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INFRATOPOLOGICAL BORNOLOGIES AND MACKEY-CONVERGENCE OF SERIES

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SYNOPSIS: In this paper some aspects of infratopological bornologies are discussed. First, it is shown that bornologies with countable basis are infratopological. Second, it is shown that the convergence of series presents some pathologies beyond this class.

We follow closely the terminology of [5] and [4], where the main topics concerning bornological spaces can be found. Roughly speaking, a bornology is infratopological when it can be defined from a topology in the usual way: the bounded sets are those which are absorbed by zero neighbourhoods. More precisely, let (E, β) be a linear bornological space, and $\tau\beta$ the Mackey closure topology associated to this space. The Von Neumann bornology of $\tau\beta$, denoted by $B\tau\beta$, is defined as follows: a subset $B \subset E$ is bounded when for every zero neighbourhood V, there is some $\lambda > 0$ such that $B \subset \lambda V$ (B is absorbed by V). The bornology β is said to be infratopological when $\beta = B\tau\beta$. A more useful characterization, which does not involve any topology is the following [1]:

<u>Proposition 1. A linear bornological space (E, β) is infratopological if and only if every subset $B \subset E$, which is absorbed for every bornivorous subset of E, is bounded in (E, β) (a subset $U \subset E$ is bornivorous when U absorbs every bounded set of (E, β)).</u>

The bounded subsets of an ordinary topological vector space (the Von Neumann bornology) form an infratopological bornology. Nevertheless, bornologies which are not infratopological can appear when we deal with equicontinuous sets.

We refer to [1], example p. 166, for an equicontinuous bornology not Kolmogorov and consequently not infratopological.

We see next that spaces with countable basis are also infratopological.

<u>Proposition 2.</u> <u>Let</u> (E, β) <u>be a separated convex bornological space and suppose that</u> β <u>admits a countable basis</u> $(B_n)_{n \geq 1}$. <u>Then</u> β <u>is infratopological.</u>

Proof. We can suppose that the B_n's are absolutely convex and that the sequence is increasing. Thus, if we denote by $|| \cdot ||_n$ the gauge of B_n, we obtain an increasing sequence of norms, each $|| \cdot ||_n$ defined in $E_n = \text{Span } B_n$. To make it easier, we define $||x||_n = \infty$ for $x \in E \setminus E_n$.

Let A be an unbounded subset of (E, β). then, for every n, we can find $x_n \in A$ with $||x_n||_n > n^2$. For every k, we define:

$$\lambda_k = \frac{1}{k} \min (||x_1||_k , \ldots, ||x_k||_k , 1),$$

and putting:

$$V = \bigcup_{k \geq 1} \lambda_k \cdot B_k ,$$

we obtain a bornivorous subset of E, and V does not absorb A, because, for $\varepsilon > 0$ arbitrary, if $n > \varepsilon^{-1}$, x_n does not belong to any $\lambda_k B_k$:

a) if $k \geq n$, we have $|| \varepsilon x_n||_k \geq \varepsilon k \lambda_k \geq \varepsilon n \lambda_k > \lambda_k$

b) if $1 \leq k < n$, we have $|| \varepsilon x_n||_k \geq || \varepsilon x_n||_n > \varepsilon n^2 > n > \lambda_k$.

Applying Proposition 1, we conclude that (E, β) is infratopological. //

People which is used to linear bornological spaces knows that such a space can fail to have the properties that one hopes to find in Functional Analysis, unless some extra assumptions are taken. Restriction to infratopological spaces is an example of extra assumption under which some patologies do not appear. We will see here that the convergence of series does not work in the usual way when we place ourselves outside the class of infratopological bornologies.

A series $\sum_{n \geq 1} x_n$ is said to be Mackey-convergent in a linear bornological space when the sequence $(\sum_{k=1}^{n} x_k)_{n \geq 1}$ of partial sums is Mackey-convergent. The following results are well-known for the topological convergence in a topological vector space. Our convergence has been considered in [6] .

