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CONTINUOUS MAPS OF THE CIRCLE WITH FINITELY MANY PERIODIC POINTS

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Abstract. Let f be a continuous map of the circle into itself. The main purpose of this paper is to study the properties of the unstable manifold associated to a periodic point of f. Let $\Omega(f)$ denote the nonwandering set of f. Suppose f has finitely many periodic points. Then, using the unstable manifolds associated to periodic points of f, three theorems are proved providing complete answers to the following three questions:

(1) Which are the possible periods of the periodic points of f?

(2) Which is the value of the topological entropy of f?

(3) If $\Omega(f)$ is finite, which are the points of $\Omega(f)$?

§1. Introduction

Let S^1 denote the circle and $C^0(S^1,S^1)$ denote the space of continuous maps of S^1 into itself. For $f \in C^0(S^1,S^1)$ let $\Omega(f)$ denote the nonwandering set of f, and let P(f) denote the set of positive integers which occur as the period of some periodic point of f. Our main results are the following (see §2 for definitions):

THEOREM A. *Let $f \in C^0(S^1,S^1)$ and suppose that f has finitely many periodic points. Then there are integers $m \geq 1$ and $n \geq 0$, such that $P(f) = \{m,2m,4m,\ldots,2^n m\}$.*

THEOREM B. *Let $f \in C^0(S^1,S^1)$ and suppose $\Omega(f)$ is finite. Then $\Omega(f)$ is the set of periodic points of f.*

THEOREM C. *Let $f \in C^0(S^1,S^1)$ and suppose that f has finitely many periodic points. Then the topological entropy of f is zero.*

THEOREM D. *Let $f \in C^0(S^1,S^1)$. Suppose f has finitely many periodic points, and all periodic points of f are fixed points of f. Then $\Omega(f)$ is the set of fixed points of f.*

A map $f \in C^0(S^1,S^1)$ is a Morse-Smale endomorphism of the circle if it satisfies the following properties (see [3] for more details):

(1) f is a continuously differentiable map.

(2) $\Omega(f)$ is finite.

(3) All periodic points of f are hyperbolic.

(4) No singularity of f is eventually periodic.

For a Morse-Smale endomorphism of the circle it was proved, by Block in [3] and [4], that Theorems A and B hold.

Theorems B,C and D were p[illegible] for a continuous map of a closed interval into itself. The proofs of Theorems B and D can easily [illegible] an arbitrary interval.

Suppose $\Omega(f)$ is finite, then the orbit of any $x \in \Omega(f)$ is finite. This implies that x is eventually periodic (i.e. some point in the orbit of x is periodic) but does not imply that x is periodic. It is possible for some $f \in C^0(S^1,S^1)$ to have points $x \in \Omega(f)$ which are eventually periodic but not periodic. In the proof of Theorem B, we show that this cannot happen when $\Omega(f)$ is finite.

We also note that for $f \in C^0(S^1,S^1)$, $\Omega(f)$ may not be the closure of the set of periodic points of f. See [2] for an example.

An example was given, by Block in [6], of a continuous map f, of a compact, connected, metrizable, one-dimensional space, for which $\Omega(f)$ consists of exactly two points, one of which is not periodic.

We conclude this section with the following theorem.

THEOREM E *(proved by Block in [4]). Let m and n be integers* $m \geq 1$, $n \geq 0$. *There is a map* $f \in C^0(S^1,S^1)$ *such that* $P(f) = \{m,2m,4m,\ldots,2^n m\}$.

In fact, Block proved that there is a Morse-Smale endomorphism f of the circle with $P(f) = \{m,2m,4m,\ldots,2^n m\}$ for any integers $m \geq 1$ and $n \geq 0$.

§2. Preliminary definitions and results

Let X be a topological space, and $C^0(X,X)$ denote the set of continuous maps of X into itself. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let f^0 denote the identity map.

Let $p \in X$. A point p is called a *fixed point* of f if $f(p) = p$. Let *Fix(f)* denote the set of fixed points of f. We say p is a *periodic point* of f, if p is a fixed point of f^n for some positive integer n. Let *Per(f)* denote the set of periodic points of f. If p is a periodic point of f, the smallest positive n with $f^n(p) = p$ is called the *period of p*. Let *P(f)* denote the set of positive integers which occur as the period of some periodic point of f.

