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LIMITS AND COLIMITS IN THE CATEGORY OF SMALL CATEGORIES* Marek Golasiński

<u>Abstract.</u> The aim of this note is to show some properties of the homotopy groups of limits and colimits in the category of small categories Cat and to give a version of Milnor's theorem in this category.

Moreover, one proves that the homotopy limit (in the sen se of Bousfield and Kan, see [1, ch XI]) of a diagram of nerves of categories is itself the nerve of a category. In fact, if $F: I \longrightarrow Cat$ is a functor and \widetilde{F} is its Eilenberg-Moore rectification (see [6]) then holim NF = N(lim \widetilde{F}).

For a similar result on the homotopy colimit see [7].

<u>1. Preliminaries.</u> Let C be a category. A cohomotopy system in C is a quadruple (P; p_0, p_1, s), where P : C \longrightarrow C is a functor, whereas $p_0, p_1 : P \longrightarrow id_{\mathbb{C}}$, s: $id_{\mathbb{C}} \longrightarrow P$ are such natural transformations that $p_0 s = p_1 s = id_{\mathbb{C}}$.

Refering to Kamps (see [4]) we can define the Hurewicz fibration and cofibration. Moreover, we also have a homotopy relation in such a category.

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This quadruple determines a sequence of functors

 $P^n : C \longrightarrow C$, where $P^O := id_C$, $P^{n+1} := P(P^n)$, for $n \ge 0$ and natural transformations

$$d_{i,n}^{\delta} := P^{i-1} P_{\delta} P^{n-i} : P^{n} \longrightarrow P^{n-1}, i = 1, \dots, n, \delta = 0, 1$$

and $s_{i,n} := P^{i-1} SP^{n+1-i} : P^{n} \longrightarrow P^{n+1}, i = 1, \dots, n+1.$

Lemma 1.1. (see [4]). A sequence of functors and natural transformations $(p^n; d_{i,n}^{\delta}, s_{i,n}) \xrightarrow{n \ge 0}$ defines a cubical object in the endofunctors category of C.

Denote by Cat (Cat*) the category of (pointed) small categories and by Set^{00P} (Set^{*00P}) the category of (pointed) cubical sets.

The cohomotopy system in Cat (Cat*) is defined in the following way. Let 3 be the category given by:

 $\dots \longrightarrow -2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2 \longleftarrow \dots$

For $C \in obCat a$ functor $\sigma : 2 \longrightarrow C$ is called finite iff there exist $m_0, n_0 \in ob2$ such that $\sigma(m) = \sigma(m_0), \sigma(n) = \sigma(n_0)$ and $\sigma(m \longrightarrow m') = id_{\sigma(m_0)}, \sigma(n \longrightarrow n') = id_{\sigma(n_0)}$ for $m, m' \leq m_0$ and $n, n' \geq n_0$. The above conditions will be written briefly as $\sigma(m_0) = \sigma(-\infty)$ and $\sigma(n_0) = \sigma(+\infty)$. The full subcategory given by finite functors of Cat(2, C) is denoted by P(C).

Then P: Cat \longrightarrow Cat is the functor and for $\mathbb{C} \in obCat$ there are functors $s(\mathbb{C}) : \mathbb{C} \longrightarrow \mathbb{P}(\mathbb{C})$, and $\mathbb{P}_{O}(\mathbb{C}), \mathbb{P}_{1}(\mathbb{C}): \mathbb{P}\mathbb{C} \longrightarrow \mathbb{C}$ defined by $s(\mathbb{C})(\mathbb{C})(\mathbb{k}) = \mathbb{C}$ for $\mathbb{C} \in ob\mathbb{C}$, $\mathbb{k} \in ob\mathbb{Z}$ and $\mathbb{P}_{\delta}(\mathbb{C})(\sigma) = \sigma((-1)^{\delta}\infty)$ for $\delta = 0, 1$.

Hence we obtain the cohomotopy system $(P; p_0, p_1, s)$ in Cat and the functor Q: Cat x Cat \longrightarrow Set^{OOP}, where $Q(C, C')_{p} =$

= $Cat(C, p^n(C))$ for $C, C' \in obCat$. In particular, for C = *we have the functor $Q : Cat \longrightarrow Set^{OOP}$.

In Cat we can define the Serre fibration (see [2]). Moreover, in Cat it is easy to define a notion of the loop functor Ω , the homotopy fibre of a map etc.

For further considerations we shall need the following

<u>Theorem 1.2.</u> (see [2]). For a functor $p : \mathbb{E} \longrightarrow \mathbb{B}$ the following conditions are equivalent:

i) $p : E \longrightarrow B$ is the Serre fibration,

ii) $Q(p) : Q(E) \longrightarrow Q(B)$ is the Kan fibration.

