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PARTIALLY ORDERED GROTHENDIECK GROUPS

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Motivation for the study of partially ordered abelian groups has come from many different parts of mathematics, for mathematical systems with compatible order and additive (or linear) structures are quite common. This is particularly evident in functional analysis, where spaces of various kinds of real-valued functions provide impetus for investigating partially ordered real vector spaces. In the past decade, the observation that a Grothendieck group (such as K_0 of a ring or algebra) often possesses a natural partially ordered abelian group structure has led to new directions of investigation, whose goals have been to develop structure theories for certain types of partially ordered abelian groups to the point where effective application to various Grothendieck groups is possible. Such recent developments in the area of partially ordered abelian groups are the subject of this note. We present a sketch of the construction of Grothendieck groups as abelian groups equipped with pre-orderings that are often partial orderings, together with brief sketches of several situations in which the theory of partially ordered abelian groups can be applied, via the Grothendieck groups K_0, to the study of certain rings and C*-algebras. This discussion is somewhat cursory, in the interest of avoiding technicalities, and for reasons of space. For

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the same reasons, we do not discuss the recent applications of partially ordered abelian groups to topological Markov chains [27, 3, 20, 21] or to positive polynomials and compact group actions [22, 23, 25, 26].

A Grothendieck group is an invariant attached to a collection of objects, such as the vector bundles on a given topological space, the finitely generated projective modules over a given ring, or the projection operators associated with a given C*-algebra. The construction only requires a collection of objects equipped with a notion of isomorphism and with some means of combining any two objects from the collection into a third object. While it is traditional to develop Grothendieck groups using short exact sequences as the means of combining objects, in many important applications this reduces to direct sums (or direct products). Since the construction process is simpler using direct sums, we shall restrict our discussion to that case.

Thus for our basic data we take a set $\mathcal{P}$ of objects in some category, such that $\mathcal{P}$ has a zero object and every finite set of objects in $\mathcal{P}$ has a direct sum (coproduct) in $\mathcal{P}$. The easiest algebraic system to build using this data is an abelian semigroup, whose elements are the isomorphism classes of objects in $\mathcal{P}$, and whose operation is addition induced from the direct sum operation in $\mathcal{P}$. However, the semigroup obtained in this manner need not have cancellation, and so cannot always be embedded in an abelian group. This problem can be circumvented either by reducing the semigroup modulo a suitable congruence relation, or, more conveniently, by using an equivalence relation on $\mathcal{P}$ slightly coarser than isomorphism, as follows.

Objects $A,B \in P$ are said to be <u>stably isomorphic</u> (in P) if and only if there is an object $C \in P$ such that $A\oplus C \simeq B\oplus C$. Stable isomorphism is an equivalence relation on P, and we write $[A]$ for the stable isomorphism class of an object $A \in P$. Let $\mathrm{Grot}(P)^+$ denote the collection of all stable isomorphism classes in P. The direct sum operation in P induces an addition operation in $\mathrm{Grot}(P)^+$, where $[A]+[B] = [A\oplus B]$ for all $A,B \in P$, and using this operation, $\mathrm{Grot}(P)^+$ becomes an abelian semigroup with cancellation. By formally adjoining additive inverses to $\mathrm{Grot}(P)^+$, we obtain an abelian group $\mathrm{Grot}(P)$, the <u>Grothendieck group of</u> P. (Reminder: if short exact sequences are available in P, the Grothendieck group constructed from P using short exact sequences may well be different from the group constructed here.) All elements of $\mathrm{Grot}(P)$ have the form $[A]-[B]$ for $A,B \in P$, and elements $[A]-[B]$ and $[C]-[D]$ in $\mathrm{Grot}(P)$ are equal if and only if $A\oplus D$ is stably isomorphic to $B\oplus C$.

For example, if P is the collection of all real (complex) vector bundles on some compact Hausdorff space X, then $\mathrm{Grot}(P)$ is the real (complex) K-group $K^0(X)$. For another example, let R be a ring with 1, and let P be the collection of all finitely generated projective right R-modules (i.e., all direct summands of free right R-modules of finite rank). In this case, $\mathrm{Grot}(P)$ is the algebraic K-group $K_0(R)$. Alternatively, $K_0(R)$ may be constructed by taking P to be the category of all rectangular matrices over R. Then the objects in P are all idempotent square matrices over R, and the direct sum of idempotent matrices e and f is the block matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. In case R is a C*-algebra, the set of idempotent matrices over R may be reduced to the set of self-adjoint idempotent matrices (see [11, Chapter 19]).

In order not to lose track of the original semigroup $\mathrm{Grot}(\mathcal{P})^+$ inside $\mathrm{Grot}(\mathcal{P})$, we make it the "positive cone" of an order relation. Namely, for $x, y \in \mathrm{Grot}(\mathcal{P})$, we define $x \leq y$ if and only if $y-x$ lies in $\mathrm{Grot}(\mathcal{P})^+$. This relation is a pre-order (i.e., a reflexive, transitive relation) which is invariant under translation (that is, $x \leq y$ implies $x+z \leq y+z$). The combined structure $(\mathrm{Grot}(\mathcal{P}), +, \leq)$ is called a pre-ordered abelian group. In case the relation $\leq$ is a partial order (i.e., an anti-symmetric pre-order), $\mathrm{Grot}(\mathcal{P})$ is a partially ordered abelian group.

Various relatively mild assumptions on $\mathcal{P}$ will force $\mathrm{Grot}(\mathcal{P})$ to be partially ordered. A common assumption is that all the objects in $\mathcal{P}$ are "directly finite", i.e., objects $A, B \in \mathcal{P}$ can satisfy $A \oplus B \simeq A$ only if B is a zero object. For example, if $\mathcal{P}$ is the collection of all finitely generated projective right modules over a unital ring R, the objects in $\mathcal{P}$ are directly finite if and only if all square matrices over R that satisfy $xy = 1$ also satisfy $yx = 1$. In particular, this holds if R is commutative, or if R is a directed union of finite-dimensional algebras, or if R is noetherian on either side. It also holds if R is a unit-regular ring, meaning that for any $x \in R$ there exists a unit (invertible element) $u \in R$ such that $xux = x$, for in that case the objects in $\mathcal{P}$ may be cancelled from direct sums [18, Theorem 2; 9, Theorem 4.5].

