم من ثلثين وتر جزء ونصف · وثلثا وتر جزء ونصف هي ا ب مطمح ، فوتر واحد أعظم من ا ب مطمح ·

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[1] **الباب الرابع**] –B.   [2] أدّاهم] اداهما : L.   [3] تذكر] نذكر : B.

فإنّ نسبة جزء واحد إلى ثلاثة أرباع جزء أعظم من نسبة وتر جزء إلى وتر ثلاثة أرباع جزء · (L-16a) لكنّ نسبة الجزء إلى ثلاثة أرباع الجزء مثل وثلث ، فوتر جزء واحد أقلّ من مثل وثلث وتر الثلاثة ارباع · لكنّ مثل وثلث وتر الثلاثة ارباع هو ا ب مط نب ، فوتر جزء واحد أقلّ من ا ب مط نب · وقد كان تبيّن أنّه أعظم من ا ب مطح[4] ، فاعتمدوا على مقدارٍ بعده من الطرفين سواء ، إهمالاً لعلّة التفاوت لأنّه من الثوالث · وانحصار وتر جزء واحد بين مقدارين مختلفين لا يعلم بعده من أحدهما وقف بهم عند هذا التقريب ·

والحقّ أنّه ليس وراء ما اعتمدوه غاية في الاجتهاد · لكنّهم لو قسموا المحيط غير قسمتهم التي هي ٣٦٠ وعدلوا إلى ٢٤٠ أو إلى ٤٨٠ ، أمكنهم استخراج وتر جزء واحد بالحقيقة ؛ لأنّ الدائرة إذا كانت ٤٨٠ كان وتر الخمس والسدس (B-17a) معلوماً ، ووتر تفاضلهما معلوماً فوتر ١٦ معلوماً ، ووتر ٨ معلوم ووتر ٤ معلوم ووتر ٢ معلوم ووتر الواحد معلوم بالحقيقة · ولو فعلوا ذلك ، لأمكنهم بعد استخراج هذه الأوتار بهذه القسمة[5] استخراج وتر كلّ واحد من أجزاء ٣٦٠ · وذلك ما أردنا أن نبيّن ·

[4] From here, L omits the rest of the text. In the margin, the copyist of L adds the following note: هاهنا الكلام مبقور قد وقع بين ساحة أوراق أو سطور "Here splitting the sentence occurs between blanks of folios or lines". [5] القسمة] + استخراج هذه الاوتار و : B.

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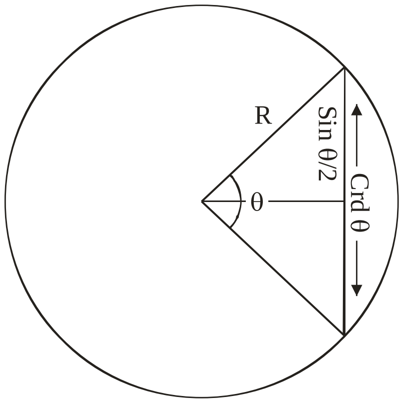

