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THE BP^*-MODULE STRUCTURE OF $BP^*(E_8)$ FOR $p = 3$.

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§0. Introduction.

Let $BP^*(-)$ be the Brown-Peterson cohomology theory with the coefficient $BP^* \simeq Z_{(p)}[v_1,\ldots]$ at an odd prime p. In this paper we shall study $BP^*(G)$ for simple simply connected Lie groups G. When G is torsion free, there is a BP^*-algebra isomorphism $BP^*(G) \simeq BP^* \otimes H^*(G)$. Hence we shall only consider the cases when G has p-torsion, i.e., the exceptional Lie groups

$$p=3 \quad F_4,\ E_6,\ E_7,\ E_8 \qquad \text{and}$$

$$p=5 \quad E_8.$$

When $p=3$ $G=F_4$, E_6 and $p=5$ $G=E_8$, the BP^*-algebra structures of $BP^*(G)$ are given in [5]. In this paper we shall determine the BP^*-module structures of $BP^*(G)$ for the rest, i.e., $p=3$ E_7, E_8 (Theorem 1.1, Therem 1.2). Because the BP^*-module structure of $BP^*(E_8)$ is so complicated we give a graph which will, hopefully, make it clearer (, see Graph 1.3).

To compute $BP^*(G)$, we use $H^*(G)$, $H^*(G;Z_3)$[1], $BP^*(G;Z_3)$[4], $K^*(G)$, $K^*(G;Z_3)$ [2],[4]. The main machine of the computation is the Atiyah-Hirzebruch type spectral sequence. Its non zero differentials are $d_{2p-1}=v_1 \otimes Q_1$, $d_{2p^2-1}=v_2 \otimes Q_2$ and $d_{4p-3}=v_1^2 \otimes$(some operation). In this paper only the proof of Theorem 1.2 ($BP^*(E_8)$) is given. Theorem 1.1 is proved by similar but easier arguments. Most parts of the computations are routine. So only the tables of results are given.

§1. The results.

In this section we give the main results.

Theorem 1.1. There is a BP*-module isomorphism for p=3

$$BP^*(E_7) \simeq BP^*(F_4) \otimes \Lambda(x_{19},x_{27},x_{35})$$

$$\simeq [BP^*\{1,y_3,y_{26}\} \oplus BP^*\{y_{19},y_{23}\}/(3y_{19}=v_1y_{23}) \oplus BP^*/(3,v_1)[x_8]/(x_8{}^3)]$$

$$\otimes \Lambda(x_{11},x_{15},x_{19},x_{27},x_{35}).$$

Theorem 1.2. There is a BP*-module isomorphism for p=3

$$BP^*(E_8) \simeq (T/R_1 \oplus F/R_2) \otimes \Lambda(x_{27},x_{35},x_{39},x_{47})$$

where

(1) $T=BP^*/(3)\otimes [(Z_3[x_8]/(x_8{}^3)\otimes Z_3[x_{20}]/(x_{20}{}^3)\otimes \Lambda(u_{27})-\{1\}-\{u_{27}x_8{}^2x_{20}{}^2\})$
$\oplus\ Z_3\{(x_8,x_8{}^2,u_{27},u_{27}x_8)\otimes(w_{43},w_{55})\}]$.

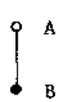
(2) $R_1 = \mathrm{Ideal}(v_1x_8-v_2x_{20}, v_1w_{43}-v_2w_{55}, v_1x_{20}, v_2abc$ where
$a,b,c\in\{x_8,x_{20},u_{27}\})$.

(3) $F = BP^*\{1,y_{23},w_{15},y_{59},w_{55},w_{43},y_{23},y_{59},y_{23}w_{55},y_{23}w_{43},s_{74},y_3,y_3y_{23},$
$w_{22},y_{62},y_{85},y_{81},y_{15},y_{38},w_{34},y_{74},y_{97},s_{93},y_3y_{15},y_{41},y_{77},y_{100}\}$.

