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ON BLOCK-QUASI-TRIDIAGONAL MATRICES

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Abstract - We intend to invert (and to study the eigenvalues of) block-quasi-tridiagonal matrices. We also mention ways for handling the problem of eigenvalues of a block-quasi-tridiagonal matrix and we obtain upper bounds for the spectral radius of a (certain) block-quasi-tridiagonal matrix which arises in the discretization of partial differential equations of elliptic type, self-adjoint case.

1. About the inversion of block-quasi-tridiagonal matrices

1.1 - In this section we interest us for inverting block-quasi-tridiagonal matrices, that is to say, matrices of the form

$$T := \begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \dots & 0 & 0 & 0 & A_{1n} \\ A_{21} & A_{22} & A_{23} & 0 & \dots & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & & & & & \\ & & & & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ A_{n1} & 0 & 0 & 0 & \dots & 0 & 0 & A_{n,n-1} & A_{nn} \end{pmatrix}$$

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where all blocks are of the same order and commute in pairs. Such matrices arise in the discretization of elliptic partial differential equations.

We need a result involving a matrix obtained from T by taking its determinant considering the (commuting) blocks as elements. Let $\Delta_T := \operatorname{dev} T$ be such a matrix; Δ_T is the formal determinant of T. It is known that $\det \Delta_T = \det T$. This result allows us to manipulate only matrices of low order, when solving large systems of linear equations and, like in our case, inverting large block-matrices.

1.2 - To invert the matrix T we shall use an hybrid method: classical partitioning in four blocks plus a recurring procedure.

Let us partition and note as follows

$$T = \left(\begin{array}{ccccc|c} & & \# & & & A_{1n} \\ & & & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ & & & & & A_{n-1,n} \\ \hline A_{n1} & 0 & 0,\ldots 0 & & A_{n,n-1} & A_{nn} \end{array}\right) := \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right)$$

where P is a block-tridiagonal matrix.

It is known that the inverse of a matrix partitioned in that manner is

$$T^{-1} := \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right)$$

with the same partitioning and where

$$K := P^{-1} - P^{-1}QM,\quad M := -NRP^{-1},\quad L := -P^{-1}QN \quad \text{and} \quad N := (S-RP^{-1}Q)^{-1}.$$

In this way we need only to invert two matrices: P and $(S-RP^{-1}Q)$. So we have to invert a large matrix P, with commuting blocks. For achieving this, one can use a recurring procedure. We obtain a matrix $P^{-1}=(X_{ij})$ partitioned in the same way as P and where the blocks X_{ij} also commute in pairs.

Let us denote by $P_{I(i,j)}$ the matrix obtained from P by replacing the j^{th} block-column with the matrix

$$\begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ block-line.} \\ \\ \\ \\ \end{array} \end{array}$$

Then, if we put

$$\Delta := \operatorname{dev} P \quad \text{and} \quad \eta_{ij} := \operatorname{dev} P_{I(i,j)}, \quad (i,j=1,2,\ldots,n-1),$$

we obtain

$$X_{ij} = \Delta^{-1} \eta_{ij}\ , \quad (i,j=1,2,\ldots,n-1)$$

1.3 - As we have seen, for inverting a block-quasi-tridiagonal matrix we need only to invert two matrices: $S-RP^{-1}Q$ and Δ: =dev P. But, in pratice, we invert two matrices of the same order, instead of inverting a low order matrix $S-RP^{-1}Q$ and a high order matrix P. We remark that the matrices $S-RP^{-1}Q$ and Δ have the order of the original blocks A_{ij}.

2. About the eigenvalues of a (certain) block-quasi-tridiagonal matrix

In this section we interest us for the eigenvalue problem in block-quasi-tridiagonal matrices. Here we get upper bounds for the absolute value of the eigenvalues by using a matricial norm:

Upper bounds for the spectral radius of a block-quasi-tridiagonal matrix, which arise in the discretization of partial differential equations of elliptic type, self-adjoint case.

Given the matrix

$$T: = \begin{pmatrix} A & B & 0 & 0 & \ldots & 0 & 0 & B \\ B & A & B & 0 & \ldots & 0 & 0 & 0 \\ 0 & B & A & B & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & B & A & B \\ B & 0 & 0 & 0 & \ldots & 0 & B & A \end{pmatrix}$$

we look for upper bounds of $\rho(T)$, where $\rho(T)$ is the spectral radius of the matrix T, that is to say, $\rho(T) := \text{Max}\, |\lambda(T)|$, where $\lambda(T)$ is any eigenvalue of T.

We take the (scalar) norm $\|\cdot\|_i^{(1)}$, $(i=1,\infty)$, of each block, so obtaining a matricial norm $M_i(T)$, $(i=1,\infty)$. It is known, that $\rho(T) \leq \rho(M_i(T))$, $(i=1,\infty)$. And it is also known that $\rho(A) \leq \|A\|_j$, $(j=1,\infty)$, for any matrix A, and any (subordinate) matrix norm $\|\cdot\|_i$.

(1) For $B = (\beta_{ij}) \in M_{r,s}(K)$, K=R (or C), we let

$$\|B\|_1 := \operatorname*{Max}_{j=1,2,\ldots,s} \{\sum_{i=1}^{r} |\beta_{ij}|\}\ , \quad \|B\|_\infty := \operatorname*{Max}_{i=1,2,\ldots,r} \{\sum_{j=1}^{s} |\beta_{ij}|\}.$$

Hence we have

$$\rho(T) \leq \|A\|_i + 2\|B\|_i, \quad (i=1,\infty)$$

a very simple upper bound for the eigenvalues of T.

Remark

The complete version of this paper is to appear in "Revista da Universidade de Coimbra".

PRINCIPAL REFERENCES

TENEUR, J. — *Étude critique de méthodes rapides pour la solution de l'équation de Poisson dans un rectangle.* (Project de fin d'études). Institut Polythécnique de Grenoble - Math. Appliquées. Grenoble. 1970.

VITÓRIA, J. — Matrices partitionnées en blocs commutatifs. *Gaz. Mat.* (Lisboa), 117-120: 49-57 (1970), MR 49 # 328; Zb 278, 15008.

——— Inversion of matrices partitioned in commuting blocks. *Rev.Cienc.Mat.* 4: 29-32 (1973), MR 52 # 3187; Zb 322, 15009.

——— *Matrizes de blocos comutativos.* Tese, Univ. Lourenço-Marques [now Maputo], Mozambique. 1974.

——— *Normas vectoriais de vectores e de matrizes.* (Report). Universidade de Lourenço-Marques [now Maputo], Mozambique. 1974.

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