A GENERALIZATION OF WRIGHT'S INEQUALITY

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Let R be a commutative ring with identity, E an R-module and $x_1, \ldots, x_r \epsilon R$ a multiplicity system on E (see [1], p.295). Then the length $\ell_R(E/_{(x_1^{n_1}, \ldots, x_r^{n_r})E})$ is finite for any positive integers $n_1, \ldots, n_r$, and Wright's inequality (see [1] p.296) says

$$\ell_R(E/_{(x_1^{n_1}, \ldots, x_r^{n_r})E}) \leq n_1 \cdots n_r \cdot \ell_R(E/_{(x_1, \ldots, x_r)E}),$$

for arbitrary $n_1, \ldots, n_r$.

This inequality can be written as

$$\ell_R H_o K(x_1^{n_1}, \ldots, x_r^{n_r}|E) \leq n_1 \cdots n_r \cdot \ell_R H_o K(x_1, \ldots, x_r|E),$$

where $K(x_1^{n_1}, \ldots, x_r^{n_r}|E)$ denotes the Koszul complex defined by E and the elements $x_1^{n_1}, \ldots, x_r^{n_r}$.

In this paper we establish that for a Noetherian module similar inequalities hold for the higher Koszul homology modules, i.e., for $i \geq 0$, $\ell H_i K(x_1^{n_1}, \ldots, x_r^{n_r}|E) \leq n_1 \cdots n_r \cdot \ell H_i K(x_1, \ldots, x_r|E)$. Moreover, the same is true for the higher Euler-Poincaré characteristics of the Koszul complexes $K(x_1^{n_1}, \ldots, x_r^{n_r}|E)$, i.e., for $i \geq 0$, we have $\chi_i(x_1^{n_1}, \ldots, x_r^{n_r}|E) \leq n_1 \cdots n_r \cdot \chi_i(x_1, \ldots, x_r|E)$, where, by definition,

$$\chi_i(x_1, \ldots, x_r|E) = \sum_{j \geq i} (-1)^{j-i} \ell H_j K(x_1, \ldots, x_r|E).$$

Actually the inequality for χ_i is an equality if i=0 (see [1] p.311).

We also prove that the functions $\ell \, H_i K(x_1^{n_1}, \ldots, x_r^{n_r} | E)$ and $\chi_i(x_1^{n_1}, \ldots, x_r^{n_r} | E)$ increase with the exponents $n_1, \ldots, n_r$.

In what follows R denotes a commutative ring with identity, E is a Noetherian module over R and $x_1, \ldots, x_r \epsilon R$ is a system of multiplicity on E. This will ensure that all the lengths which appear are indeed finite, though some of the results would also hold without assuming the lengths to be finite. We denote the length by ℓ or ℓ_R.

Lemma 1. Let E be an R-module and $x_1, \ldots, x_r, y$ be elements of R. Then, for any $i \geqslant 0$,

$$\ell_R H_i K(x_1, \ldots, x_r | E) \leqslant \ell_R H_i K(x_1 y, x_2, \ldots, x_r | E).$$

Proof. The inequality follows from the exact sequence (cf. [2] p.IV-2)

$$0 \longrightarrow \frac{H_i K(x_2, \ldots, x_r | E)}{x_1 H_i K(x_2, \ldots, x_r | E)} \longrightarrow H_i K(x_1, x_2, \ldots, x_r | E) \longrightarrow$$

$$\longrightarrow (0:x_1)_{H_{i-1}K(x_2, \ldots, x_r | E)} \longrightarrow 0,$$

and the corresponding one for $H_i K(x_1 y, x_2, \ldots, x_r | E)$, by observing that both $(0:x_1)_{H_{i-1}K(x_2, \ldots, x_r | E)} \subseteq (0:x_1 y)_{H_{i-1}K(x_2, \ldots, x_r | E)}$

and $x_1 H_i K(x_2, \ldots, x_r | E) \supseteq x_1 y H_i K(x_2, \ldots, x_r | E)$. #

Bearing in mind that the Koszul homology modules do not depend on the order of the elements defining it, we get the following

Proposition 2. For any $i \geqslant 0$, the mapping from $\mathbb{N}^r$ to $\mathbb{N}$ defined by

$$(n_1, \ldots, n_r) \longmapsto \ell H_i K(x_1^{n_1}, \ldots, x_r^{n_r} | E)$$

is _increasing_, i.e., $n_1 \leqslant m_1, \ldots, n_r \leqslant m_r$ imply

$$\ell H_i K(x_1^{n_1}, \ldots, x_r^{n_r} | E) \leqslant \ell H_i K(x_1^{m_1}, \ldots, x_r^{m_r} | E). \quad \#$$

Lemma 3. If $a \in R$, then we have:

i) $\ell_E(0 : a^n) \leqslant n \cdot \ell_E(0 : a)$, and

ii) $\ell(E/_{a^n E}) \leqslant n \cdot \ell(E/_{aE})$.