Figure 1: The relation between the chord and the sine

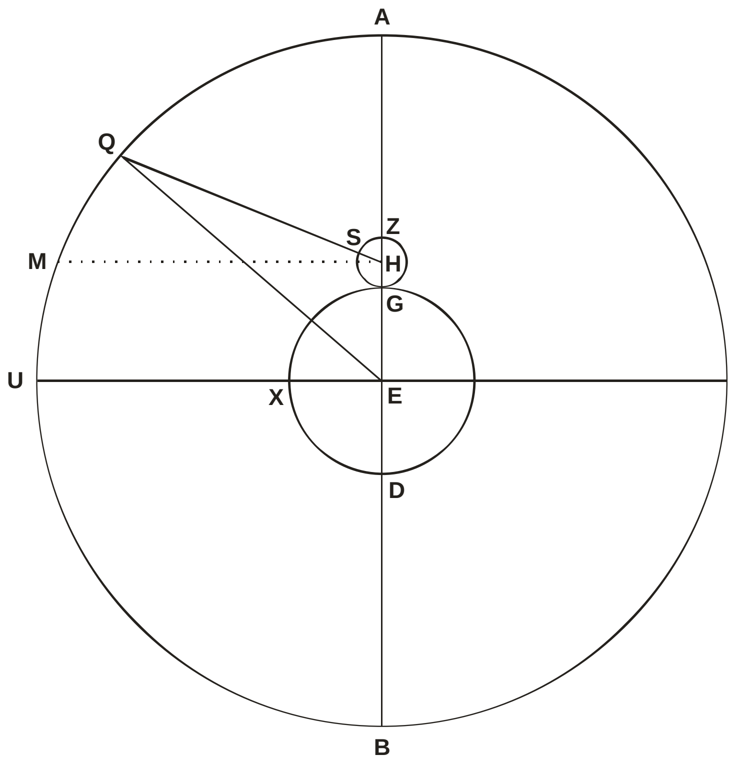

Figure 2: Determining the true altitude of a star

Table Constructed by al-Samaw'al on the Correction of the Altitude of the Stars				
Argument	Chord		Difference	Corrections
1	0;47,8	[+1]	[+1]	0;0,15
2	1;34,15	[+1]	[+1]	0;0,30
3	2;21,20			0;0,45
4	3;8,25			0;1,0
5	3;55,29	[+2]	[+3]	0;1,15
6	4;42,29	[+2]	[+3]	0;1,30
7	5;29,26	[+2]	[+2]	0;1,45
8	6;16,18			0;2,0
9	7;3,7	[-1]		0;2,15
10	7;49,53	[-1]		0;2,30
11	8;36,34			0;2,45
12	9;23,10			0;3,0
13	10;9,38	[-1]		0;3,15
14	10;56,2	[-1]		0;3,30
15	11;42,19	[-1]		0;3,45
16	12;28,29			0;4,0
17	13;14,29	[-2]		0;4,15
18	14;0,21	[-3]	[-1]	0;4,30
19	14;46,8	[-1]	[+1]	0;4,45
20	15;31,45			0;5,0
21	16;17,10	[-1]		0;5,15
22	17;2,26	[-1]	[+1]	0;5,30
23	17;47,32	[-1]	[+1]	0;5,45
24	18;32,28			0;6,0
25	19;17,9[11]	[-2]		0;6,15
26	20;1,39	[-3]		0;6,30
27	20;45,59	[-2]	[+1]	0;6,45
28	21;30,8	[+1]	[+1]	0;7,0
29	22;13,58	[-2]		0;7,15
30	22;57,37	[-3]		0;7,30
31	23;41,2	[-3]	[-1]	0;7,45
32	24;24,15			0;8,0
33	25;7,4	[-7]	[+5]	0;8,15
34	25;49,47[12]	[-3]		0;8,30
35	26;32,12	[-2]	[+1]	0;8,45
36	27;14,22			0;9,0
37	27;56,10	[-3]		0;9,15
38	28;37,43	[-3]	[+1]	0;9,30
39	[13]			0;9,45
40	30;0,0			0;10,0

Table 1: Al-Samaw'al's 480-part chord table

[11] Both manuscripts have the scribal error 19;7,9.
[12] **B** has 29;49,47.
[13] Both manuscripts have the incorrect 29;18,6.

Argument	Chord	Difference	Corrections
41	30;40,37 [-2]	[+1]	0;10,15
42	31;20,56 [-4]		0;10,30
43	32;0,58 [-3]		0;10,45
44	32;40,42		0;11,0
45	33;20,0 [-3]		0;11,15
46	33;59,0 [-4]		0;11,30
47	34;37,30 [-3]		0;11,45
48	35;16,2		0;12,0
49	35;53,56 [-2]	[+2]	0;12,15
50	36;31,28 [-4]	[+1]	0;12,30
51	37;8,42 [-2]	[+2]	0;12,45
52	37;45,34 [+1]	[+1]	0;13,0
53	38;21,56 [-3]	[+1]	0;13,15
54	38;57,57 [-4]	[+1]	0;13,30
55	39;33,36 [-3]	[+1]	0;13,45
56	40;8,52		0;14,0
57	40;43,37 [-4]		0;14,15
58	41;17,59 [-6]		0;14,30
59	41;51,58 [-5]	[-1]	0;14,45
60	42;25,35		0;15,0
61	42;58,37 [-4]	[+1]	0;15,15
62	[14]		0;15,30
63	[15]		0;15,45
64	44;35,19		0;16,0
65	45;6,33 [-4]	[+1]	0;16,15
66	45;37,22 [-6]		0;16,30
67	46;7,45 [-5]	[-1]	0;16,45
68	46;7,43 [-1]	[-1]	0;17,0
69	47;7,3 [-5]		0;17,15
70	47;35,58 [-6]	[+1]	0;17,30
71	48;4,26 [-5]		0;17,45
72	48;32,27 [-1]	[-1]	0;18,0
73	48;59,49 [-6]	[-1]	0;18,15
74	49;26,45 [-6]	[+1]	0;18,30
75	49;53,12 [-5]		0;18,45
76	50;19,13		0;19,0
77	[16]		0;19,15
78	51;9,23 [-7]		0;19,30
79	51;33,46 [-6]		0;19,45
80	51;57,41		0;20,0

Table 1, continued

[14] Both manuscripts have the incorrect 43;39,39.
[15] Both manuscripts have the incorrect 44;2,57.
[16] Both manuscripts have the incorrect 50;45,18.