In the case that $\mathcal{P}$ is the collection of all finitely generated projective right modules over a unital ring R, the module R plays a special role in $\mathcal{P}$, for every object in $\mathcal{P}$ is isomorphic to a direct summand of a finite direct sum of copies of R. As a consequence, $[R]$ plays a special role in $K_0(R)$: given any $x \in K_0(R)$ there exists a positive integer n such that $x \leq n[R]$.

By virtue of this property, $[R]$ is called an <u>order-unit</u> of $K_0(R)$.

We may make K_0 into a functor from the category of rings with unit (and unital ring homomorphisms) into a category whose objects are all pairs (G,u) where G is a pre-ordered abelian group and u is an order-unit in G. The appropriate morphisms in this latter category are <u>normalized positive homomorphisms</u> $f : (G,u) \to (H,v)$, that is, group homomorphisms $f : G \to H$ such that $f(G^+) \subseteq H^+$ and $f(u) = v$. We call this category <u>the category of pre-ordered abelian groups with order-unit</u>.

Any unital ring homomorphism $\phi : R \to S$ induces a functor $(-)\otimes_R S$ from the category of finitely generated projective right R-modules to the category of finitely generated projective right S-modules. Since this functor preserves direct sums, it in turn induces a positive homomorphism $K_0(\phi) : K_0(R) \to K_0(S)$, such that

$$K_0(\phi)([A]-[B]) = [A\otimes_R S]-[B\otimes_R S]$$

for all finitely generated projective right R-modules A and B. As $K_0(\phi)([R]) = [S]$, we see that $K_0(\phi)$ is a normalized positive homomorphism from $(K_0(R),[R])$ to $(K_0(S),[S])$. Since the appropriate functorial properties are clear, we obtain a functor from the category of rings with unit to the category of pre-ordered abelian groups with order-unit, given by the assignments $R \mapsto (K_0(R),[R])$ and $\phi \mapsto K_0(\phi)$.

The construction of $K_0(R)$ for a ring R without 1 is not based on some class of non-unital projective modules but instead is obtained from the "unitification" of R, namely the ring R^1 based on the abelian group $\mathbb{Z}\times R$, with multiplication given by the rule $(m,r)(n,s) = (mn,ms+nr+rs)$. The original ring R may be identified with the set $\{0\}\times R$, which is an ideal of R^1, and then $R^1/R \cong \mathbb{Z}$.

Then $K_0(R)$ is defined to be the kernel of K_0 of the natural map $R^1 \to \mathbb{Z}$, that is, the subgroup of $K_0(R^1)$ consisting of those elements $[A]-[B]$ in $K_0(R^1)$ for which A/AR and B/BR are (stably) isomorphic abelian groups, and $K_0(R)$ is equipped with the pre-ordered abelian group structure inherited from $K_0(R^1)$. In general, $K_0(R)$ need not have an order-unit. To take the place of an order-unit, we may use the subset $D(R)$ of $K_0(R)^+$ consisting of those elements $x \in K_0(R)$ for which $0 \leq x \leq [R^1]$. In the context of the category of not-necessarily-unital rings, K_0 may be viewed either as a functor into the category of pre-ordered abelian groups and positive homomorphisms, or as a functor into a category whose objects are all pairs (G,D) where G is a pre-ordered abelian group and D is a suitable subset of G^+.

Grothendieck groups having been constructed, two basic meta-questions arise: What sort of information about the data $\mathcal{P}$ is stored in $\mathrm{Grot}(\mathcal{P})$, and how may this information be retrieved? For instance, since $\mathrm{Grot}(\mathcal{P})$ is an invariant of $\mathcal{P}$, there can be situations in which it can be proved that data $\mathcal{P}$ and $\mathcal{P}'$ are not equivalent by showing that $\mathrm{Grot}(\mathcal{P})$ and $\mathrm{Grot}(\mathcal{P}')$ are not isomorphic. Also, by its construction $\mathrm{Grot}(\mathcal{P})$ reflects the arrangement of direct sum decompositions in $\mathcal{P}$, and we may ask how much of the direct sum decomposition structure of $\mathcal{P}$, and what other structural information about $\mathcal{P}$, may be recovered from $\mathrm{Grot}(\mathcal{P})$. By way of illustration, we shall discuss a number of situations in which these questions have been successfully answered. Due to the author's bias, these examples are K_0's of certain rings and C*-algebras, but the patterns of these examples are to be expected in Grothendieck groups of other mathematical systems.

□ IRRATIONAL ROTATION ALGEBRAS. Let T be the unit circle in the plane, and let $C(T)$ denote the algebra of all continuous complex-valued functions on T. We may view the functions in $C(T)$ as bounded linear operators on the Hilbert space $L^2(T)$ (where a function from $C(T)$ acts on functions from $L^2(T)$ by multiplication). Given a positive real number α, let $\rho_\alpha : T \to T$ be counterclockwise rotation through the angle $2\pi\alpha$. The rule $\rho_\alpha^*(f) = f\rho_\alpha$ then defines a bounded linear operator ρ_α^* on $L^2(T)$, and we let A_α denote the norm-closed self-adjoint subalgebra of bounded linear operators on $L^2(T)$ generated by $C(T)$ and ρ_α^*. This algebra is known as "the transformation group C*-algebra of the rotation ρ_α".

