Pub. Mat. UAB
Vol. 29 Nº 2-3 Nov. 1985

ON THE MEAN VALUES OF AN INTEGRAL FUNCTION REPRESENTED BY DIRICHLET SERIES

Poonam Sharma*

1. Consider the Dirichlet series

$$(1.1) \qquad f(s) = \sum_{n=1}^{\infty} a_n \exp[s\lambda_n],$$

where $s = \sigma + it$, $\lambda_1 \geqslant 0$, $\lambda_n < \lambda_{n+1} \longrightarrow \infty$, as $n \to \infty$

and

$$(1.2) \qquad \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D < \infty.$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c = \infty$ and σ_a will also be infinite, since according to a known result ([1], p.4) a Dirichlet series which satisfies (1.2) has its abscissa of convergence equal to its abscissa of absolute convergence, and so, $f(s)$ is an integral function.

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*This work has been suported by J.R.F. of C.S.I.R. (New Delhi).

Let $E = \{f(s)$: defined by (1.1), satisfying (1.2) and $\sigma_c = \infty\}$. Let the maximum modulus of $f(s)$, $f(s) \in E$, over a vertical line be

$$M(\sigma) = \underset{-\infty < t < \infty}{\text{l. u. b}} |f(\sigma + it)|.$$

The (Ritt) order ρ and the lower order λ are defined by Ritt ([2], p.78) as

(1.3) $$\lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix}$$

and he also defined the type τ and the lower type t as

(1.4) $$\lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \begin{matrix} \tau \\ t \end{matrix}.$$

Let the mean values of $|f(s)|$, $f(s) \in E$, be

(1.5) $$I_\delta(\sigma, f) = I_\delta(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta \, dt$$

and

(1.6) $$m_{\delta,k}(\sigma, f) = m_{\delta,k}(\sigma) = \frac{2}{e^{k\sigma}} \int_0^\sigma I_\delta(x) e^{kx} dx,$$

where $\delta, k \in R^+$, the set of positive real numbers.

Some properties of these mean values were also studied by Rizvi, M.I. [3] and he obtained a number of valuable results.

In this paper we have obtained some inequalities, growth properties and asymptotic relations involving $m_{\delta,k}(\sigma,f)$ and $I_\delta(\sigma,f)$ for $f(s) \epsilon E$. The results obtained are best possible. The results of ([4],pp. 43-48) and ([5],pp. 51.56) follow from ours.

Dikshit [6] in 1972 established the following theorems.

<u>Theorem</u> A. $I_\delta(\sigma)$ increases steadily with σ and log $I_\delta(\sigma)$ is a convex function of σ for $\sigma > \sigma_0 = \sigma_0(f) > 0$.

<u>Proof</u>: - Let $0 < \sigma_0 < \sigma_1 < \sigma_2 < \sigma_3$ and $h(t)$ and $F(s)$ be defined as

$$h(t)\ \{f(\sigma_2 + it)\}^\delta = |f(\sigma_2 + it)|^\delta, \quad (-\infty < t < \infty)$$

and

$$F(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{f(s+it)\}^\delta\, h(t)\, dt.$$

Then $F(s)$ is regular for $0 < \sigma_0 \leq \text{Re}(s) < \sigma_3$ and its least upper bound is obtained on its boundary say at $s = \sigma_3 + it_3$. Hence,

$$I_\delta(\sigma_2) = F(\sigma_2) \leq |f(\sigma_3 + it_3)| \leq I_\delta(\sigma_3),$$

which proves the first part. To prove the second part we choose α, such that

$$e^{\alpha\sigma_1} I_\delta(\sigma_1) = e^{\alpha\sigma 3} I_\delta(\sigma_3).$$

Therefore,

$$e^{\alpha\sigma_2} I_\delta(\sigma_2) = e^{\alpha\sigma_2} F(\sigma_2) \leqslant \underset{-\infty<t<\infty}{\text{l.u.b.}} | e^{\alpha s} f(s)| \leqslant$$

$$\leqslant e^{\alpha\sigma_1} I_\delta(\sigma_1) = e^{\alpha\sigma_3} I_\delta(\sigma_3)$$

which gives on eliminating α,

$$\log I_\delta(\sigma_2) \leqslant \frac{\sigma_3-\sigma_2}{\sigma_3-\sigma_1} \log I_\delta(\sigma_1) \quad \frac{\sigma_2-\sigma_1}{\sigma_3-\sigma_1} \log I_\delta(\sigma_3)$$

and the result follows.