<u>Corollary 1.3.</u> For any $\mathbb{C} \in ob\mathbb{C}at$ the cubical set $\mathbb{Q}(\mathbb{C})$ satisfies the Kan extension condition.

One can prove that for any $C, C' \in obCat$ the cubical set Q(C, C') also satisfies the Kan extension condition.

2. The homotopy groups and Milnor's theorem. Following the paper [2] for $\mathbb{C} \in \text{obCat}^*$ we put $\pi_n(\mathbb{C},*) := \pi_n(\mathbb{Q}(\mathbb{C}),*)$, where $\pi_n(\mathbb{Q}(\mathbb{C}),*)$ is the n-th homotopy group of the cubical set $\mathbb{Q}(\mathbb{C})$ (see [3]).

<u>Theorem 2.1.</u> (see [2]) i) A map $f : \mathbb{C} \longrightarrow \mathbb{D}$ induces the long exact sequence

 $\dots \longrightarrow \pi_{n}(\mathbb{C}, *) \longrightarrow \pi_{n}(\mathbb{D}, *) \longrightarrow \pi_{n-1}(f_{n}^{-1}(*), *) \longrightarrow \dots,$

Let I be a small category. Thomason (see [7]). proved that for a functor $F : I \longrightarrow Cat$, the classifying space of the Grothendieck construction B(IfF), is homotopy equivalent to

the realization of the Bousfield-Kan homotopy colimit hocolim NF(. There also exists a relation between the homotopy groups of hocolim F and colim F.

Let $F : I \longrightarrow Cat$ be a functor. The Grothendieck construction on F, $I \neq F$, is the category with objects: the pairs (i,X) with i an object of I and X an object of F(i), and with morphisms $(\alpha, x) : (i_1, X_1) \longrightarrow (i_0, X_0)$ given by a morphism $\alpha : i_1 \longrightarrow i_0$ in I and $\alpha x : F(\alpha)(X_1) \longrightarrow X_0$ in $F(i_0)$. The composition is defined by $(\alpha, x)(\alpha', x') = (\alpha \alpha', xF(\alpha)(x'))$.

For $F: I \longrightarrow Cat^*$ let $p: I \int F \longrightarrow colim^*F$ be the functor given by p(i,X) = X, then $p^{-1}(*) = I$. Moreover, for any $C \in ob \ colim^*F$ we have the pair of functors $p^{-1}(C) \longrightarrow p_{/C}$ and $p^{-1}(C) \longrightarrow C_{/p}$ given in the obvious way, where $p_{/C}$ and $C_{/p}$ are comma categories. It is not difficult to see that $p^{-1}(C)^P \longrightarrow p_{/C}$ has a left adjoint and $p^{-1}(C) \longrightarrow C_{/p}$ has a right adjoint. Hence p is the Serre fibration (see [5]).

Therefore, following the Thomason's result we have

<u>Corollary 2.2.</u> For a functor $F : I \longrightarrow Cat$ there is the long exact sequence

 $\dots \longrightarrow \pi_n(\mathbf{I}, *) \longrightarrow \pi_n(\operatorname{holim} \mathbf{F}, *) \longrightarrow \pi_n(\operatorname{colim}^* \mathbf{F}, *) \longrightarrow \dots$

In particular, if I is a contractible category (for instance, a left of right filtering category) then

 π_n (hocolim F,*) $\simeq \pi_n$ (colim* F,*) for $n \ge 0$.

On the base of the proof of Theorem 3.1 from [1,ch.IX] and with references to the fact that the cubical set Q(C,C') satisfies the Kan extension condition, we obtain

<u>Theorem 2.3.</u> (Milnor's theorem). Let \mathbb{I} be a countable small right filtering category. If $F : \mathbb{I}^{OP} \longrightarrow Cat^*$ and $F' : \mathbb{I} \longrightarrow Cat^*$ are such functors that for any map $\alpha : i \longrightarrow i^*$ in $\mathbb{E} F(\alpha) : F(i') \longrightarrow F(i)$ is the Hurewicz fibration and $F'(\alpha) : F'(i) \longrightarrow F'(i')$ is the Hurewicz cofibration then there is the short exact sequence of pointed sets

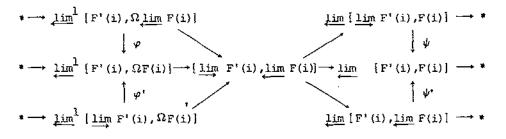
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$$\longrightarrow \underline{\lim}^{1} [F'(i), \mathcal{L}F(i)] \longrightarrow [\underline{\lim} F'(i), \underline{\lim} F(i)] \longrightarrow \underline{\lim} [F'(i), F(i)] \longrightarrow \mathbb{L}$$

where [,] denotes the set of homotopy classes of maps and $\underline{\lim}^1$ - the 1-th derived functor of $\underline{\lim}$.