For any $p \in X$ we define the *orbit of p* by $orb(p) = \{f^n(p): n = 0,1,2,\dots\}$. The orbit of any periodic point will be called a *periodic orbit*. We say a point $p \in X$ is *eventually periodic* if orb(p) is finite (or equivalently if some element of orb(p) is periodic).

A point $p \in X$ is said to be *wandering* if for some neighborhood V of p, $f^n(V) \cap V = \emptyset$ for all $n > 0$. The set of points which are not wandering is called the *nonwandering set* and is denoted $\Omega(f)$.

Let X be a compact topological space. For $f \in C^0(X,X)$ let *ent(f)* denote the topological entropy of f (see [1] for a definition).

Let a and b be two distinct points of S^1. We will use the notation (a,b) (respectively [a,b]) to denote the *open* (respectively *closed*) *arc* from a counterclockwise to b. Similarly, we will define the arcs (a,b] and [a,b). The point a (respectively b) is called the *left* (respectively *right*) *endpoint* of the arc.

Let X denote an arbitrary interval of the real line. Let $f \in C^0(X,X)$ (respectively $f \in C^0(S^1,S^1)$) and let p be a periodic point of f. We define the *unstable manifold* $W^u(p,f)$ and *one-sided unstable manifolds* $W^u(p,f,+)$ and $W^u(p,f,-)$ as follows. Let $x \in W^u(p,f)$ if for every neighborhood V of p, $x \in f^n(V)$ for some positive integer n. Let $x \in W^u(p,f,+)$ if for every closed interval (respectively arc) K with left endpoint p, $x \in f^n(K)$ for some positive integer n. Let $x \in W^u(p,f,-)$ if for every closed interval (respectively arc) K with right endpoint p, $x \in f^n(K)$ for some positive integer n.

In Lemma 1, we compile some properties of the unstable manifold. See [6] for proofs. Although proofs are given for a mapping of a closed interval, they can easily be modified to a mapping of the circle or to a mapping of an arbitrary interval.

LEMMA 1. *Let X be either an arbitrary interval of the real line or the circle, and let* $f \in C^0(X,X)$.

i) Let $p \in Fix(f)$. *Then* $W^u(p,f)$, $W^u(p,f,+)$ *and* $W^u(p,f,-)$ *are connected.*

Let $p \in Per(f)$.

ii) $W^u(p,f) = W^u(p,f,+) \cup W^u(p,f,-)$.

iii) If $p_1 = p$ *and* $orb(p) = \{p_1,\dots,p_n\}$, *then*

$$W^u(p_1,f) = W^u(p_1,f^n) \cup \dots \cup W^u(p_n,f^n).$$

iv) $f(W^u(p,f)) = W^u(p,f)$.

v) Let $J = W^u(p,f)$ *and let* $\overline{J}$ *denote the closure of J. If the set* $\overline{J} - J$ *is nonempty, then any element of* $\overline{J} - J$ *is periodic.*

vi) Suppose $\Omega(f)$ *is finite. Let* $x \in \Omega(f)$ *and suppose* $x \notin Per(f)$. *Then for some* $p \in Per(f)$, *there exists* $z \in W^u(p,f)$ *such that* $f(z) = p$ *and* $z \notin Per(f)$.

LEMMA 2. *Let X be either an arbitrary interval of the real line or the circle. Suppose* $f \in C^0(X,X)$ *and* $\{p_1,\ldots,p_n\}$ *is a periodic orbit of f. If* $f(p_i)=p_j$, *then* $f(W^u(p_i,f^n)) = W^u(p_j,f^n)$.

Proof. Let $x \in W^u(p_i,f^n)$. We shall show that $f(x) \in W^u(p_j,f^n)$. To prove this, let V be any neighborhood of p_j. There is a neighborhood W of p_i, with $f(W) \subset V$. Now for some $m>0$, $x \in f^{nm}(W)$. Hence $f(x) \in f(f^{nm}(W)) = f^{nm}(f(W)) \subset f^{nm}(V)$. Since V was arbitrary, $f(x) \in W^u(p_j,f^n)$. This proves that $f(W^u(p_i,f^n)) \subset W^u(p_j,f^n)$.

