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SEIFERT ORBIFOLDS AND THEIR UNITARY TANGENT SPACE

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Introduction

In this paper we define Seifert orbifolds as geometric structures on surfaces, allowing singularities of angles $\frac{2\pi\beta_i}{n_i}$ i.e. in a sector

$$P = \{ \begin{matrix} re^{i\theta} \\ r \text{ small} \end{matrix} \quad / \quad 0 \leqslant \theta \leqslant \frac{2\pi\beta_i}{n_i} \}$$

we identify two parts re^{i0} and $re^{i\frac{2\pi\beta_i}{n_i}}$ and we call this cocient $^{P}/G_i$ (G_i is the group generated by f_i, given as

$$f_i(x) = x \cdot e^{i\frac{2\pi\beta_i}{\alpha_i}}).$$

We prove that these structures can be defined, in a natural way, from polygons in H^2, $\mathbb{R}^2$ or S^2 with the same angles and that the unitary tangent bundle to a polygon projects to a Seifert orbifold. This paper deals with a geometric interpretation of the construction of foliations on Seifert bundles as it appears in "Foliations on Seifert Bundles", Ph.D. Dissertation, University of Chicago 1977 of the author; and a preprint of the same title in 1980, with some corrections on the computations and more general results. The same problem of existence of transverse foliations is approached with different methods in the paper "Transverse Foliations of Seifert Bundles and Self Homeomorphisms of the Circle", D. Eisenbud, U. Hirsch, W. Newmann, Coment. Math. Helv 56, 1981, 638, 660.

I.- Seifert orbifolds and their unitary tangent space

1.1. Definition.- A Seifert orbifold is a closed surface of genus g, B_g, with the following additional structure:

a) B_g is covered by open sets $\{U_i\}$, where each U_i is associated to a homeomorphism

$$\psi_i : U_i \longrightarrow {}^{P_i}/_{G_i} ,$$

where $P_i = \{re^{i\theta},\ 0 \leqslant \theta \leqslant \frac{2\Pi\beta_i}{n_i},\ r > 0$ being a small number$\}$ for some pair of integers (β_i, n_i) with $0 < \beta_i \leqslant n_i$, and G_i is the group generated by the rotation of angle $\frac{2\Pi\beta_i}{n_i}$.

b) Whenever $U_i \quad U_j$, there is an injective homomorphism

$$f_{ij} : G_i \longrightarrow G_j \qquad \text{and}$$

an isometric embedding preserving the canonical orientation

$$\psi_{ij} : P_i \to P_j$$

so that the following diagram commutes:

$$
\begin{tikzcd}
P_i \arrow[r, "\psi_{ij}"] \arrow[d] & P_j \arrow[d] \\
P_i/G_i \arrow[r] & P_j/f_{ij}(G_i) \\
U_i \arrow[r] \arrow[u, "\psi_i"] & U_j \arrow[u]
\end{tikzcd}
$$

Given a Seifert orbifold structure on B_g, we can choose an open cover $\{U_\alpha\}$ of B_g and a subcollection $\{U_i\}_{i=1,\dots,N}$, such that $n_\alpha = 1$ for $\alpha \not\in \{1,\dots,N\}$, $n_i \neq 1$ for $i = 1,\dots,N$.

We denote the Seifert orbifold by $B_{(g,\{(n_i,\beta_i)\},N)}$, with the convention that g will be positive if B_g is orientable and g will be negative if B_g is not orientable.

Let us suppose that P is a geodesic polygon, that can be realized in H^2, $\mathbb{R}^2$ or S^2, with sides $\{S_i\}_{i=1,\dots,2N+4g}$ positively oriented and angles α_i, between S_i and S_{i+1}, with values

$$\{\alpha_i\} = \{\frac{2\Pi\beta_1}{n_1}, \frac{2\Pi}{4g+N}, \frac{2\Pi\beta_2}{n_2}, \dots, \frac{2\Pi\beta_N}{n_N}, \frac{2\Pi}{4g+N}, \dots\}$$

for pairs of integers (n_i,β_i) with $0 < \beta_i < n_i$.