(4) $R_2 = \mathrm{Ideal}(v_1{}^2y_{23}-3w_{15}, v_1y_{59}-3w_{55}, v_2y_{59}-3w_{43}, v_1w_{43}-v_2w_{55},$
$v_1y_{23}w_{55}-3s_{74}, v_1y_3y_{23}-3w_{22}, v_1y_{85}-3s_{81}, v_1y_{38}-3w_{34}, v_1y_{97}-3s_{93})$.

The above theorem appears to be too complicated to be understood. Hence the following graph may be useful.

In the graph, one line means $v_1A=B$

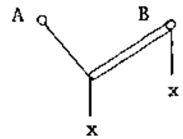


A double line means multiplication by v_2. A dotted line means multiplication by 3. An X-mark means zero is the result of multiplication.

Graph 1.3.

(1) 3-torsion parts

(1.1)

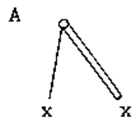


This means

$v_1A=v_2B$, $v_1^2A=0$ and $v_1B=0$.

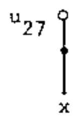
Here $(A,B)=(x_8a,x_{20}a)$ where $a=1,x_8,u_{27}$ or $(A,B)=(w_{43}b,w_{55}b)$ where $b=x_8,x_8^2$, u_{27}, $u_{27}x_8$.

(1.2)

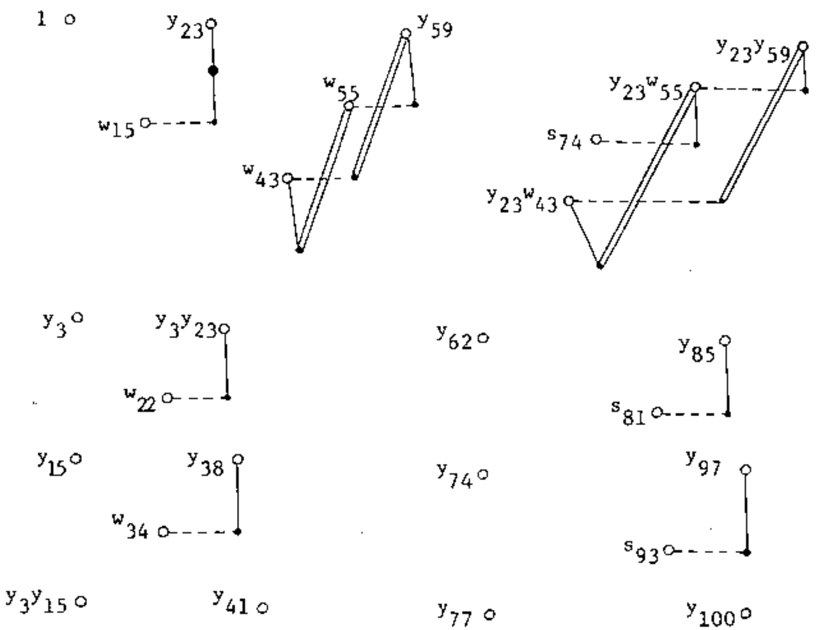


$A=(x_8^2x_{20},x_{20}^2x_8,x_{20}^2)(1,u_{27})$,
$x_8^2x_{20}^2$, $x_8^2u_{27}$, $x_{20}x_8u_{27}$.

(1.3)



(2) Torsion free parts



Remark 1.4. If $\Sigma v_j x_j = 0$ in $BP^*(E_8)$ then there exists $y \in H^*(E_8;Z_p)$ such that $Q_j(y) = i(x_j)$ where $i : BP \to KZ_p$ is the natural map. This is expressed by the graph;

$$\begin{tikzcd}[column sep=small] x_1 \arrow[dr, "v_1"'] & & x_2 \arrow[dl, Rightarrow, "v_2"] \\ & \circ & \end{tikzcd} \qquad \text{implies} \qquad \begin{tikzcd}[column sep=small] i(x_1) & & -i(x_2) \\ & y \arrow[ul, "Q_1"] \arrow[ur, Rightarrow, "Q_2"'] & \end{tikzcd}$$

Results generalizing this fact are discussed in [6].

§2. Preliminary results.