It is fairly easy to distinguish among the algebras A_α for rational α, and it is also easy to distinguish the rational cases from the irrational cases. However, distinguishing among the irrational cases is a subtler problem, which was only solved when Pimsner, Voiculescu, and Rieffel calculated K_0 of these algebras. Namely, for any irrational number $\alpha \in (0,1)$, there is an isomorphism (in the category of pre-ordered abelian groups with order-unit) from $(K_0(A_\alpha),[A_\alpha])$ onto $(\mathbb{Z}+\alpha\mathbb{Z},1)$, where the subgroup $\mathbb{Z}+\alpha\mathbb{Z}$ of $\mathbb{R}$ is given the usual ordering [28, Corollary 2.6; 29, Corollary 1; 30, Theorem 1].

Thus if $A_\alpha \simeq A_\beta$ for some irrational numbers $\alpha,\beta \in (0,1)$, there must be an isomorphism f of $(\mathbb{Z}+\alpha\mathbb{Z},1)$ onto $(\mathbb{Z}+\beta\mathbb{Z},1)$. (Since the topologies in $\mathbb{Z}+\alpha\mathbb{Z}$ and $\mathbb{Z}+\beta\mathbb{Z}$ may be defined in terms of the ordering, f must also be a homeomorphism.) After approximating elements of $\mathbb{Z}+\alpha\mathbb{Z}$ and $\mathbb{Z}+\beta\mathbb{Z}$ by rational numbers, clearing denominators, and using the relation $f(1) = 1$, it is easily seen that $\mathbb{Z}+\alpha\mathbb{Z} = \mathbb{Z}+\beta\mathbb{Z}$. From this a quick computation leads to the conclusion that either $\beta = \alpha$ or $\beta = 1-\alpha$, whence either $\rho_\beta = \rho_\alpha$ or $\rho_\beta = \rho_\alpha^{-1}$.

□ <u>ULTRAMATRICIAL ALGEBRAS</u>. Fix a field F. A <u>matricial</u> F-algebra is any F-algebra that is isomorphic to a finite direct product of full matrix algebras over F. (In case F is algebraically closed, the matricial F-algebras are exactly the finite-dimensional semisimple F-algebras.) An <u>ultramatricial</u> F-algebra is any F-algebra that is a union of a countable ascending sequence of matricial subalgebras (equivalently, any F-algebra that is isomorphic to a direct limit of a countable sequence of matricial F-algebras and F-algebra homomorphisms). It is easily checked that matricial algebras are unit-regular. Hence, any unital ultramatricial F-algebra R is unit-regular, and so $K_0(R)$ is a partially ordered abelian group. Using F-algebra unitifications, it follows that K_0 of any ultramatricial F-algebra is partially ordered.

These partially ordered abelian groups may be used to classify ultramatricial F-algebras, following a method of Elliott [7]. If R and S are unital ultramatricial F-algebras, then $R \simeq S$ if and only if there exists an isomorphism of $(K_0(R),[R])$ onto $(K_0(S),[S])$ in the category of pre-ordered abelian groups with order-unit [7, Theorem 4.3; 9, Theorem 15.26]. If R and S are non-unital ultramatricial F-algebras, then $R \simeq S$ if and only if there exists an ordered group isomorphism of $K_0(R)$ onto $K_0(S)$ mapping $D(R)$ onto $D(S)$ [7, Theorem 4.3].

The partially ordered abelian groups which can appear as K_0 of ultramatricial F-algebras are just those which are isomorphic (as ordered groups) to direct limits of countable sequences of finite products of copies of $\mathbb{Z}$ [7, Theorems 5.1, 5.5]. However, it is usually impossible to check directly whether a given partially ordered abelian group is isomorphic to such a direct limit. Some obvious properties of these direct limits are that they are countable, they are directed (upward and downward), and they are unperforated (any x satisfying $nx \geq 0$ for some $n \in \mathbb{N}$ also satisfies $x \geq 0$). A more fundamental property, also easily checked, is the Riesz interpolation property: given any x_1, x_2, y_1, y_2 such that $x_i \leq y_j$ for all i,j, there exists z such that $x_i \leq z \leq y_j$ for all i,j. The direct limits of (sequences of) finite products of copies of $\mathbb{Z}$ were characterized by Effros, Handelman, and Shen as exactly those (countable) partially ordered abelian groups which are directed and unperforated and which satisfy the Riesz interpolation property [5, Theorem 2.2; 11, Corollary 21.8].

Partially ordered abelian groups with the latter three properties are now called dimension groups. Thus, given a partially ordered abelian group G and an order-unit $u \in G$, there exists an isomorphism of (G,u) onto $(K_0(R),[R])$ for some unital ultramatricial F-algebra R if and only if G is a countable dimension group. For the non-unital case, replace the order-unit u by an upward directed subset $D \subseteq G^+$ such that every element of G^+ is a sum of elements from D, and such that any element of G^+ which lies below an element of D must lie in D. Then there exists an ordered group isomorphism of G onto K_0 of some ultramatricial F-algebra R, with D mapping onto $D(R)$, if and only if G is a countable dimension group. (These results are obtained by combining Elliott's results [7, Theorems 5.1, 5.5] with those of Effros, Handelman, and Shen [5, Theorem 2.2].

For example, the subgroup $\{a/2^n \mid a \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$ of $\mathbb{Q}$

appears as K_0 of a direct limit of matrix algebras

$$M_2(F) \to M_4(F) \to M_8(F) \to \cdots$$

using block diagonal maps $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. The lexicographic product of $\mathbb{Z}$ with itself appears as K_0 of a direct limit

$$F \times M_2(F) \to F \times M_3(F) \to F \times M_4(F) \to \cdots$$

using maps $(x,y) \mapsto \left(x, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right)$. The subgroup $\mathbb{Z}+\sqrt{2}\,\mathbb{Z}$ of $\mathbb{R}$ appears as K_0 of a direct limit

$$F \times M_2(F) \to M_3(F) \times M_4(F) \to M_7(F) \times M_{10}(F) \to \cdots$$

using maps

$$(x,y) \mapsto \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}\right).$$

An algorithm for obtaining $\mathbb{Z}+\alpha\mathbb{Z}$ (where α is any positive irrational number) as K_0 of an ultramatricial algebra, by using the partial fraction decomposition of α, was developed by Effros and Shen [6, Theorem 3.2].