By renumbering we may assume that $f(p_i) = p_{i+1}$ for $i=1,\ldots,n-1$ and $f(p_n) = p_1$. Therefore
$f^n(W^u(p_1,f^n)) \subset f^{n-1}(W^u(p_2,f^n)) \subset \ldots \subset f(W^u(p_n,f^n)) \subset W^u(p_1,f^n)$.
By iv) of Lemma 1, we have that $f^n(W^u(p_1,f^n)) = W^u(p_1,f^n)$. Hence $f(W^u(p_n,f^n)) = W^u(p_1,f^n)$. Q.E.D.

The following lemma is a simple consequence of Bolzano's Theorem.

LEMMA 3. *Let* $f \in C^0(\mathbb{R},\mathbb{R})$. *If K is a closed interval such that* $K \subset f(K)$, *then f has a fixed point in K.*

Let $f \in C^0(S^1,S^1)$ and let X be a subset of S^1. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let $p\colon \mathbb{R} \longrightarrow S^1$ be the natural projection. Since p is a covering map, if g is the restriction of f to X there exists a continuous map $\overline{g}\colon X \longrightarrow \mathbb{R}$ such that $g = p\circ\overline{g}$. From now on for a given continuous map $g\colon X \longrightarrow S^1$, $\overline{g}\colon X \longrightarrow \mathbb{R}$ will denote the continuous map such that $g = p\circ\overline{g}$.

The following lemma follows immediately from Lemma 3.

LEMMA 4. *Let* $f \in C^0(S^1,S^1)$ *and suppose* $K \subset S^1$ *is a closed arc such that either* $K \subset f(K)$ *and* $f(K) \neq S^1$ *or* $K \subset \overline{f}(K)$. *Since*

$S^1 = \mathbb{R}/\mathbb{Z}$, *we may assume* $K \subset (0,1)$. *Then* f *has a fixed point in* K.

§3. Some results for $f \in C^0(S^1,S^1)$ with finite periodic set

We shall use the two following Lemmas, which are proved in [6] (see Lemma 6 and Theorem 7 of [6]).

LEMMA 5. *Let* X *be an arbitrary interval of the real line, and let* $f \in C^0(X,X)$. *Suppose* $Per(f)$ *is finite, and* $p \in Fix(f)$. *Let* $x \in W^u(p,f)$. *If* $x > p$, *then* $x \in W^u(p,f,+)$. *If* $x < p$, *then* $x \in W^u(p,f,-)$.

LEMMA 6. *Let* X *be an arbitrary interval of the real line, and let* $f \in C^0(X,X)$. *Suppose* $Per(f)$ *is finite, and* $p \in Fix(f)$. *If* $x \in W^u(p,f)$ *and* $f(x) = p$, *then* $x = p$.

By a *partition* of S^1, we mean a finite set of points of S^1, $\{x_1,\dots,x_n\}$ such that for $i = 1,\dots,n-1$, $(x_i,x_{i+1}) \cap \{x_1,\dots,x_n\} = \emptyset$.

THEOREM 7. *Let* $f \in C^0(S^1,S^1)$. *Suppose* $Per(f)$ *is finite and* $\{p_1,\dots,p_n\}$ *is a periodic orbit of* f *with period* $n \geq 2$. *If* $W^u(p_i,f) \neq S^1$ *and* $j \neq i$, *then* $p_j \notin W^u(p_i,f^n)$.

Proof. Suppose p_i and p_j are distinct elements of $\{p_1,\dots,p_n\}$ with $p_j \in W^u(p_i,f^n)$. By Lemma 2, we have that for each $k = 1,\dots,n$, $W^u(p_k,f^n)$ contains an element of $\{p_1,\dots,p_n\} - \{p_k\}$.

By renumbering, we may assume that $\{p_1,\dots,p_n\}$ is a partition of S^1. By i) of Lemma 1, either $p_2 \in W^u(p_1,f^n)$ or $p_n \in W^u(p_1,f^n)$. Without loss of generality we can suppose that $p_2 \in W^u(p_1,f^n)$. Let $J = W^u(p_1,f^n) \cup W^u(p_2,f^n)$. We separate the proof into two cases.