<u>Proposition 3.</u> (i) <u>Let</u> (E, β) <u>be a Mackey-complete convex bornological space,</u> $(x_n)_{n \geq 1}$ <u>a bounded sequence in</u> (E, β), <u>and</u> $(a_n)_{n \geq 1}$ <u>a sequence of real numbers such that</u> $\sum_{n \geq 1} |a_n|$ <u>is finite. Then,</u> $\sum_{n \geq 1} a_n x_n$ <u>is Mackey-convergent.</u>

(ii) <u>Let</u> (E, β) <u>be an infratopological convex bornological space, and</u> $(x_n)_{n \geq 1}$, <u>a sequence such that, for every sequence of real numbers</u> $(a_n)_{n \geq 1}$ <u>with</u> $\sum_{n \geq 1} |a_n|$ <u>finite, the series</u> $\sum_{n \geq 1} a_n \cdot x_n$ <u>is Mackey-convergent. Then</u> $(x_n)_{n \geq 1}$ <u>is bounded.</u>

Proof. (i) . Take a bounded disk B containing $(x_n)_{n\geq 1}$, and denote by $\|\cdot\|_B$ its gauge. The sequence of partial sums $(\sum_{k=1}^{n} a_k x_k)_{n\geq 1}$ is a Cauchy sequence with respect to $\|\cdot\|_B$, and, (E, β) being Mackey-complete, it is Mackey-convergent in (E, β).

(ii) If $(x_n)_{n\geq 1}$ is unbounded, there is a bornivorous subset V which does not absorb this sequence. Replacing $(x_n)_{n\geq 1}$ by a subsequence if necessary, we can suppose that $x_n \notin n^2 V$ for every n. Then $(n^{-2}. x_n)_{n\geq 1}$ is not Mackey-convergent to zero. //

As we have previously announced, be will show, through a counterexample, that the assumption on (E, β) in part (ii) of the preceding Proposition is not superfluous.

Example 4. Let E be the space of measurable real functions on the unit interval $I = [0,1]$ (with the standard identification), provided with the order bornology: a subset $A \subset E$ is bounded if there is some $g \in E$, with $g \geq 0$ and $|f| \leq g$ for every $f \in A$. This bornology is convex, but not the associated infratopological bornology. So E is not infratopological. Details on this fact can be found in [3] . Moreover the Mackey-convergence relative to the order bornology coincides with the almost everywhere convergence.

We consider the sequence $(f_n)_{n\geq 1}$ defined as follows: (we denote by χ_T the characteristic function of a subset T of I):

$f_1 = \chi_I$

$f_2 = 2.\chi_{[0,\frac{1}{2}]}$ $\qquad f_3 = 2.\chi_{(\frac{1}{2},1]}$

$f_4 = 3.\chi_{[0,1/4]}$ $\qquad f_5 = 3.\chi_{(\frac{1}{4},\frac{1}{2}]}$, and so on.

Obviously, $(f_n)_{n \geq 1}$ is not bounded in the order bornology. Nevertheless,

$\sum_{n \geq 1} a_n f_n$ converges almost everywhere when $\sum_{n \geq 1} |a_n|$ converges.

It suffices to consider the case in which $a_n \geq 0$ for all n. Moreover, it suffices, in this case, to prove that $\sum_{n \geq 1} a_n f_n$ converges in measure, because for an increasing sequence, both types of convergence are equivalent.

For each n, we denote:

$$b_n = \sum_{2^{n-1} \leq j < 2^n} a_j \qquad g_n = \sum_{2^{n-1} \leq j < 2^n} a_j f_j$$

Take a fixed n. On each interval $\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right]$, the function g_n takes the constant value na_j for some j, $2^{n-1} \leq j < 2^n$. Therefore, g_n exceeds b_n on this interval if and only if $a_j > n^{-1} b_n$. But this happens for at most n-1 of these intervals. Thus:

$$m(\{ x : g_n(x) > b_n \}) \leq \frac{n}{2^{n-1}}$$

Take now an arbitrary $\varepsilon > 0$, and choose N such that:

$$\sum_{n \geq N} b_n < \varepsilon, \qquad \sum_{n \geq N} \frac{n}{2^{n-1}} < \varepsilon.$$

Then:

$$m(\{x: \sum_{n \geq N} g_n(x) > \varepsilon\}) \leq \sum_{n \geq N} m(\{x: g_n(x) > b_n\}) \leq \sum_{n \geq N} \frac{n}{2^{n-1}} < \varepsilon.$$

This argument proves that $\sum_{n \geq 1} g_n$ converges in measure, and so does $\sum_{j \geq 1} a_j f_j$.//

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