The ability to realize any countable dimension group as K_0 of an ultramatricial algebra is an aid to constructing examples, since dimension groups are easier to construct than ultramatricial algebras. For instance, we may use this method to construct algebras with various sorts of ideal-theoretic structure, for this corresponds directly to ideal-theoretic structure in K_0. (An <u>ideal</u> in a partially ordered abelian group G is any directed subgroup H of G such that whenever $x_1, x_2 \in H$ and $y \in G$ with $x_1 \leq y \leq x_2$, then $y \in H$.) For any unital ultramatricial algebra R (or, more generally, for any unit-regular ring), the lattice of two-sided ideals of R is isomorphic to the lattice of ideals of $K_0(R)$ [9, Corollary 15.21]. Given any countable ordinal α, it is easy to construct a countable dimension group, with an order-unit, whose lattice of ideals is isomorphic to the interval $[1,\alpha]$ (see [10, Proposition 1.3]). Consequently, there exists a unital ultramatricial F-algebra whose lattice of two-sided ideals is isomorphic to $[1,\alpha]$. Similarly, there exists a unital ultramatricial F-algebra whose lattice of two-sided ideals is anti-isomorphic to $[1,\alpha]$. (These ideal-theoretic examples are the basic ingredients in two corresponding module-theoretic examples constructed by the author [10, Corollaries 1.6, 1.7].)

Any family of ultramatricial algebras defined by properties which are reflected in K_0 may be classified by a corresponding family of countable dimension groups. For example, since an ultramatricial algebra R is simple if and only if $K_0(R)$ is simple (i.e., there are exactly two ideals in $K_0(R)$, namely $K_0(R)$ and $\{0\}$), the family of simple unital ultramatricial F-algebras is classified by the family of countable simple dimension groups with order-unit. The task of constructing all (countable) simple dimension groups was initiated by Effros, Handelman, and Shen [5, Lemmas 3.1, 3.2, Theorem 3.5] and completed by the author and Handelman [15, Theorem 4.11].

□ <u>APPROXIMATELY</u> <u>FINITE</u>-<u>DIMENSIONAL</u> <u>COMPLEX</u> <u>C</u>*-<u>ALGEBRAS</u>. A complex C*-algebra is <u>approximately</u> <u>finite</u>-<u>dimensional</u> (abbreviated <u>AF</u>) if it is isomorphic (as a complex C*-algebra) to a direct limit of a countable sequence of finite-dimensional complex C*-algebras. Since all finite-dimensional C*-algebras are semisimple, any complex AF C*-algebra A contains a dense ultramatricial complex *-subalgebra R. Moreover, the inclusion map $R \to A$ induces an ordered group isomorphism of $K_0(R)$ onto $K_0(A)$ (see [11, Corollary 19.10] for the unital case). Hence, K_0 of any complex AF C*-algebra is a countable dimension group. Bratteli proved that complex AF C*-algebras are determined up to isomorphism by their dense ultramatricial complex

-subalgebras [2, Theorem 2.7; 11, Theorem 20.7]. Consequently, complex AF C-algebras are classified in terms of countable dimension groups, via K_0, in exactly the same manner as ultramatricial algebras [7, Theorems 4.3, 5.1, 5.5; 5, Theorem 1.2; 11, Theorems 20.7, 21.10].

□ DIRECT LIMITS OF FINITE-DIMENSIONAL SEMISIMPLE REAL ALGEBRAS. The family $\mathcal{R}$ of unital direct limits of countable sequences of finite-dimensional semisimple $\mathbb{R}$-algebras is not as easy to classify as the family of unital direct limits of countable sequences of finite-dimensional semisimple $\mathbb{C}$-algebras, for while the latter family consists exactly of the unital ultramatricial $\mathbb{C}$-algebras, $\mathcal{R}$ contains, in addition to the unital ultramatricial $\mathbb{R}$-algebras, direct limits in which matrix algebras over $\mathbb{C}$ and $\mathbb{H}$ may appear. The invariant $(K_0(-),[-])$ does not contain enough information to distinguish among the algebras in $\mathcal{R}$ even at the most basic level, since for any division ring D we have $(K_0(D),[D]) \cong (\mathbb{Z},1)$. Thus a more complicated invariant is needed. We may construct such an invariant for algebras $R \in \mathcal{R}$ by using $K_0(R)$ together with K_0 of the complexifications R^c and K_0 of the quaternionifications R^h. Specifically, for each $R \in \mathcal{R}$ let $k(R)$ denote the diagram

$$(K_0(R),[R]) \rightarrow (K_0(R^c),[R^c]) \rightarrow (K_0(R^h),[R^h]),$$

where the maps $K_0(R) \rightarrow K_0(R^c) \rightarrow K_0(R^h)$ are obtained by applying the functor K_0 to the natural maps $R \rightarrow R^c \rightarrow R^h$. Then k becomes a functor from $\mathcal{R}$ to a category whose objects are all diagrams of the form

$$(G_1,u_1) \rightarrow (G_2,u_2) \rightarrow (G_3,u_3)$$

within the category of pre-ordered abelian groups with order-unit.

Since K_0 of any semisimple ring is isomorphic (as an ordered group) to a finite direct product of copies of $\mathbb{Z}$ [9, Lemma 15.22], it follows that K_0 of any algebra in $\mathcal{R}$ is a countable dimension group. For any $R \in \mathcal{R}$, the algebras R^c and R^h also lie in $\mathcal{R}$. Thus all three pre-ordered abelian groups appearing in the diagram $k(R)$ are countable dimension groups.