<u>Corollary 2.4.</u> i) If F(i) = F for any $i \in ob \mathbb{I}$ then $* \longrightarrow \underline{\lim}^{1} [F'(i), \Omega F] \longrightarrow [\underline{\lim} F'(i), F] \longrightarrow \underline{\lim} [F'(i), F] \longrightarrow *$ is the Milnor's sequence.

ii) If F'(i) = F' for any $i \in ob I$ then $* \longrightarrow \underline{\lim}^{1}[F', \Omega F(i)] \longrightarrow [F', \underline{\lim} F(i)] \longrightarrow \underline{\lim}[F', F(i)] \longrightarrow *$ is the Vogt-Cohen's sequence.

Remark that the following diagram



is commutative. From the "Snake Lemma" we have that coker $\varphi =$ = ker ψ and coker φ ' = ker ψ '.

Hence we obtain

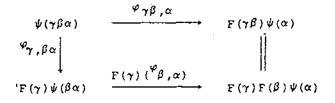
Corollary 2.5. There are the following exact sequences:

i) •
$$\longrightarrow \lim_{i \to \infty} \left[F'(i), \Omega \lim_{i \to \infty} F(i) \right] \longrightarrow \lim_{i \to \infty} \left[F'(i), \Omega F(i) \right] \longrightarrow$$

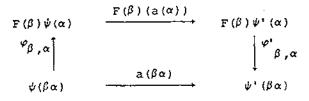
 $\longrightarrow \lim_{i \to \infty} \left[F'(i), \lim_{i \to \infty} F(i) \right] \longrightarrow \lim_{i \to \infty} \left[F'(i), F(i) \right] \longrightarrow$
ii) • $\longrightarrow \lim_{i \to \infty} \left[\lim_{i \to \infty} F'(i), \Omega F(i) \right] \longrightarrow \lim_{i \to \infty} \left[F'(i), \Gamma(i) \right] \longrightarrow$
 $\longrightarrow \lim_{i \to \infty} \left[\lim_{i \to \infty} F'(i), F(i) \right] \longrightarrow \lim_{i \to \infty} \left[F'(i), F(i) \right] \longrightarrow$

<u>3. Homotopy limit in Cat.</u> Let I be a small category and Set^{Δ op} - the category of simplicial sets. A.K. Bousfield and D.M. Kan defined for F : I \longrightarrow Set^{Δ op} the homotopy limit - holim F. We will prove that the homotopy limit of a diagram of nerves of categories is itself the nerve of a category.

For $F : I \longrightarrow Cat$ we define the functor $\tilde{F} : I \longrightarrow Cat$ (the Eilenberg-Moore rectification or Street "second construction", see [6]). For $i \in ob I$ $\tilde{F}(i)$ is the category whose objects are pairs (ψ, φ) , where ψ is a function assigning each $\alpha : i \longrightarrow i'$ in I with source i an object $\psi(\alpha)$ of F(i'); and φ assigns each string $i \xrightarrow{\alpha} i' \xrightarrow{\beta} i''$ a map $\varphi_{\beta} = \alpha : \psi(\beta \alpha) \longrightarrow F(\beta)\psi(\alpha)$ in F(i''), subject to



commute. A map a : $(\psi, \varphi) \longrightarrow (\psi', \varphi')$ is a function which assigns to each α : $i \longrightarrow i'$ in I a map of F(i'), $a(\alpha) : \psi(\alpha) \longrightarrow \psi'(\alpha)$; subject to, for $i \xrightarrow{\alpha} i' \xrightarrow{\beta} i''$, that



commutes. The composition is given by $\overline{a} \cdot a(\alpha) = \overline{a}(\alpha)a(\alpha)$. For $\delta : i \longrightarrow \overline{i}$, $\widetilde{F}(\delta) : \widetilde{F}(1) \longrightarrow \widetilde{F}(\overline{i})$ is given on objects by $\widetilde{F}(\delta)(\psi, \varphi) = (\psi^{\delta}, \varphi^{\delta})$, where $\psi^{\delta}(\alpha) = \psi(\alpha\delta)$, $\varphi^{\delta}_{\beta,\alpha} = \varphi_{\beta,\alpha\delta}$; and on morphisms by $\widetilde{F}(\delta)(a) = a^{\delta}$, $a^{\delta}(\alpha) = a(\alpha\delta)$.

Then we have the following

<u>Theorem 3.1.</u> For a functor $F : \mathbf{I} \longrightarrow \mathbf{Cat}$ there is a natural isomorphism holim NF $\simeq N \lim_{i \to \infty} \widetilde{F}$, where N is the nerve functor.

The proof is straightforward.

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