Proof. By induction on n. From the exact sequence

$$R/_{aR} \xrightarrow{\ .a^{n-1}\ } R/_{a^n R} \longrightarrow R/_{a^{n-1} R} \longrightarrow 0,$$

if we apply $\mathrm{Hom}_R(\ , E)$, we get i), for $\mathrm{Hom}_R(R/_{a^n R}, E) \cong 0 :_E a^n$,

and if we apply $. \otimes_R E$, we get ii), for $(R/_{a^n R}) \otimes_R E \cong E/_{a^n E}$. $\#$

Proposition 4. The following inequality holds for all $i \geqslant 0$,

$$\ell H_i K(x_1^n, x_2, \ldots, x_r | E) \leqslant n \cdot \ell H_i K(x_1, x_2, \ldots, x_r | E).$$

Proof. From the exact sequences (see [2] p.IV-2)

$$0 \longrightarrow H_0 K(a | H_i K(x_2, \ldots, x_r | E)) \longrightarrow H_i K(a, x_2, \ldots, x_r | E)$$

$$\longrightarrow H_1 K(a | H_{i-1} K(x_2, \ldots, x_r | E)) \longrightarrow 0,$$

with $a = x_1$ or x_1^n , we get

$$\ell H_i K(x_1^n, x_2, \ldots, x_r | E) = \ell \left(\frac{H_1 K(x_2, \ldots, x_r | E)}{x_1^n H_i K(x_2, \ldots, x_r | E)} \right) + \ell(0 : x_1^n)_{H_{i-1}(x_2, \ldots, x_r | E)}$$

$$\leqslant n \cdot \ell \left(\frac{H_i K(x_2, \ldots, x_r | E)}{x_1 H_i K(x_2, \ldots, x_r | E)} \right) + n \cdot \ell(0 : x_1) H_{i-1} K(x_2, \ldots, x_r | E) =$$

$$= n \cdot \ell H_i K(x_1, x_2, \ldots, x_r / E),$$

the inequalities being by virtue of lemma 3. #

Again by the independence of the Koszul homology modules with respect to the order of the elements, Proposition 4 yields the following theorem which generalizes Wright's inequality.

Theorem 5. For any $i \geqslant 0$, and any $n_1, \ldots, n_r \geqslant 0$, we have

$$\ell H_i K(x_1^{n_1}, \ldots, x_r^{n_r} | E) \leqslant n_1 \ldots n_r \cdot \ell H_i K(x_1, \ldots, x_r | E). \text{ #}$$

Let us consider now the higher Euler-Poincaré characteristics. Observe first that $\chi_0(x_1^{n_1}, \ldots, x_r^{n_r} | E) = n_1 \ldots n_r \cdot \chi_0(x_1, \ldots, x_r | E)$ and that $\chi_0(x_1, \ldots, x_r | E) \geqslant 0$ (cf. [1] p.311). For the higher characteristics we have

Proposition 6. For all $i \geqslant 0$, the mapping

$$(n_1, \ldots, n_r) \longmapsto \chi_i(x_1^{n_1}, \ldots, x_r^{n_r} | E)$$

from $\mathbb{N}^r$ to $\mathbb{N}$ is increasing, i.e., $n_1 \leqslant m_1, \ldots, n_r \leqslant m_r$, imply

$$\chi_i(x_1^{n_1}, \ldots, x_r^{n_r} | E) \leqslant \chi_i(x_1^{m_1}, \ldots, x_r^{m_r} | E).$$

Proof. By [2] p.IV-56, we have

$$\chi_i(a, x_2, \ldots, x_r | E) = \ell H_1 K(a | H_{i-1} K(x_2, \ldots, x_r | E)) +$$

$$+ x_0(a|x_i(x_2,\ldots,x_r|E)).$$

Setting $a = x_1$ or x_1y and using the multiplicativity of x_0
(see [1] p.309 Thm.7), we deduce

$$x_i(x_1,x_2,\ldots,x_r|E) \leqslant x_i(x_1y,x_2,\ldots,x_r|E).$$

From this we get, for $n \leqslant m$, that

$$x_i(x_1^n,x_2,\ldots,x_r|E) \leqslant x_i(x_1^m,x_2,\ldots,x_r|E).$$

Proceeding equally with the other variables (x_i does not
depend on the order of the elements), we get the result. #

We finish with a theorem on higher Euler-Poincaré
characteristics similar to theorem 5.

Theorem 7. For any $i \geqslant 0$, and any $n_1,\ldots,n_r \geqslant 0$, we have

$$x_i(x_1^{n_1},\ldots,x_r^{n_r}|E) \leqslant n_1\ldots n_r \cdot x_i(x_1,\ldots,x_r|E).$$

Proof. It is enough to prove

$$x_i(x_1^n,x_2,\ldots,x_r|E) \leqslant n \cdot x_i(x_1,x_2,\ldots,x_r|E),$$

and this can be done by considering the formula used in the
proof of the preceding proposition with $a=x_1$ or x_1^n. We get

$$x_i(x_1^n,x_2,\ldots,x_r/E) =$$

$$= \ell(0:x_1^n)_{H_{i-1}K(x_2,\ldots,x_r/E)} + x_0(x_1^n|x_i(x_2,\ldots,x_r|E))$$

$$\leqslant n \cdot \ell(0:x_1)_{H_{i-1}K(x_2,\ldots,x_r|E)} + n \cdot x_0(x_1|x_i(x_2,\ldots,x_r|E)) =$$

$$= n \cdot x_1(x_1, x_2, \ldots, x_r | E),$$

the inequality being justified by lemma 3. #

References

[1]   Northcott, D.G.   Lessons on Rings, Modules and Multiplicities, Cambridge U.P.(1968).

[2]   Serre, J-P.   Algèbre locale: Multiplicités, Lecture Notes in Math. No.11, Springer, Berlin(1975).

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