For example, k of the $n\times n$ matrix algebras $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$ are isomorphic to the following diagrams:

$$(\mathbb{Z},n) \xrightarrow{1} (\mathbb{Z},n) \xrightarrow{1} (\mathbb{Z},n)$$

$$(\mathbb{Z},n) \xrightarrow{\text{diag}} (\mathbb{Z},n) \times (\mathbb{Z},n) \xrightarrow{\text{sum}} (\mathbb{Z},2n)$$

$$(\mathbb{Z},n) \xrightarrow{2} (\mathbb{Z},2n) \xrightarrow{2} (\mathbb{Z},4n)$$

[16, Propositions 2.2, 2.6].

That k contains enough information to classify the algebras in $\mathcal{R}$ was proved by the author and Handelman [16, Theorem 5.1]: If $R,S \in \mathcal{R}$, then $R \simeq S$ if and only if $k(R) \simeq k(S)$, and in fact any isomorphism of $k(R)$ onto $k(S)$ may be obtained as k of an isomorphism of R onto S.

The question of exactly which diagrams appear as k of algebras in $\mathcal{R}$ has remained open. However, the form of k of those algebras in $\mathcal{R}$ which can be constructed using matrix rings over only one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ is known (provided, in the second case, that only $\mathbb{C}$-algebra maps are allowed). Namely, an algebra $R \in \mathcal{R}$ is isomorphic to (a) an ultramatricial $\mathbb{R}$-algebra, (b) an ultramatricial $\mathbb{C}$-algebra, or (c) a direct limit of a countable sequence of finite direct products of full matrix rings over $\mathbb{H}$, if and only if $k(R)$ is isomorphic to

(a) $$(G,u) \xrightarrow{1} (G,u) \xrightarrow{1} (G,u)$$

(b) $$(G,u) \xrightarrow{\text{diag}} (G,u) \times (G,u) \xrightarrow{\text{sum}} (G,2u)$$

(c) $$(G,u) \xrightarrow{2} (G,2u) \xrightarrow{2} (G,4u)$$

for some countable dimension group (G,u) with order-unit [16, Theorems 7.1, 7.2, 7.5].

As an application, consider an algebra $R \in \mathcal{R}$ whose center contains a copy of $\mathbb{C}$. With some computation, it may be shown that $k(R)$ has the form (b), where $(G,u) = (K_0(R),[R])$, as done in [16, Proposition 2.4]. Since G is a countable dimension group, there exists a unital ultramatricial $\mathbb{C}$-algebra S such that $(K_0(S),[S]) \simeq (G,u)$. Then $k(S)$ has the form (b), whence $k(R) \simeq k(S)$ and so $R \simeq S$. Thus the only complex algebras in $\mathcal{R}$ are the unital ultramatricial complex algebras [16, Theorem 7.2].

□ <u>APPROXIMATELY FINITE-DIMENSIONAL REAL C*-ALGEBRAS</u>. In parallel with complex AF C*-algebras, we may define a real C*-algebra to be <u>AF</u> if it is isomorphic (as a real C*-algebra) to a direct limit of a countable sequence of finite-dimensional real C*-algebras. Any unital real AF C*-algebra contains a dense unital real *-subalgebra from the family $\mathcal{R}^*$ of unital direct limits of countable sequences of those finite-dimensional semisimple real *-algebras for which the involution arises from the conjugate transpose involution on matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. In parallel with the classification of the unital direct limits of countable sequences of finite-dimensional semisimple real algebras, the functor k classifies both the algebras in $\mathcal{R}^*$ and the real AF C*-algebras. In particular, for real AF C*-algebras A and B we have $A \simeq B$ (as real C*-algebras) if and only if $k(A) \simeq k(B)$ [16, Theorem 9.1]. The form of k of real AF C*-algebras constructed using matrix rings over only one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ also follows the same pattern as in the previous section. For instance, a real AF

C*-algebra R is isomorphic to a complex AF C*-algebra if and only if $k(R)$ is isomorphic to a diagram of the form

$$(G,u) \xrightarrow{\text{diag}} (G,u) \times (G,u) \xrightarrow{\text{sum}} (G,2u),$$

for some countable dimension group (G,u) with order-unit.

□ <u>PSEUDO-RANK FUNCTIONS</u>. A ring R is <u>(von Neumann) regular</u> provided that for each $x \in R$ there exists $y \in R$ such that $xyx = x$. (This ensures a large supply of idempotents, for whenever $xyx = x$, the elements xy and yx are idempotent.) A (normalized) <u>pseudo-rank function</u> on a regular ring R with 1 is any map N from R to the unit interval $[0,1]$ such that (a) $N(1) = 1$; (b) $N(xy) \leq N(x)$ and $N(xy) \leq N(y)$ for all $x,y \in R$; (c) $N(e+f) = N(e)+N(f)$ for all orthogonal idempotents $e,f \in R$. For example, if R is the ring of all $n\times n$ matrices over a field F, then normalized matrix rank defines a pseudo-rank function N on R (that is, $N(x) = \operatorname{rank}(x)/n$ for all matrices $x \in R$). For another example, if R is a ring of subsets of some set X, and $X \in R$, then a pseudo-rank function on R is just a nonnegative finitely additive measure μ on R such that $\mu(X) = 1$.

Pseudo-rank functions on R correspond to normalized positive homomorphisms from $(K_0(R),[R])$ to $(\mathbb{R},1)$. In general, given a pre-ordered abelian group (G,u) with order-unit, a <u>state on</u> (G,u) is any morphism $(G,u) \to (\mathbb{R},1)$ in the category of pre-ordered abelian groups with order-unit. There is a canonical bijection between the set of states on $(K_0(R),[R])$ and the set of pseudo-rank functions on R [14, Proposition 2.4; 9, Proposition 17.12]. Thus questions of existence and/or uniqueness for pseudo-rank functions may be translated into questions of existence and/or uniqueness for states.