<u>Theorem</u> B. $e^{k\sigma} I_\delta(\sigma)$ is a convex function of $e^{k\sigma} m_{\delta,k}(\sigma)$ and $\log m_{\delta,k}(\sigma)$ is a convex function of σ, for $\sigma > \sigma_0 = \sigma_0(f)>0$.

<u>Proof</u>: - We have

$$\frac{d\{e^{k\sigma} I_\delta(\sigma)\}}{d\{e^{k\sigma} m_{\delta,k}(\sigma)\}} = \frac{k\, e^{k\sigma} I_\delta(\sigma) + e^{k\sigma} I_\delta'(\sigma)}{2\, I_\delta(\sigma) e^{k\sigma}}$$

$$= \frac{1}{2}\{ k + \frac{I_\delta'(\sigma)}{I_\delta(\sigma)} \}$$

for $\sigma > \sigma_0 = \sigma_0(f) > 0$ and increases with σ by Theorem A, since $\log I_\delta(\sigma)$ is a convex function of σ. This proves the first part. To prove the second part, we have

$$\frac{d}{d(\sigma)} \{\log m_{\delta,k}(\sigma)\} = \frac{2I_\delta(\sigma) - k\, m_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)}$$

$$= \{ \frac{2I_\delta(\sigma)}{m_{\delta,k}(\sigma)} - k \}$$

for $\sigma > \sigma_0 = \sigma_0(f) > 0$ and increases with σ, since $e^{k\sigma} I_\delta(\sigma)$ is a convex function of $e^{k\sigma} m_{\delta,k}(\sigma)$.

2. In theorem 1 we estimate the ratio of $m_{\delta,k}(\sigma)$ for any two positive values of σ in terms of the ratio of $I_\delta(\sigma)$ and $m_{\delta,k}(\sigma)$ for those values of σ.

<u>Theorem</u> 1. If $f(s) \epsilon E$ and $0<\sigma_0 < \sigma_1 < \sigma_2$, then

$$(2.1)\quad \frac{I_\delta(\sigma_1)}{m_{\delta,k}(\sigma_1)} - \frac{k}{2} \leqslant \frac{1}{2(\sigma_2-\sigma_1)} \log\frac{m_{\delta,k}(\sigma_2)}{m_{\delta,k}(\sigma_1)} \leqslant \frac{I_\delta(\sigma_2)}{m_{\delta,k}(\sigma_2)} - \frac{k}{2}\,.$$

<u>Proof</u>. From (1.6), we have

$$(2.2)\quad \{\log e^{k\sigma} m_{\delta,k}(\sigma)\} = \log\{e^{k\sigma_0} m_{\delta,k}(\sigma_0)\} + 2\int_{\sigma_0}^{\sigma} \frac{I_\delta(x)}{m_{\delta,k}(x)}\,dx\,.$$

Therefore,

$$\log \{e^{k(\sigma_2-\sigma_1)} \frac{m_{\delta,k}(\sigma_2)}{m_{\delta,k}(\sigma_1)}\} = 2\int_{\sigma_1}^{\sigma_2} \frac{I_\delta(x)}{m_{\delta,k}(x)}\,dx$$

$$\geqslant \frac{2\, I_\delta(\sigma_1)}{m_{\delta,k}(\sigma_1)}\,(\sigma_2-\sigma_1),\ \sigma_1>\sigma_0$$

and

$$\log \{e^{k(\sigma_2-\sigma_1)} \frac{m_{\delta,k}(\sigma_2)}{m_{\delta,k}(\sigma_1)} \leqslant \frac{2\, I_\delta(\sigma_2)}{m_{\delta,k}(\sigma_2)}\,(\sigma_2-\sigma_1),\ \sigma_1 > \sigma_0,$$

From Theorem B, we have $e^{k\sigma} I_\delta(\sigma)$ is convex function of $e^{k\sigma} m_{\delta,k}(\sigma)$, and therefore $\{ \frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \}$ increase for $\sigma>\sigma_0$.