Argument	Chord		Difference	Corrections
81	52;20,54	[-5]		0;20,15
82	52;43,38	[-6]	[+1]	0;20,30
83	53;5,52	[-5]		0;20,45
84	53;27,36	[-1]	[-1]	0;21,0
85	53;48,39	[-6]	[-1]	0;21,15
86	54;9,11	[-7]		0;21,30
87	54;29,14	[-5]	[+1]	0;21,45
88	54;48,46			0;22,0
89	55;7,33[17]	[-6]		0;22,15
90	55;25,51	[-7]		0;22,30
91	55;43,28	[-15]	[-9]	0;22,45
92	56;0,55	[+2]	[+2]	0;23,0
93	56;17,15	[-14]	[-8]	0;23,15
94	56;33,23	[-8]		0;23,30
95	56;48,51	[-6]		0;23,45
96	57;3,48			0;24,0
97	57;17,59	[-5]	[+1]	0;24,15
98	57;31,38	[-7]	[+1]	0;24,30
99	57;44,45	[-5]	[+1]	0;24,45
100	57;57,20			0;25,0
101	58;9,8	[-6]		0;25,15
102	58;20,24	[-8]		0;25,30
103	58;31,8	[-6]		0;25,45
104	58;41,24	[+4]		0;26,0
105	58;50,44	[-6]		0;26,15
106	58;59,35	[-8]	[+1]	0;26,30
107	59;7,53	[-7]	[-1]	0;26,45
108	59;15,41			0;27,0
109	59;22,39	[-6]		0;27,15
110	59;29,4	[-8]		0;27,30
111	59;34,57	[-6]		0;27,45
112	59;40,17			0;28,0
113	59;44,48	[-6]		0;28,15
114	59;48,45	[-9]	[-1]	0;28,30
115	59;52,12	[-6]		0;28,45
116	59;55,4			0;29,0
117	59;57,8	[-5]	[+1]	0;29,15
118	59;58,33	[-13]	[-5]	0;29,30
119	59;59,35	[-6]		0;29,45
120	60;0,0			0;30,0

Table 1, continued

[17] Both manuscripts have the scribal error 55;9,33.

Argument	Chord		Difference	Corrections
121	59;59,35	[-6]		0;30,15
122	59;58,33	[-13]	[-5]	0;30,30
123	59;57,8	[-5]	[+1]	0;30,45
124	59;55,4			0;31,0
125	59;52,12	[-6]		0;31,15
126	59;48,44	[-10]	[-2]	0;31,30
127	59;44,48	[-6]		0;31,45
128	59;40,17			0;32,0
129	59;34,57	[-6]		0;32,15
130	59;29,4	[-8]		0;32,30
131	59;22,39	[-6]		0;32,45
132	59;15,41			0;33,0
133	59;7,53	[-7]	[-1]	0;33,15
134	58;59,35	[-8]	[+1]	0;33,30
135	58;50,44	[-6]		0;33,45
136	58;41,20			0;34,0
137	58;31,8	[-6]		0;34,15
138	58;20,24	[-8]		0;34,30
139	58;9,8	[-6]		0;34;45
140	57;57,20			0;35,0
141	57;44,45	[-5]	[+1]	0;35,15
142	57;31,38	[-7]	[+1]	0;35,30
143	57;17,59	[-5]	[+1]	0;35,45
144	57;3,48			0;36,0
145	56;48,43	[-14]	[-8]	0;36,15
146	56;33,23[18]	[-8]		0;36,30
147	56;17,26	[-3]	[+3]	0;36,45
148	56;0,53			0;37,0
149	55;43,37	[-6]		0;37,15
150	[19]			0;37,30
151	55;7,33[20]	[-6]		0;37,45
152	54;48,46			0;38,0
153	54;29,13	[-6]		0;38,15
154	54;9,11	[-7]		0;38,30
155	53;48,39	[-6]	[-1]	0;38,45
156	53;27,37			0;39,0
157	53;5,23	[-34]	[-29]	0;39,15
158	52;43,22	[-22]	[-15]	0;39,30
159	52;21,53	[+54]	[+59]	0;39,45
160	51;57,41			0;40,0

Table 1, continued

[18] **B** has 56;33,43.
[19] Both manuscripts have the incorrect 55;26,34.
[20] Both manuscripts have the scribal error 55;9,33.

