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ON BLOCK-QUASI-TRIDIAGONAL MATRICES

José Vitória

Dpto. de Matemática Universidade de Coimbra

Abstract - We intend to invert (and to study the eigenvalues of) block-quasi-tridiagonal matrices. We also mention ways for handling the problem of eigenvalues of a block-quasi-tridiagonal matrix and we
obtain upper bounds for the spectral radius of a (certain) block-quasi-tridiagonal matrix which arises in the discretization of
partial differential equations of elliptic type, self-adjoint case.

## 1. About the inversion of block-quasi-tridiagonal matrices

1.1 - In this section we interest us for inverting block-quasi-tridiagonal matrices, that is to say, matrices of the form

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where all blocks are of the same order and commute in pairs. Such matrices arise in the discretization of elliptic pertial differential equations.

We need a result involving a matrix obtained from T by taking its determinant considering the (commuting) blocks as elements. Let  $\Delta_T$ : = dev T be such a matrix;  $\Delta_T$  is the formal determinant of T. It is known that det  $\Delta_T$  = det T. This result allows us to manipulate only matrices of low order, when solving large systems of linear equations and, like in our case, inverting large block-matrices.

1.2 - To invert the matrix T we shall use an hybrid method: classical partitioning in four blocks plus a recurring procedure.

Let us partition and note as follows

$$T = \begin{pmatrix} & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & &$$

where P is a block-tridiagonal matrix.

It is known that the inverse of a matrix partitioned in that manner is

$$T^{-1}$$
: =  $\left(\begin{array}{c|c} -K & -1 & -L \\ -M & -1 & N \end{array}\right)$ 

with the same partitioning and where

$$K:=P^{-1}-P^{-1}QM$$
,  $M:=-NRP^{-1}$ ,  $L:=-P^{-1}QN$  and  $N:=(S-RP^{-1}Q)^{-1}$ .

In this way we need only to invert two matrices: P and  $(S-RP^{-1}Q)$ . So we have to invert a large matrix P, with commuting blocks. For achieving this, one can use a recurring procedure. We obtain a matrix  $P^{-1}=(X_{ij})$  partitioned in the same way as P and where the blocks  $X_{ij}$  also commute in pairs.

Let us denote by  $P_{I(i,j)}$  the matrix obtained from P by replacing the  $j^{th}$  block-column with the matrix  $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$   $j^{th}$  block-line.

Then, if we put

$$\Delta$$
: = dev P and  $\eta_{ij}$ : = dev P<sub>I(i,j)</sub>, (i,j=1,2,...,n-1),

we obtain

$$X_{ij} = \Delta^{-1} \eta_{ij}$$
, (i,j=1,2,...,n-1)

1.3 - As we have seen, for inverting a block-quasi-tridiagonal matrix we need only to invert two matrices: S-RP $^{-1}$ Q and  $\Delta$ : =dev P. But, in pratice, we invert two matrices of the same order, instead of inverting a low order matrix S-RP $^{-1}$ Q and a high order matrix P. We remark that the matrices S-RP $^{-1}$ Q and  $\Delta$  have the order of the original blocks  $A_{4.4}$ .

## 2. About the eigenvalues of a (certain) block-quest-tridiagonal matrix

In this section we interest us for the eigenvalue problem in block--quasi-tridiagonal matrices. Here we get upper bounds for the absolute value of the eigenvalues by using a matricial norm:

Upper bounds for the spectral radius of a block-quasi-tridiagonal matrix, which arise in the discretization of partial differential equations of elliptic type, self-adjoint case.

Given the matrix

we look for upper bounds of  $\rho(T)$ , where  $\rho(T)$  is the spectral radius of the matrix T, that is to say,  $\rho(T)$ : = Max  $|\lambda(T)|$ , where  $\lambda(T)$  is any eigenvalue of T.

We take the (scalar) norm  $\|\cdot\|_1^{\{1\}}$ ,  $(i=1,\infty)$ , of each block, so obtaining a matricial norm  $M_1^{\{T\}}$ ,  $(i=1,\infty)$ . It is known, that  $p(T) \leq p(M_1^{\{T\}})$ ,  $(i=1,\infty)$ . And it is also known that  $p(A) \leq \|A\|_{\hat{J}}$ ,  $(j=1,\infty)$ , for any matrix A, and any (subordinate) matrix norm  $\|\cdot\|_{\hat{J}}$ .

(1) For 
$$B = (\beta_{ij}) \in M_{r,s}(K)$$
,  $K = R$  (or C), we let 
$$||B||_1 := \max_{\substack{i=1,2,\dots,s\\i=1,2,\dots,s}} \{\sum_{i=1}^r |\beta_{ij}|\}, ||B||_{\infty} := \max_{\substack{i=1,2,\dots,r\\i=1,2,\dots,r}} \{\sum_{j=1}^s |\beta_{ij}|\}.$$

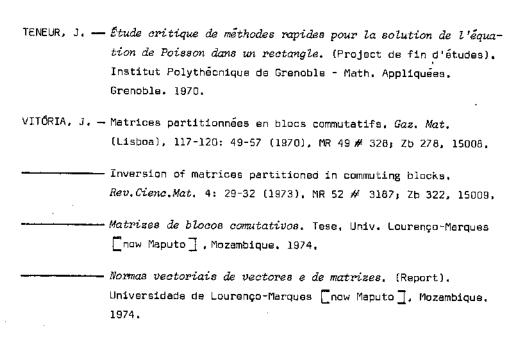
$$\rho(T) \le ||A||_1 + 2||B||_1$$
, (1=1,\infty)

a very simple upper bound for the eigenvalues of I.

### Remark

The complete version of this paper is to appear in "Revista da Universidade de Coimbra".

#### PRINCIPAL REFERENCES



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