For example, the author and Handelman showed that any nonzero partially ordered abelian group with an order-unit has at least one state [14, Corollary 3.3; 9, Corollary 18.2]. Since K_0 of any nonzero unit-regular ring is nonzero and partially ordered, it follows that any nonzero unit-regular ring has at least one pseudo-rank function [14, Corollary 3.5; 9, Corollary 18.5]. The author and Handelman also developed various criteria for a nonzero partially ordered abelian group G with an order-unit u to have a unique state. For instance, this happens if and only if there exist integers $s > t > 0$ such that given any $x,y \in G^+$ with $x+y = u$, there is some $n \in \mathbb{N}$ for which either $ntx \leq nsy$ or $nty \leq nsx$. As a consequence, a nonzero unit-regular ring R possesses a unique pseudo-rank function if and only if there exist integers $s > t > 0$ such that given any orthogonal idempotents $e,f \in R$ with $e+f = 1$, there is some $n \in \mathbb{N}$ for which either the direct sum of nt copies of eR embeds in the direct sum of ns copies of fR or the direct sum of nt copies of fR embeds in the direct sum of ns copies of eR [14, Theorem 4.6; 9, Theorem 18.6].

□ PSEUDO-RANK FUNCTION SPACES AND TRACE SPACES. The collection $\mathbb{P}(R)$ of all pseudo-rank functions on a regular ring R (with 1) can be viewed as a subset of the real vector space $\mathbb{R}^R$ of all real-valued functions on R. If $\mathbb{R}^R$ is given the product topology, it is a locally convex Hausdorff linear topological space, and $\mathbb{P}(R)$ is a compact convex subset of $\mathbb{R}^R$ [9, Proposition 16.17]. In fact, $\mathbb{P}(R)$ is a rather special kind of compact convex set known as a "Choquet simplex" [9, Theorem 17.5]. (Choquet simplices are infinite-dimensional analogs of classical finite-dimensional simplices, and may be characterized as exactly those compact convex subsets of locally convex Hausdorff linear topological spaces which arise as inverse limits of finite-dimensional simplices.)

In a similar fashion, the collection $S(G,u)$ of all states on a pre-ordered abelian group (G,u) with order-unit, called the <u>state space of</u> (G,u), is a compact convex subset of the product space $\mathbb{R}^G$ [9, Proposition 17.11]. If G is a dimension group (more generally, if G satisfies the Riesz interpolation property), then $S(G,u)$ is a Choquet simplex [17, Theorem I.2.5; 5, Proposition 1.7].

For the case of K_0 of the regular ring R, the canonical bijection between the state space $S(K_0(R),[R])$ and the pseudo-rank function space $\mathbb{P}(R)$ is an affine homeomorphism [9, Proposition 17.12], i.e., an isomorphism in the category of compact convex sets. Hence, to realize a given Choquet simplex K as $\mathbb{P}(R)$ for some regular ring R, it suffices to realize K as $S(K_0(R),[R])$. In the metrizable case, this was done by the author [8, Theorem 5.1; 9, Theorems 17.19, 17.23]. We sketch an easier proof of this result, taking advantage of our ability to realize any countable dimension group as K_0 of an ultramatricial algebra.

Thus let K be an arbitrary metrizable Choquet simplex, and let $\mathrm{Aff}(K)$ be the partially ordered real Banach space of all affine (i.e., convex-combination-preserving) continuous real-valued functions on K. (The ordering in $\mathrm{Aff}(K)$ is the pointwise ordering of functions, and the norm is the supremum norm.) From the metrizability of K, it follows that $\mathrm{Aff}(K)$ is separable. Also, since K is a Choquet simplex, $\mathrm{Aff}(K)$ satisfies the Riesz interpolation property [4, Théorème; 32, Theorem 5]. Consequently, we may construct a countable dense additive subgroup G of $\mathrm{Aff}(K)$ such that G contains the constant function 1 and G has the Riesz interpolation property.

Then G is a dimension group and 1 is an order-unit in G. Since G is dense in $\mathrm{Aff}(K)$, the restriction map $S(\mathrm{Aff}(K),1) \to S(G,1)$ is an affine homeomorphism. On the other hand, a standard folklore result is that the evaluation map $K \to S(\mathrm{Aff}(K),1)$ is an affine homeomorphism, and thus K is affinely homeomorphic to $S(G,1)$. As there exists a unital ultramatricial algebra R for which $(K_0(R),[R]) \cong (G,1)$, we conclude that $\mathbb{P}(R)$ is affinely homeomorphic to K. By using the strict ordering on $\mathrm{Aff}(K)$ (under which $f < g$ only if $f(x) < g(x)$ for all $x \in K$), we can ensure that the dimension group G is simple, whence R is a simple algebra.

Parallel procedures can be used to realize metrizable Choquet simplices as trace spaces of unital C*-algebras. Given a unital complex C*-algebra A, the set A_{sa} of self-adjoint elements of A becomes a partially ordered real vector space with positive cone

$$A_{sa}^{+} = \{x \in A_{sa} \mid \mathrm{spectrum}(x) \subseteq \mathbb{R}^{+}\}$$

and order-unit 1 [11, Proposition 6.1]. A <u>state on</u> A is any linear functional $A \to \mathbb{C}$ which restricts to a state on $(A_{sa},1)$. Since all states on $(A_{sa},1)$ extend uniquely to states on A, the collection of all states on A may be identified with the state space of $(A_{sa},1)$. A <u>tracial state</u> on A is any state t such that $t(xx^*) = t(x^*x)$ for all $x \in A$, and a <u>normalized finite trace</u> on A is the restriction to A_{sa}^{+} of any tracial state. The <u>trace space</u> of A is the collection $T(A)$ of all normalized finite traces on A. We identify $T(A)$ with the collection of all tracial states on A, which is a compact convex subset of $S(A_{sa},1)$. In fact, $T(A)$ is a Choquet simplex [31, Theorem 3.1.18].