Argument	Chord		Difference	Corrections
161	51;33,46	[-6]		0;40,15
162	51;9,23	[-7]		0;40,30
163	50;44,32	[-5]		0;40,45
164	50;19,13			0;41,0
165	49;53,13[21]	[-4]	[+1]	0;41,15
166	49;26,45	[-6]	[+1]	0;41,30
167	48;59,49	[-6]	[-1]	0;41,45
168	48;32,27	[-1]	[-1]	0;42,0
169	48;4,26	[-5]		0;42,15
170	47;35,58	[-6]	[+1]	0;42,30
171	47;7,3	[-5]		0;42,45
172	46;37,43[22]	[-1]	[-1]	0;43,0
173	46;7,45	[-5]	[-1]	0;43,15
174	45;37,22	[-6]		0;43,30
175	45;6,33	[-4]	[+1]	0;43,45
176	44;35,19			0;44,0
177	[23]			0;44,15
178	[24]			0;44,30
179	42;58,37	[-4]	[+1]	0;44,45
180	42;25,35			0;45,0
181	41;51,58	[-5]	[-1]	0;45,15
182	41;17,59	[-6]		0;45,30
183	40;43,37	[-4]		0;45,45
184	40;8,52			0;46,0
185	39;33,36	[-3]	[+1]	0;46,15
186	38;57,57	[-4]	[+1]	0;46,30
187	38;21,56	[-3]	[+1]	0;46,45
188	37;45,34	[+1]	[+1]	0;47,0
189	37;8,42	[-2]	[+2]	0;47,15
190	36;31,28	[-4]	[+1]	0;47,30
191	35;53,55	[-3]	[+1]	0;47,45
192	35;16,2			0;48,0
193	34;37,40	[-3]		0;48,15
194	33;59,0	[-4]		0;48,30
195	33;20,0	[-3]		0;48,45
196	32;40,42			0;49,0
197	32;0,58	[-3]		0;49,15
198	31;20,56	[-4]		0;49,30
199	30;40,37	[-2]	[+1]	0;49,45
200	30;0,0			0;50,0

Table 1, continued

[21] **B** has 49;53,3.
[22] Both manuscripts have the scribal error 46;37,3.
[23] Both manuscripts have the incorrect 44;2,57.
[24] Both manuscripts have the incorrect 43;30,9.

Argument	Chord	Difference	Corrections
201	[25]		0;50,15
202	28;37,43 [-3]	[+1]	0;50,30
203	27;56,10 [-3]		0;50,45
204	27;14,22		0;51,0
205	26;32,12 [-2]	[+1]	0;51,15
206	25;49,47 [-3]		0;51,30
207	25;7,4 [-7]	[-5]	0;51,45
208	24;24,15		0;52,0
209	23;41,2 [-3]	[-1]	0;52,15
210	22;57,37 [-3]		0;52,30
211	22;13,58 [-2]		0;52,45
212	21;30,8 [+1]		0;53,0
213	20;45,59 [-2]	[+1]	0;53,15
214	20;1,39 [-3]		0;53,30
215	19;17,9 [-2]		0;53,45
216	18;32,28		0;54,0
217	17;47,32 [-1]	[+1]	0;54,15
218	17;2,26 [-1]	[+1]	0;54,30
219	16;17,10 [-1]		0;54,45
220	15;31,45		0;55,0
221	14;46,8 [-1]	[+1]	0;55,15
222	14;0,21 [-3]	[-1]	0;55,30
223	13;14,29 [-2]		0;55,45
224	12;28,29		0;56,0
225	11;42,19 [-1]		0;56,15
226	10;56,2 [-1]		0;56,30
227	10;9,38 [-1]		0;56,45
228	9;23,10		0;57,0
229	8;36,34	[+1]	0;57,15
230	7;49,53 [-1]		0;57,30
231	7;3,7 [-1]		0;57,45
232	6;16,18		0;58,0
233	5;29,24		0;58,15
234	4;42,27	[+1]	0;58,30
235	3;55,27	[+1]	0;58,45
236	3;8,25		0;59,0
237	2;21,20		0;59,15
238	1;34,14		0;59,30
239	0;47,6 [-1]	[-1]	0;59,45
240	0;0,0		1;0,0

Table 1, continued

[25] Both manuscripts have the incorrect 29;18,6.