Corollary. If $f(s) \in E$, other than a constant and $0 < \alpha < 1$, then

(2.3) $$\lim_{\sigma\to\infty} \left\{ \frac{m_{\delta,k}(\alpha\sigma)}{e^{k\sigma}\, m_{\delta,k}(\sigma)} \right\} = 0.$$

If we put $\sigma_1 = \alpha\sigma$ and $\sigma_2 = \sigma$ in (2.1), then

$$\exp\left[-2\left\{\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}\right\}\sigma(1-\alpha)\right] \leqslant \frac{m_{\delta,k}(\alpha\sigma)}{m_{\delta,k}(\sigma)} \cdot \exp[k\sigma(\alpha-1)]$$

$$\leqslant \exp\left[-2\left\{\frac{I_\delta(\alpha\sigma)}{m_{\delta,k}(\alpha\sigma)}\right\}\sigma(1-\alpha)\right]$$

The result follows on taking limits of both the sides after dividing by $e^{k\alpha\sigma}$.

3. Theorem 2. If $f(s) \in E$ and is of Ritt order $\rho\,(0<\rho<\infty)$, type τ and lover type t, then

(3.1) $$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\left\{\frac{I_{\delta(\sigma)}}{m_{\delta,k(\sigma)}}\right\}}{e^{\rho\sigma}} \leqslant \begin{matrix} e\,\rho\,\delta\,\tau/2 \\ e\,\rho\,\delta\, t/2 \end{matrix}$$

Proof. From (2.2) we have for $h > 0$

$$\log\{e^{k(\sigma+h)}\, m_{\delta,k}(\sigma+h)\} = O(1) + 2\int_{\sigma_0}^{\sigma+h} \frac{I_\delta(x)}{m_{\delta,k}(x)}\,dx$$

$$> 2\int_{\sigma}^{\sigma+h} \frac{I_\delta(x)}{m_{\delta,k}(x)}\,dx\,, \qquad \sigma > \sigma_0,$$

$$\geqslant 2\,\frac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)}\,h.$$

Hence,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\left\{ \dfrac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\}}{e^{\rho\sigma}} \leqslant \frac{e^{\rho h}}{2h} \lim_{\sigma\to\infty} \frac{\log m_{\delta,k}(\sigma+h)}{e^{\rho\cdot(\sigma+h)}}$$

$$\leqslant \frac{\delta\ e^{\rho h}}{2h} \lim_{\sigma\to\infty} \frac{\log M(\sigma+h)}{e^{\rho\,(\sigma+h)}}$$

$$= \begin{matrix}\delta\ e^{\rho h}\ \tau/2h\\ \delta\ e^{\rho h}\ t/2h\end{matrix}$$

Taking $h = \frac{1}{\rho}$, we get (3.1).

4. Let $L(e^\sigma)$ be a slowly changing function, i.e.

(i) $L(e^\sigma) > 0$ and is continuous for $\sigma > \sigma_0$,

(ii) $L(\ell e^\sigma) \sim L(e^\sigma)$ as $\sigma \longrightarrow \infty$ for every constant $\ell > 0$.

Let, for $0 < \rho < \infty$,

$$(4.1) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}(\sigma)}{e^{\rho\sigma}\ L(e^\sigma)} = \begin{matrix}T\\ t\end{matrix}, \quad (0 < t \leqslant T < \infty);$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\left\{ \dfrac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\}}{e^{\rho\sigma} L(e^\sigma)} = \begin{matrix}p\\ q\end{matrix}, \quad (0 < q \leqslant p < \infty).$$

4. <u>Theorem</u> 3. If $f(s) \in E$ and is of Ritt order $\rho\,(0 < \rho < \infty)$, then

(i) $\frac{2q}{\rho} \leqslant t \leqslant T \leqslant \frac{2p}{\rho}$

(ii) $t \leqslant \frac{2q}{\rho} \log \left(\frac{ep}{q}\right)$, and

(iii) $T \geqslant \frac{2p}{e\rho} e^{q/p}$

<u>Proof</u>. Writing (2.2) as

$$\log \{ e^{k(\sigma+h)} m_{\delta,k}(\sigma+h) \}$$

$$= 0(1) + 2 \int_{\sigma_0}^{\sigma} \frac{I_\delta(x)}{m_{\delta,k}(x)} dx + 2 \int_{\sigma}^{\sigma+h} \frac{I_\delta(x)}{m_{\delta,k}(x)} dx, \quad (\sigma > \sigma_0)$$

$$< 0(1) + 2(p+\epsilon) \int_{\sigma_0}^{\sigma} e^{\rho x} L(e^x) dx + 2 \frac{I_\delta(\sigma+h)}{m_{\delta,k}(\sigma+h)} h$$

$$= 0(1) + 2(p+\epsilon) \int_{e^{\sigma_0}}^{e^{\sigma}} x^{\rho-1} L(x) dx + 2 \frac{I_\delta(\sigma+h)}{m_{\delta,k}(\sigma+h)} h$$

$$\sim 2(p+\epsilon) \frac{e^{\rho\sigma}}{\rho} L(e^{\sigma}) + 2 \frac{I_\delta(\sigma+h)}{m_{\delta,k}(\sigma+h)} h,$$

by ([7], Lemma 5).