In this section, we recall known results which are needed to compute $BP^*(E_8)$. First recall the mod $p=3$ ordinary cohomology group [1]

(2.1) $A_p = H(E_8;Z_3) \cong Z_3[x_8,x_{20}]/(x_8^3,x_{20}^3) \otimes \Lambda(x_3,x_7,x_{15},x_{19},x_{27},x_{35},x_{39},x_{47})$

where $Q_0x_7 = x_8$, $Q_0x_{19} = x_{20}$, $Q_1x_3 = x_8$, $Q_1x_{15} = x_{20}$, $Q_2x_3 = x_{20}$.

For ease of notation, let $B' = \Lambda(x_{27},x_{35},x_{39},x_{47})$.

Secondly, consider the cohomology group $H^*(E_8)$. Note that E_8 has no higher 3-torsion ([3]). Consider the Bockstein exact sequence

$$\begin{tikzcd}[column sep=small] H^*(E_8) \otimes Z_{(p)} \arrow[rr, "3"] & & H^*(E_8) \otimes Z_{(p)} \arrow[dl, "i"] \\ & H^*(E_8;Z_3). \arrow[ul, "\delta"] & \end{tikzcd}$$

By the fact that $i\delta = Q_0$, we have

$$\begin{aligned} (2.2) \quad A &= H^*(E_8) \otimes Z_{(3)} \\ &\cong [(Z_3[x_8]/(x_8^3) \otimes Z_3[x_{20}]/(x_{20}^3) \otimes \Lambda(\{x_8x_{19}-x_7x_{20}\}) - \{1\} - \\ &\qquad (x_8^2x_{20}^2)\{x_8x_{19}-x_7x_{20}\}) \oplus \Lambda(x_8^2x_7, x_{20}^2x_{19})] \otimes \Lambda(x_3,x_{15}) \otimes B' \end{aligned}$$

Using the Atiyah-Hirzebruch spectral sequence

$${}^{P}E_2 = H^*(E_8;Z_3) \otimes BP^* \Longrightarrow BP^*(E_8;Z_3),$$

in [4], $BP^*(E_8;Z_3)$ is computed

(2.3) $BP^*(E_8;Z_3) \simeq \big(BP^*/(3)\{1,w_{15},w_{74}\} \oplus BP^*/(3) \otimes Z_3\{w_{55},w_{43},x_{20},x_{20}^2\} \otimes Z_3\{1, x_8,x_8^2\}/\mathrm{Ideal}(v_1w_{43}-v_2w_{55},\ v_1x_8-v_2x_{20},\ v_1x_{20},\ v_2x_8^2x_{20})]\big) \otimes \Lambda(x_7,x_{19}) \otimes B'$.

Let $K^*(-)$ be the K-theory. By Hodgkin [2][4], $K^*(E_8)$ is torsion free and

(2.4) $K^*(E_8) \simeq \Lambda(\alpha,\beta,\gamma,\delta) \otimes B'$,

$$K^*(E_8)\otimes Z_3 \simeq K^*(E_8;Z_3) \simeq K(1)^*\otimes\Lambda(\{x_3x_8^2\},\{x_{15}x_{20}^2\})\otimes\Lambda(x_7,x_{19})\otimes B'.$$

On the other hand

$$K^*(E_8)\otimes Q \simeq K\otimes H^*(E_8)\otimes Q \simeq K^*\otimes\Lambda(\{x_8^2x_7\},\{x_{20}^2x_{19}\})\otimes\Lambda(x_3,x_{15})\otimes B'.$$

§3. The differential d_{2p-1}.

In this section we begin the computation of $BP^*(E_8)$. In this section we compute the E_{2p}-term of the Atiyah-Hirzebruch spectral sequence.