For a matrix algebra $M_n(\mathbb{C})$, the trace space $T(M_n(\mathbb{C}))$ is a singleton, as is the state space of $(K_0(M_n(\mathbb{C})),[M_n(\mathbb{C})])$. Using the observation that the functors $T(-)$ and $S(K_0(-),[-])$ convert finite products to finite coproducts and convert direct limits to inverse limits, it follows that the trace space of any unital complex AF C*-algebra A is affinely homeomorphic to $S(K_0(A),[A])$ [1, Corollary 3.2]. Given any metrizable Choquet simplex K, there is a countable simple dimension group (G,u) with order-unit such that $S(G,u)$ is affinely homeomorphic to K, as indicated above. Hence, by choosing a simple unital complex AF C*-algebra A for which $(K_0(A),[A]) \cong (G,u)$, we obtain Blackadar's result that any metrizable Choquet simplex K is affinely homeomorphic to $T(A)$ for some simple unital complex AF C*-algebra A [1, Theorem 3.9].

□ METRICALLY COMPLETE REGULAR RINGS. A norm-like function N^* may be defined on any regular ring R (with 1) by setting $N^*(x)$ equal to the supremum of the values $N(x)$ for $N \in \mathbb{P}(R)$. It is easily checked that the rule $d(x,y) = N^*(x-y)$ then defines a pseudo-metric d on R [12, Lemma 1.2]. In case d is a metric and R is complete with respect to d, we say that R is N*-complete. For instance, if there exists a positive integer n such that all nilpotent elements $x \in R$ satisfy $x^n = 0$, then $N^*(y) \geq 1/n$ for all nonzero elements $y \in R$, and so R is N*-complete [12, Theorem 1.3]. This occurs, for instance, if R can be embedded in a direct product of $n \times n$ matrix rings over division rings. The function N^* on R corresponds to a norm-like function on $K_0(R)$, defined using states in place of pseudo-rank functions, as follows.

Given any pre-ordered abelian group (G,u) with order-unit, we may define

$$||x|| = \sup\{|s(x)| : s \in S(G,u)\}$$

for all $x \in G$. Alternatively, $||x||$ may be computed as

$$||x|| = \inf\{k/n \mid k,n \in \mathbb{N} \text{ and } -ku \leq nx \leq ku\}$$

[17, Lemma I.6.1]. Then $||\cdot||$ is a nonnegative real-valued function on G such that $||mx|| = |m|\cdot||x||$ and $||x+y|| \leq ||x||+||y||$ for all $m \in \mathbb{Z}$ and all $x,y \in G$ [ibid]. If the pseudo-metric d' on G defined by $d'(x,y) = ||x-y||$ is actually a metric, and if G is complete with respect to d', we say that (G,u) is <u>norm-complete</u>.

Because of the canonical bijection between $\mathbb{P}(R)$ and $S(K_0(R),[R])$, it follows that $||[xR]|| = N^*(x)$ for all $x \in R$, and so N^*-completeness of R is related to norm-completeness of $K_0(R)$. Specifically, in case the regular ring R is N^*-complete, the author proved that $(K_0(R),[R])$ is an archimedean norm-complete dimension group [12, Theorem 2.11]. (For a <u>partially</u> ordered abelian group G to be <u>archimedean</u> means that whenever $x,y \in G$ with $nx \leq y$ for all positive integers n, then $x \leq 0$.) Consequently, some structure theory for N^*-complete regular rings may be obtained from corresponding structure theory for archimedean norm-complete dimension groups.

For example, there exists a complete representation of any archimedean norm-complete dimension group (G,u) in terms of affine continuous real-valued functions on its state space. The only restrictions placed on the functions appearing in this representation are the values allowed at discrete extremal states. (A state s on (G,u) is <u>discrete</u> if $s(G)$ is a discrete subgroup of $\mathbb{R}$. A point x in a convex set K is <u>extremal</u> if x does not lie in the interior of any line segment within K.) Setting

$$A = \{q \in \mathrm{Aff}(S(G,u)) \mid q(s) \in s(G) \text{ for all discrete extremal states } s\},$$

the author and Handelman proved that the evaluation map $G \to \mathrm{Aff}(S(G,u))$ gives an isomorphism of (G,u) onto $(A,1)$ (as ordered groups with order-unit) [15, Theorem 5.1].

This affine continuous function representation for archimedean norm-complete dimension groups in turn provides an affine continuous function representation for K_0 of the N*-complete regular ring R. Under the canonical affine homeomorphism between $S(K_0(R),[R])$ and $\mathbb{P}(R)$, a discrete extremal state s on $(K_0(R),[R])$ corresponds to an extremal pseudo-rank function P with a discrete range of values. Specifically, if $s(K_0(R)) = (1/m)\mathbb{Z}$ for some $m \in \mathbb{N}$, then $P(R) = \{0,1/m,2/m,\ldots,1\}$, and this occurs if and only if $R/\ker(P)$ is isomorphic to an $m\times m$ matrix ring over a division ring. Set $B_P = (1/m)\mathbb{Z}$ in this case, and for all other extremal pseudo-rank functions P set $B_P = \mathbb{R}$. Then there is a natural isomorphism of $(K_0(R),[R])$ onto $(B,1)$, where

$$B = \{q \in \mathrm{Aff}(\mathbb{P}(R)) \mid q(P) \in B_P \text{ for all extremal pseudo-rank functions } P\}$$

[12, Theorem 4.11].

To give an easy application of this affine continuous function representation of $K_0(R)$, assume, for some fixed positive integer t, that all simple artinian factor rings of R (if there are any) are $t\times t$ matrix rings (over some other rings, not necessarily over division rings). Then if P is an extremal pseudo-rank function and $R/\ker(P)$ is isomorphic to an $m\times m$ matrix ring over a division ring, t must divide m, whence $1/t \in B_P$. As a result, the constant function $1/t$ belongs to the group B given above. From the isomorphism of $(K_0(R),[R])$ onto $(B,1)$, it follows that $[R] = t[C]$ for some $[C] \in K_0(R)^+$. Since R is unit-regular [12, Theorem 2.3], the module R is isomorphic to a direct sum of t copies of C, whence the ring R is isomorphic to a $t\times t$ matrix ring (over the endomorphism ring of

C) [12, Corollary 4.14]. In particular, if R has no simple artinian factor rings, then R is a $t\times t$ matrix ring for all positive integers t.