Let us denote the polygon by $P_{(g,(n_1,\beta_1),\dots,(n_N,\beta_N)}$ and let $\{f_i\}_{i=1,\dots,N}$ be an isometry sending S_{2i-1} to S_{2i} while we shall denote by g_i the isometry sending $S_{2N+4j+i}$ to $S^{-1}_{2N+4j+i+2}$, $j=1,\dots,g$, $i = 1,2$.

Similarly we can consider a polygon $P_{(g,(n_1,\beta_1),\dots,(n_N,\beta_N))}$

with angles $\{\alpha_i\} = \{\frac{2\Pi\beta_1}{n_1}, \frac{2\Pi}{2g+N}, \frac{2\Pi\beta_2}{n_2}, \frac{2\Pi}{2g+N}, \ldots, \frac{2\Pi\beta_N}{n_N}, \frac{2\Pi}{2g+N} \ldots\}$ $i=1,\ldots N+2g$. The $\{f_i\}$ are defined as above and g_i sends $S_{2N+2j+1}$ to $S_{2N+2j+2}$ $j=0,\ldots,g-1$.

Let G be the group generated by $\{f_i, g_i\}$ for a given polygon $P_{(g;(n_1,\beta_1),\ldots,(n_N,\beta_N))}$. The isometries f_i are conjugate with rotations and the isometries g_i are hyperbolic functions; they can be considered also as diffeomorphisms of the circle restricting to the boundary of H^2 (in the case of S^2 or $\mathbb{R}^2$ they are rotations). Therefore they act on the boundary of P and on the fibre of these points i.e. G acts on the unitary tangent space of the polygon $T_1(P_{(g;(n_1,\beta_1),\ldots,(n_N,\beta_N)})$.

1.2. <u>Theorem</u>.-

Given a polygon $P(g,(n_1,\beta_1),\ldots,(n_N,\beta_N)$ we have the following conmutative diagram:

$$\begin{array}{ccc} T_1(P) & \longrightarrow & \dfrac{T_1(P)}{G} \\ \Big\downarrow & & \Big\downarrow \rho \\ P & \longrightarrow & \dfrac{P}{G} \end{array}$$

1) p is a Seifert bundle with Seifert invariants, [3],

$$(0,\ o,\ g,\ N-2+2g,\ n_1,\ \beta_1,\ldots,n_N,\ \beta_N)$$

for $m_1\beta_i + \tau_i n_i = 1, \quad 0 < \beta_i < n_i \quad$ if $\quad g \geqslant 0$.

1') p is a Seifert bundle with Seifert invariants:

$(0,\ n_{\cdot}\ -|g|\ ,\ N-2+|g|\ ,\ n_1\beta_1,\dots,\beta_N)$ for $m_i\beta_i + r_i n_i = 1,$

$$0 < _{\cdot\, i} < n_i \quad \text{if} \quad g < 0.$$

2) Π gives to ${}^{P}/_{G}$ an structure of Seifert orbifold ${}^{B}(g,\{(n_i,\beta_i)\},\ N)$

3) $\frac{T_1(P)}{G} \xrightarrow{\ P\ } \frac{P}{G}$ admits a connection on the Seifert bundle.

1.3. <u>Definiton</u>.- A connection in a Seifert bundle $\xi : M \longrightarrow B$, $g \geqslant 0$, is a one for $\bar{\theta} \in A'(M)$ satisfying:

$$1)\ \hat{\theta}_x \circ \lambda_x = id$$

$$2)\ Rg^* \hat{\theta} = \hat{\theta}$$

where R_g is given by the action of g on M, and $\lambda_x : \mathbb{R} \longrightarrow T_xM$ is the induced map on the tangent spaces of the map $S^1 \longrightarrow M$, given by the fiber in a point.

If B is a non-orientable closed surface, $\Pi : B' \longrightarrow B$ is the oriented double cover with group G of covering transformations, then a connection for a Seifert bundle $\xi(M \xrightarrow{P} B)$ is a one form $\hat{\theta}$ with twisted coeficients $A'(M;\mathbb{R}[G])$ such that $\Pi^*(\hat{\theta})$ is a connection for the double Seifert bundle of ξ over B'.

We call the image of λ_x vertical vectors of the connection and the kernel of θ_x horizontal vectors of the connection.