Consider the Atiyah-Hirzebruch spectral sequences

$$\begin{array}{ccc} E_2 = H^*(E_8)\otimes BP^* & \Longrightarrow & BP^*(E_8) \\ \downarrow{\scriptstyle i(E)} & & \downarrow{\scriptstyle i} \\ {}^pE_2 = H^*(E_8;Z_3)\otimes BP^* & \Longrightarrow & BP^*(E_8;Z_3) \end{array}.$$

Notice that for dimensional reasons, the first non zero differential d_r is d_{2p-1}. Recall that

$$\mathrm{Image}\ d_r \subset \mathrm{Torsion}\ E_r .$$

Let $d_rx=v_1\otimes y\neq 0$, for $x,y\in A$. Then $v_1y\in\mathrm{Tor}\ E_r$ and so $y\in\mathrm{Tor}(A)$. Since all elements in $\mathrm{Tor}(A)$ are 3-torsion (no higher torsion), $i(E)(y)\neq 0$. This implies;

Lemma 3.1. In the above spectral sequence E_{2p-1}, $d_{2p-1}x=y\neq 0$ if and only if $d_{2p-1}(i(E)(x))=i(E)(y)\neq 0$ in ${}^pE_{2p-1}$.

In the mod p spectral sequence ${}^pE_{2p-1}$

$$d_{2p-1} = v_1 \otimes Q_1.$$

The homology group of the differential Q_1 is computed by using (2.1) and the fact Q_1 is a derivation. The following table describes the Q_1 homology.

Table 1.

(a) Z_3-module in A

	1	x_8	x_8^2	x_{20}	x_{20}^2	x_8x_{20}	$x_8^2x_{20}$	$x_8x_{20}^2$	$x_8^2x_{20}^2$
1	-	I	I	I	I	II	II	II	II
$x_8x_{19}-x_7x_{20}$	0	I	II	I	II	II	II	II	-
x_3	-	X	0	X	X	X	I	X	I
x_{15}	-	X	X	X	0	X	X	I	I
x_3x_{15}	-	X	X	X	X	X	X	X	0
$x_3(x_8x_{19}-x_7x_{20})$	X	X	I	X	X	X	I	-	-
$x_{15}(x_8x_{19}-x_7x_{20})$	X	X	X	X	I	X	0	I	-
$x_3x_{15}(x_8x_{19}-x_7x_{20})$	X	X	X	X	X	X	0	0	-
$x_8x_{15}-x_3x_{20}$	I	I	-	I	-	I			
$(x_8x_{15}-x_3x_{20})(x_8x_{19}-x_7x_{20})$	I	I	-	I	-	I			

(b) torsion free nodule in A

	1	$x_8^2x_7$	$x_{20}^2x_{19}$	$x_8^2x_7x_{20}^2x_{19}$
1	0	0	0	0
x_3	X	0	X	0
x_{15}	X	X	0	0
x_3x_{15}	X	X	X	0

The entries in this table have the following meaning,

- ; there is no element corresponding to this entry

$(x_8x_{20}{}^2x_3(x_8x_{19}-x_7x_{20})$ is express as $x_3(x_8x_{19}-x_7x_{20})x_8x_{20}{}^2+$

$x_8x_{20}(x_8x_{15}-x_3x_{20})(x_8x_{19}-x_7x_{20}))$

O ; in ker Q_1 and not in image Q_1, X ; Q_1-image is non zero ,

I ; in image Q_1, II ; in double Q_1-image from two entries.

From the above Table 1, the E_{2p}-term is expressed as

(3.2) $E_{2p} \simeq BP^* \otimes Z\{0 \text{ in } (b)\}$

$\oplus\ BP^*\{x_3x_{20}^2x_{19}-x_{15}(x_8x_{19}-x_7x_{20})x_{20},\ x_{15}x_8^2x_7-x_3(x_8x_{19}-x_7x_{20})x_8\}$

$\oplus\ BP^*\{3x_3, 3x_{15}, 3x_3x_{15}, 3x_8^2\cdot x_7x_3x_{15}, 3x_{20}^2x_{19}x_3x_{15}\}$

$\oplus\ BP^*/(3) \otimes Z_3\{0 \text{ in } (a)\}$

$\oplus\ BP^*/(3,v_1) \otimes Z_3\{\text{I and II in } (a)\}$.

§4. The differential d_{4p-3}.

The next non zero differential is d_{4p-3}.