Dividing by $e^{\rho\sigma} L(e^{\sigma})$, taking limits and using (4.1), we get

$$e^{\rho h} T \leqslant \frac{2p}{\rho} + 2h e^{\rho h} p, \tag{4.2}$$

$$e^{\rho h} t \leqslant \frac{2p}{\rho} + 2h e^{\rho h} q, \tag{4.3}$$

Similarly, we obtain

$$e^{\rho h} T \geqslant \frac{2q}{\rho} + 2hp, \tag{4.4}$$

(4.5) $$e^{\rho h}\ t \geqslant \frac{2q}{\rho} + 2hq.$$

It can be seen that minima of the right hand expressions of (4.2) and (4.3) occur at $h = 0$ and $e^{\rho h} = p/q$. Substituting $h = 0$ in (4.2) and $e^{\rho h} = p/q$ in (4.3) we get second part of (i) and (ii) respectively. Taking $h = (\frac{(p-q)}{\rho p})$ in (4.4) and $h = 0$ in (4.5), we get (iii) and first part of (i) respectively.

5. Theorem 4. If $\log m_{\delta,k}(\sigma) \sim T\ e^{\rho\sigma}\ L(e^{\sigma})$, then

$$\frac{I_{\delta}(\sigma)}{m_{\delta,k}(\sigma)} \sim \frac{T\rho}{2}\ e^{\rho\sigma}\ L(e^{\sigma}).$$

Proof. Suppose now $T = t$. If $0 < \eta < 1$, we have from (2.2) for $\sigma > \sigma_0$.

$$\frac{I_{\delta}(\sigma)\eta}{m_{\delta,k}(\sigma)} < \int_{\sigma}^{\sigma+\eta} \frac{I_{\delta}(x)}{m_{\delta,k}(x)}\, dx$$

$$= \frac{1}{2}\log\ e^{k(\sigma+\eta)} m_{\delta,k}(\sigma+\eta) - \frac{1}{2}\log\ \{e^{k\sigma}\ m_{\delta,k}(\sigma)\}$$

$$= \frac{T}{2}\ e^{\rho\sigma}\{1 + \rho^{\eta} + O(\eta^2)\}\ \{1 + O(1)\}\ L(e^{\sigma}) -$$

$$- \frac{T}{2}\ e^{\rho\sigma}\ L(e^{\sigma}) + O(e^{\rho\sigma}\ L(e^{\sigma})).$$

Hence,

$$\lim_{\sigma\to\infty} \sup \frac{\{\frac{I_{\delta}(\sigma)}{m_{\delta,k}(\sigma)}\}}{e^{\rho\sigma}\ L(e^{\sigma})} \leqslant \frac{T}{2}\ (\rho + H\eta),$$

where H is a constant. Since η is arbitrary, we get

$$\lim_{\sigma\to\infty} \sup \frac{\left\{ \dfrac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\}}{e^{\rho\sigma} L(e^\sigma)} \leqslant \frac{T\rho}{2} .$$

Considering $\frac{1}{2}\log\{e^{k\sigma} m_{\delta,k}(\sigma)\} - \frac{1}{2}\log\{e^{k(\sigma-\eta)} m_{\delta,k}(\sigma-\eta)\}$ amb proceeding as above, we get

$$\lim_{\sigma\to\infty} \inf \frac{\left\{ \dfrac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\}}{e^{\rho\sigma} L(e^\sigma)} \geqslant \frac{T\rho}{2},$$

and hence, $I_\delta(\sigma) \sim \frac{T\rho}{2} e^{\rho\sigma} L(e^\sigma)$.

<u>Corollary</u>. If $\dfrac{I_\delta(\sigma)}{m_{\delta,k}(\sigma)} \sim p\, e^{\rho\sigma} L(e^\sigma)$, then

$$\log m_{\delta,k}(\sigma) \sim \frac{2p}{\rho} e^{\rho\sigma} L(e^\sigma).$$

From (i) of Theorem 3, if $p = q$, $T = t = 2p/\rho$.

It is my privilege to thank Dr. S.N. Srivastava for his valuable suggestions and guidance in the preparation of this paper.

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Department of Mathematics and Astronomy
Lucknow University
Lucknow

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