□ <u>ALEPH-NOUGHT-CONTINUOUS REGULAR RINGS</u>. In any regular ring R, the collection $L(R_R)$ of principal right ideals forms a lattice, with finite intersections for finite infima and finite sums for finite suprema [9, Theorems 1.1, 2.3]. The ring R is said to be <u>$\aleph_0$-continuous</u> if the lattice $L(R_R)$ is $\aleph_0$-continuous in the sense that (a) every countable subset of $L(R_R)$ has an infimum and a supremum in $L(R_R)$; (b) whenever $A \in L(R_R)$ and $B_1 \leq B_2 \leq \ldots$ in $L(R_R)$, then $A \wedge (\bigvee B_i) = \bigvee (A \wedge B_i)$; (c) whenever $A \in L(R_R)$ and $B_1 \geq B_2 \geq \ldots$ in $L(R_R)$, then $A \vee (\bigwedge B_i) = \bigwedge (A \vee B_i)$. (Since $L(R_R)$ is anti-isomorphic to the lattice of principal left ideals of R [9, Theorem 2.5], this definition is left-right symmetric.) Equivalently, R is $\aleph_0$-continuous if and only if given any countably generated right (left) ideal I of R, there exists a principal right (left) ideal $J \supseteq I$ such that every nonzero right (left) ideal contained in J has nonzero intersection with I [9, Corollary 14.4].

Handelman proved that every $\aleph_0$-continuous regular ring is unit-regular [19, Theorem 3.2; 9, Theorem 14.24], and the author proved that every $\aleph_0$-continuous regular ring is N*-complete [12, Theorem 1.8]. Hence, the structure theories for archimedean norm-complete dimension groups and N*-complete regular rings yield a structure theory for $\aleph_0$-continuous regular rings. However, a structure theory for $\aleph_0$-continuous regular rings was first derived from a structure theory for monotone σ-complete dimension groups, as follows. (A partially ordered set P is <u>monotone σ-complete</u> provided that every ascending (descending) sequence $x_1 \leq x_2 \leq \ldots$ ($x_1 \geq x_2 \geq \ldots$) in P which is bounded above (below) in P has a supremum (infimum) in P.)

Handelman, Higgs, and Lawrence proved that K_0 of any $\aleph_0$-continuous regular ring R is a monotone σ-complete dimension group [24, Proposition 2.1], and that such groups are archimedean [24, Theorem 1.3]. Since $K_0(R)$ is archimedean, they obtained

$$\cap\{\ker(P) \mid P \in \mathbb{P}(R)\} = \{0\},$$

from which it follows that the intersection of the maximal two-sided ideals of R is zero [24, Theorem 2.3]. If M is any maximal two-sided ideal of R, then the existence of a state on $(K_0(R/M),[R/M])$ implies the existence of a pseudo-rank function P on R/M, and $\ker(P) = \{0\}$ because R/M is a simple ring. As a consequence, R/M contains no uncountable direct sums of nonzero principal right or left ideals, and using this countability condition, Handelman proved that R/M is a right and left self-injective ring [19, Corollary 3.2]. Thus, since the intersection of the maximal two-sided ideals of R is zero, R is a subdirect product of simple right and left self-injective rings.

A structure theory for monotone σ-complete dimension groups was developed by the author, Handelman, and Lawrence [17] and applied to $K_0(R)$. For example, the affine continuous function representation for such groups led to a complete representation of $K_0(R)$ in terms of affine continuous functions on $\mathbb{P}(R)$ [17, Theorem II.15.1]. As a consequence, if all simple factor rings of R are $t\times t$ matrix rings (for some fixed positive integer t), then R is a $t\times t$ matrix ring [17, Theorem II.15.3].

□ FINITE RICKART C*-ALGEBRAS. A Rickart C*-algebra is a C*-algebra A in which the right annihilator of any element x (that is, the right ideal $\{a \in A \mid xa = 0\}$) equals the principal right ideal generated by some projection p (that is, $p = p^* = p^2$). This is a generalization of the concept of an AW*-algebra, which is a C*-algebra in which the right annihilator of any subset is a principal right ideal generated by a projection. In particular, all von Neumann algebras (W*-algebras) are Rickart C*-algebras. A finite C*-algebra is a unital C*-algebra A such that all elements $x \in A$ satisfying $xx^* = 1$ also satisfy $x^*x = 1$.

The K-theory of a finite Rickart C*-algebra A can be investigated with the aid of an auxiliary $\aleph_0$-continuous regular ring R which is also *-regular, i.e., there is an involution $*$ on R such that every principal right ideal of R is generated by a projection. Handelman proved that A is a *-subring of an $\aleph_0$-continuous *-regular ring R such that the only projections in R are those in A [19, Theorem 2.1]. The ring R is essentially unique (up to a *-ring isomorphism which is the identity on A), and is called the regular ring of A. Handelman also proved that the inclusion map $A \to R$ induces an isomorphism of $(K_0(A),[A])$ onto $(K_0(R),[R])$. (A proof for the case that A has no one-dimensional representations is given in [13, Theorem 5.2].)

In particular, $K_0(A)$ is a monotone σ-complete dimension group, and the structure theory for such groups yields a corresponding structure theory for A, in exactly the same manner as for $\aleph_0$-continuous regular rings. (However, this structure theory was first derived from the structure theory for $\aleph_0$-continuous regular rings, via the regular ring of A.) For example, A is a subdirect product of simple AW*-algebras, and so A can be embedded in a finite AW*-algebra [24, Theorem 3.1]. For another example, if the dimension of every finite-dimensional irreducible representation of A (if there are any) is divisible by a fixed positive integer t, then A is a $t \times t$ matrix ring over some other finite Rickart C*-algebra [17, Theorem III.16.8].

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