Assume that $d_{4p-3}(y)=v_1{}^2 \otimes x \otimes b \neq 0$ where $y,x\in A$, $b\in B$. Then x is 3-torsion in A and $v_1x\neq 0$ in E_{2p}. Hence x must be a sum of the generators of type 0 in Table 1 (a). It is easily checked that such generators except for $\{x_8x_{19}-x_7x_{20}\}$ and $x_8{}^2x_{20}{}^2x_{15}x_7$ are non zero elements in

$$H(A_p,Q_1) \simeq \Lambda(x_8{}^2x_3, x_{20}{}^2x_{15}) \otimes \Lambda(x_7,x_{19}) \otimes B'$$

Moreover d_{4p-3} is always zero in ${}^PE_{4p-3}$. So the generators are not in the image of d_{4p-3} in E_{4p-3}. Therefore only the generators $\{x_8x_{19}-x_7x_{20}\}$ and $x_8^2x_{20}^2x_{15}$ must be checked.

First consider the case $x=\{x_8x_{19}-x_7x_{20}\}$. Since $\delta(x_7x_{19})=x$, x is an infinite cycle in E_* and let $u_{27}=x$ in $BP^*(E_8)$. Then from (2.3) $i(v_1^2u_{27})=0$ in $BP^*(E_8;Z_3)$ This implies $v_1^2u_{27}=pa$ for some a in $BP^*(E_8)$.

Assume that there does not exist y such that

(4.1) $d_{4p-3}y=v_1{}^2 \otimes x$.

Then $\dim(\mathrm{Filt}(a)) < 27$ and $p^2a=pv_1{}^2u_{27}=p\delta(x_7x_{19})=0$. Since $BP^*(E_8)\otimes Q \cong BP^*\otimes H^*(E_8)\otimes Q$, torsion free elements in E_{4p-3} do not express a. From Table 1 and dimensional reasons, only $v_1{}^s\otimes x_3x_8{}^2$ in E_{4p-3} can be a in $BP^*(E_8)$. But from (2.4), $v_1{}^s\otimes x_3x_8{}^2$ represents a torsion free element and this is a contradiction. Therefore there exists y such as (4.1).

The fact that dimension $|y|=18$ and y is BP^*-free or $BP^*/3$-free in E_{4p-3}, implies $y=\lambda\{3x_3x_{15}\}$, $\lambda\neq 0 \bmod p$, i.e., we can take

(4.2) $\quad d_{4p-3}(\lambda'\{3x_3x_{15}\}) = v_1{}^2\{x_8x_{19}-x_7x_{20}\}$.

Next consider the case $x=\{x_8{}^2x_{20}^2x_{15}x_7\}$. First assume that

(4.3) $\quad d_{r'}y = v_1{}^r z\otimes x$

where $z=x_{27}x_{35}x_{39}x_{47}$. It is easily checked that elements of $\dim(\mathrm{Filt}) > \dim(\mathrm{Filt}(x\otimes z))$ and of Z_3-modules in E_* are free $BP_*/3$-modules or free $BP^*/(3,v_1)$-modules in ${}^PE_\infty$. See the right down side of Table 1 (a) and Lemma 4.7 in [8]. These elements are not in the image $d_{r'}$, $r' > 2p-1$ in ${}^PE_\infty$. This fact implies that all elements of $\dim(\mathrm{Filt}) > \dim(\mathrm{Filt}(z\otimes x))$ are infinite cycles.

Denote by $zy_{59}y_{23}$ the element in $BP^*(E_8)$ which corresponds to $\{zx_{20}^2x_{19}x_8{}^2x_7\}$. Since $i(v_1zy_{59}y_{23})=v_1zx_8{}^2x_{20}{}^2x_{19}x_7=0$ in $BP^*(E_8;Z_3)$, we can write

$$v_1zy_{59}y_{23} = p^sa\ , \quad s\geqq 1.$$

Then $\dim(\mathrm{Filt}(a)) < 59+|z|$, $\dim a=55+|z|+23$ and a must be a Z_3-module in E_*. These facts imply(, see Table 1,) that a corresponds $\{zx_8^2x_{20}^2x_{19}x_7\}=zx$, ie.,

(4.4) $\quad v_1zy_{59}y_{23} = p^szx$.