<u>proof of 1.2</u>.-

a) Suppose that $\Sigma\frac{\beta_i}{n_i} \leqslant N - 2 + 2g(\Sigma\frac{\beta_i}{n_i} \leqslant N - 2 + |g|)$ there fore $\underline{P}$ can be realized in hyperbolic or euclidean geometry.

G is a subgroup of Moebius transformations of H^2 and therefore theirs restriction to the boundary is a subgroup of $TOPS^1$ (homeomorphisms of S^1).

i) We claim that $[\tilde{g}_{2g},\tilde{g}_{2g-1}]\circ\cdots\circ[\tilde{g}_2\circ\tilde{g}_1]\circ\cdots\circ\tilde{f}_2\circ\tilde{f}_1$ is a translation by $N-2+2g$ if $0\leq\tilde{f}_i(x)-x<1$, $0\leq\tilde{g}_i(x)-x<1$.

ii) $\dfrac{T_1(P)}{G}$ is a Seifert bundle with Seifert invariants

$$(0,\ o,\ b,\ n_1,\ \beta_1,\ldots,\beta_N).$$

iii) $b=[\tilde{g}_{2g},\tilde{g}_{2g-1}]\circ\cdots\circ\tilde{f}_2\circ\tilde{f}_1$

To prove ii) observe that, if B_ϵ is a small ball in the vertex $\dfrac{2\Pi\beta_i}{n_i}$, then $B_\epsilon\cap P$ is isometric to a small ball in the center of H^2 with points $re^{i\theta}$, $0\leq\theta\leq\dfrac{2\Pi\beta_i}{n_i}$. Let $\sim$ the equivalence relation

$$(re^{i\theta},e^{i\psi})\sim(re^{i(\theta+\frac{2\Pi\beta_i}{n_i})},\ e^{i(\psi+\frac{2\Pi\beta_i}{n_i})}).$$

We construct a covering map from a solid torus to $T_1(B_\epsilon\cap P)$ as follows:

$$T\longrightarrow\frac{(B_\epsilon\cap P)\times S^1}{\sim}$$

$$(re^{i\theta},e^{i\psi})\longrightarrow(re^{i\theta'\beta_i},\ e^{i(\psi-\frac{2\Pi k\beta_i}{n_i})})$$

with $\theta=\theta'+\dfrac{2\Pi k}{n_i}$ $0\leq\theta'<\dfrac{2\Pi}{n_i}$.

The group of covering transformations is generated by:

$$\rho_{(n_i,m_i)}(re^{i\theta}, e^{i\psi}) = (re^{i(\theta + \frac{2\Pi m_i}{n_i})}, e^{i(\psi + \frac{2\Pi}{n_i})})$$

where $\beta_i m_i + \tau_i n_i = 1$ for integers $1 \leqslant m_i < n_i$.

b) If $\frac{\beta i}{n_i} > N - 2 + 2g$ ($\frac{\beta i}{n_i} > N - 2 + |g|$)

then the polygon can be realized in S^2, and the proof works as in case a).

The proof of i) and iii) is just a trivial geometric argument. Then we have proved 1) or 1'). 2) is trivial and 3) is given by the proyection of the standard connection of H^2, $\mathbb{R}^2$ or S^2 on B_g.

1.4. <u>Definition</u>.- In the conditions of theorem 1, 2, we call $T_1(P)/G$ the unit tangent bundle to the Seifert orbifold $B_{(g,\{(n_i,\beta_i)\},N)}$.

If $g=0$, $N=1$, or $g=0$, $N=2$, $n_1 \neq n_2$ we define the unitary tangent bundle to $B_{(0,\{(n_i,\beta_i)\},N)}$ as the Seifert bundle over S^2 with Seifert invariants

$$(0, o, o, -1, (n_1,\beta_1)) \text{ and } (0,o,o,o,(n_1,\beta_1),(n_2,\beta_2))$$

1.5. <u>Definition</u>.- A Seifert orbifold $B_{(g,\{(n_i,\beta_i)\},N)}$ has an associated number

$$-(N-2+2g-\Sigma\frac{\beta_i}{n_i}),\ g \geq 0.$$

$$-(N-2+|g|-\Sigma\frac{\beta_i}{n_i}),\ g < 0.$$

that we call Euler chracteristic of the Seifert orbifold.

Observe that 1.5 is well defined by theorem 5 in Seifert's paper |3|.

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