Now consider tha assumption (4.3). By it,

$$v_1{}^r\ zx = \lambda b$$

where $\lambda\in BP^*$ and $\dim(\mathrm{Filt}(b)) > |zx|$. From (4.4), $p^sb = v_1^{r+1}zy_{59}y_{23}\neq 0$. But except for $y_{59}y_{23}z$ all elements in $BP^*(E_8)$ of $\dim(\mathrm{Filt}) > |zx|$ are either torsion free elements which are of form such that 4m-3 or 4m (, check Table

3 (a) and note that the dimension of each ring generator is 4m or 4m-1,) or torsion elements in $BP^*(E_8)$, e.g., $x_{20}^2 x_{15}(x_8x_{19}-x_7x_{20})=\delta(w_{55}x_7x_{19})$. This is a contradiction. Hence there is no y such as (4.3).

It does not occur that

$$d_{r'}z = v_1^{\,s}\, x \otimes z' \neq 0$$

where $z' \in \Lambda(x_{27},x_{35},x_{39},x_{47})$, because if it occurs, then $|d_{r'}z|=4m+1$, $|x|=4m-2$ and so $|z'|=4m-1$ which shows z'=one of $x_{27},x_{35},x_{39},x_{47}$, and this contradicts the dimensions $|z|=148$ and $|x|=78$. Hence we have $d_{r'}z=0$.

If there is y such that

(4.6) $\quad d_{r'}y = v_1^{\,s}x$,

then $d_{r'}(y\otimes z)=v_1^{\,s}x\otimes z$ and this contradicts the non existence of (4.4). Therefore there is no y such as (4.6).

§5. The differential $d_{2(p^2-1)+1}$.

First notice that in ${}^PE_{2(p^2-1)+1}$

$$d_{2p^2-1}x = v_2Q_2(x) \bmod (v_1).$$

We compute $H(\ker Q_1, Q_2)$. See the following Table 2.

From Table 2, the E_{2p^2}-term is expressed as

(5.1) $\quad E_{2p^2} \simeq [BP^*\,Z\{0,\ 30,\ 90,\ a \text{ in } (b)\}$

$\oplus\ BP^*\cdot(3x_8x_8^{\,2}x_7+v_1x_3x_8^{\,2}x_7,\ 3x_{15}x_8^{\,2}x_7+v_1b)$

$\oplus\ BP^*/(3)\{0 \text{ in } (a),\ v_1x_3x_8^{\,2}\}$

$\oplus\ BP^*/(3,v_1)\{I \text{ in } (a)\}$

$\oplus\ BP^*/(p,v_1,v_2)\{W \text{ in } (a)\}] \otimes B.$

Now we compare E_{2p^2} and ${}^PE_\infty$. See Lemma 4.7 and Lemma 4.8 in [4].

(5.2) $\quad {}^PE_\infty = {}^PE_{2p^2} \simeq [BP^*/(3)\{v_1x_3x_8^{\,2},x_3x_8^{\,2}x_{15}x_{20}^{\,2},x_{15}x_{20}^{\,2}\}$

$\oplus\ BP^*/(3,v_1)\{x_3x_8^2x_{20}^2,x_{15}x_{20}^2(x_8,x_8^2),\ (x_3x_{20}-x_8x_{15})(x_{20},x_{20}x_8),$

$x_8,x_8^2,x_{20},x_8x_{20}\}$

$\oplus\ BP^*/(3,v_1,v_2)\{x_{20}^2, x_8x_{20}^2, x_8^2x_{20}, x_8^2x_{20}^2\}] \otimes \Lambda(x_7,x_{19}) \otimes B'$.

Let x be the generators in Table 2. From (5.2) and Table 2, we can easily check that if x is $BP^*/(3)$-free(, $BP^*/(3,v_1)$-free, $BP^*/(3,v_1,v_2)$-free,) in $E_{2p}2$, then $i(E)(x)$ is also $BP^*/(3)$-free(, $BP^*/(3,v_1)$-free, $BP^*/(3,v_1,v_2)$-free respectively) except for $x=x_8x_{19}-x_7x_{20}$, $x_8^2x_{20}^2x_{15}x_7$. Moreover the exceptional x are $BP^*/(3,v_1)$-free in ${}^PE_{2p}2$. Hence there is no y such that

$$d_r y = \lambda x \qquad \text{where } \lambda\in BP^*,\ r \geq 2p^2+1.$$

This implies that

(5.3) $\quad E_{2p}2 \cong E_\infty$.

Table 2.

(a) Z_3-module in $E_{2(p^2-1)+1}$.

	1	x_8	x_8^2	x_{20}	x_{20}^2	x_8x_{20}	$x_8^2x_{20}$	$x_8x_{20}^2$	$x_8^2x_{20}^2$
1	-	I	I	I	W	I	W	W	W
$x_8x_{19}-x_7x_{20}$	0	I	W	I	W	W	W	W	-
x_3	-	-	0X	-	-	-	X	-	I
x_{15}	-	-	-	-	0	-	-	I	I
x_3x_{15}	-	-	-	-	-	-	-	-	0
$x_3(x_8x_{19}-x_7x_{20})$	-	-	X	-	-	-	I	-	-
$x_{15}(x_8x_{19}-x_7x_{20})$	-	-	-	-	I	-	0	I	-
$x_3x_{15}(x_8x_{19}-x_7x_{20})$	-	-	-	-	-	-	0	0	-

$x_8x_{15}-x_3x_{20}$	X	X	-	I	-	I
$(x_8x_{15}-x_3x_{20})(x_8x_{19}-x_7x_{20})$	X	X	-	I	-	I

(b) torsion free module in $E_{2(p^2-1)+1}$.

	1	$x_8^2x_7$	$x_{20}^2x_{19}$	$x_8^2x_7x_{20}^2x_{19}$
1	0	0	0	0
x_3	30	x	a0	0
x_{15}	30	bx	0	0
x_3x_{15}	90	30	30	0

The entries in this table have the following meaning

– ; empty, 0 ; 0 in Table 1, in ker Q_2 and not in image Q_2,

30, 90 ; this entry corresponds to the element multiplied by 3 or 9,

X ; Q_2-image is non zero (OX ; moreover O in Table 1. (a)),

I ; I or II in Table 1 and not in image Q_2, W ; in image Q_2 ,

a ; $x_3x_{20}^2x_{19}-x_{15}(x_8x_{19}-x_7x_{20})x_{20}$, b ; $x_{15}x_8^2x_7-x_3(x_8x_{19}-x_7x_{20})x_8$.

§6. The BP*-module structure of $BP^*(E_8)$.

In this section, we determine the BP*-module structure of $BP^*(E_8)$. The results are expressed in Table 3. Our notation follows that of Table 2.

First consider the torsion free elements in $BP^*(E_8)$.

(1) From the definition

$$v_1y_3y_{23} = 3w_{22},\ v_1y_{38} = 3w_{34} .$$

(2) That $i(v_1^2y_{23})=v_1^2x_8^2x_7=0$ in $BP^*(E_8;Z_3)$ implies $v_1^2y_{23}=pa$. Since $x_8^2x_7$ is BP*-free in E_∞. $\dim(Filt(a)) < 23$. From $K(1)^*$-theory (2.4),

$$v_1^2y_{23} = p^s\ v_1x_3x_8^2 = p^sw_{15} .$$

Since $v_1x_3x_8^2$ is a Z_3-module in E_∞ we have

$$3\ v_1x_3x_8^2 = b \quad \text{for} \quad \text{dimension}(Filt(b)) > 15.$$

These facts imply that s=1, i.e.,

$$v_1^2y_{23} = 3w_{15} .$$

(3) From (4.4) and arguments similar to (2), we can take

$$v_1 y_{59} = 3w_{55} ,$$

$$v_2 y_{59} = 3w_{43} .$$

(4) From (3) we have

$$v_1 y_{23} y_{59} = 3y_{23} w_{55} , \quad v_2 y_{23} y_{59} = 3y_{23} w_{43} .$$

(5) In $BP^*(E_8)$ mod $(v_1, v_2, \ldots)$, we have

$$3s_{81} = 3\{x_3 x_8{}^2 x_{20}{}^2 x_{15} x_7\} = \{3x_3\}\{x_8{}^2 x_{20}{}^2 x_{15} x_7\} = y_3 w_{55} y_{23} .$$

Hence in $BP^*(E_8)$ mod$(v_1{}^2, v_2, \ldots)$, we have

$$3v_1 y_{85} = v_1 y_3 y_{23} y_{59} = 3y_3 y_{23} w_{55} = 9s_{81} .$$

Similarly we can chose

$$3v_1 y_{96} = 9s_{93} , \quad v_1{}^2 y_{23} w_{55} = 3v_1 s_{74} .$$

Table 3.

(a) Z_3-module in E_∞.

	1	x_8	x_8^2	x_{20}	x_{20}^2	$x_8 x_{20}$	$x_8^2 x_{20}$	$x_8 x_{20}^2$	$x_8^2 x_{20}^2$
1									
$x_8 x_{19} - x_7 x_{20}$	u_{27}								
x_3			w_{15}						$x_8^2 w_{43}$
x_{15}					w_{55}			$x_8 w_{55}$	$x_8^2 w_{55}$
$x_3 x_{15}$									s_{74}
$x_3(x_8 x_{19} - x_7 x_{20})$							$u_{43} y_{23}$		
$x_{15}(x_8 x_{19} - x_7 x_{20})$					$w_{55} u_{27}$		$y_{23} w_{55}$	$x_8 w_{55} u_{27}$	
$x_3 x_{15}(x_8 x_{19} - x_7 x_{20})$							s_{81}	s_{93}	
$x_8 x_{15} - x_3 x_{20}$				w_{43}		$x_8 w_{43}$			
$(x_8 x_{15} - x_3 x_{20})(x_8 x_{19} - x_7 x_{20})$				$w_{43} u_{27}$		$x_8 w_{43} u_{27}$			

(b) torsion free module in E_∞.

	1	$x_8^2x_7$	$x_{20}^2x_{19}$	$x_8^2x_7x_{20}^2x_{19}$
1	1	y_{23}	y_{59}	$y_{23}y_{59}$
x_3	y_3	y_3y_{23}, w_{22}	y_{62}	y_{85}
x_{15}	y_{15}	y_{38}, w_{34}	y_{74}	y_{97}
x_3x_{15}	y_3y_{15}	y_{41}	y_{77}	y_{100}

Here we note that the rest of elements in (a) in Table 3, are elements which contain x_8, x_{20} or u_{27} as factors. The fact that $\delta x_7=x_8$, $\delta x_{19}=x_{20}$, $\delta x_7x_{19}=u_{27}$ implies that the rest are Z_3-modules also in $BP^*(E_8)$, i.e., all torsion elements in $BP^*(E_8)$ are Z_3-modules (not higher 3-torsion).

(5) That $i(v_1v_{23}w_{55})=0$ in $BP^*(E_8\cdot Z_3)$ implies that there is a such that

$$v_1v_{23}w_{55} = 3a .$$

From (5) $3v_1(a-s_{74})=0$. Hence $a=s_{74}$ is a torsion element and a Z_3-module so $3(a-s_{74})=0$ and this implies

$$3s_{74}=3a = v_1y_{23}w_{55} .$$

Similarly we have

$$v_1s_{85}= 3s_{81}, \quad v_1y_{97}= 3s_{93} .$$

Next consider Z_3-modules in $BP^*(E_8)$.

Lemma 6.6. If $z\in BP^*(E_8)$ is a torsion element and $i(z)=0$ in $BP^*(E_8;Z_3)$ then $z=0$ also in $BP^*(E_8)$.

Proof. Let $i(z)=0$. Then $z=pa$ in $BP^*(E_8)$. Hence $p^2a=0$ and $z=pa=0$.

From this lemma we have

$$v_1^2u_{27}=0 , \quad \text{since } v_1^2(x_8x_{19}-x_7x_{20})=0 \text{ in } BP^*(E_8) ,$$

$$v_1x_8= v_2x_{20}, \quad v_1x_{20}=0 ,$$

$$v_2abc=0 \quad \text{where } a,b,c\in\{x_8,x_{20},u_{27}\}.$$

Moreover we have $v_2w_{55}=v_1w_{43}$.

By Table 2 and Table 3, we can check that there is no relation other than these.

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