Case 1. $\overline{J} \neq S^1$.

Therefore $\overline{J}$ is a closed arc. By iv) of Lemma 1, $f^n(\overline{J}) = \overline{J}$. Let g be the restriction of f^n to $\overline{J}$. Then $W^u(p_i,f^n) = W^u(p_i,g)$, for $i = 1,2$. Of course, either $p_1 \in W^u(p_2,g)$ or $p_3 \in W^u(p_2,g)$. Suppose $p_1 \in W^u(p_2,g)$. By Lemma 5, $p_2 \in W^u(p_1,g,+)$ and $p_1 \in W^u(p_2,g,-)$. Since $[p_1,p_2] \subset W^u(p_1,g)$, it follows from Lemma 6, that for all $x \in (p_1,p_2)$, $g(x)$ belongs to some arc of the form (p_1,y). Because $p_2 \in W^u(p_1,g,+)$, for some $x \in (p_1,p_2)$, $g(x) = p_2$. Let $z = \inf\{x \in (p_1,p_2)\colon g(x) = p_2\}$. Then $z \in (p_1,p_2)$ and $g(z) = p_2$. Let $a \in (p_1,z)$. Then $g([a,z])$ contains an arc of the form $[b,p_2]$. Since $p_1 \in W^u(p_2,g,-)$ $p_1 \in g^m([b,p_2])$ for some $m > 0$. This implies that $p_1 \in g^{m+1}([a,z])$. Since $g^{m+1}([a,z])$ is an arc containing p_1 and p_2, $g^{m+1}([a,z]) \supset [a,z]$. By Lemma 4, g has a periodic point in $[a,z]$. Since a was an arbitrary point with $a \in (p_1,z)$, g has infinitely many periodic points. This is a contradiction, and so $p_1 \notin W^u(p_2,g)$. Hence $p_3 \in W^u(p_2,g)$. That is, $p_3 \in W^u(p_2,f^n)$.

By the same argument, it follows that $p_{i+1} \in W^u(p_i,f^n)$, for $i = 1,\ldots,n-1$, and $p_1 \in W^u(p_n,f^n)$. Then $[p_i,p_{i+1}] \subset W^u(p_i,f^n)$, for $i = 1,\ldots,n-1$, and $[p_n,p_1] \subset W^u(p_n,f^n)$. By iii) of Lemma 1, we have that $W^u(p_i,f) = S^1$, for $i = 1,\ldots,n$, a contradiction.

Case 2. $\overline{J} = S^1$.

Since $W^u(p_i,f) \neq S^1$, by iii) of Lemma 1, J is homeomorphic to $\mathbb{R}$. By iv) of Lemma 1, $f^n(J) = J$. Let h be the restriction of f^n to J. Then $W^u(p_i,f^n) = W^u(p_i,h)$, for $i = 1,2$, and the proof is identic to the above case. Q.E.D.

LEMMA 8. *Let $f \in C^0(S^1,S^1)$ and let $\{p_1,\ldots,p_n\}$ be a periodic orbit of f with period $n \geq 2$. Suppose $Per(f)$ is finite and $W^u(p_1,f) = S^1$. If $(p_i,p_j) \cap \{p_1,\ldots,p_n\} = \emptyset$, $x \in (p_i,p_j)$ and*

$x \notin Per(f)$, *then either* $x \in W^u(p_i,f^n)$ *or* $x \in W^u(p_j,f^n)$.

Proof. Suppose $x \notin W^u(p_i,f^n)$ and $x \notin W^u(p_j,f^n)$. By v) of Lemma 1, $x \notin \overline{W^u(p_i,f^n)}$ because $x \notin Per(f)$. Therefore $\overline{W^u(p_i,f^n)} \neq S^1$. By Lemma 2, $\overline{W^u(p_k,f^n)} \neq S^1$ for $k=1,\dots,n$. Since $W^u(p_1,f) = S^1$, by iii) of Lemma 1, $x \in W^u(p_k,f^n)$ for some $k \in \{1,\dots,n\} - \{i,j\}$. Let $J = \overline{W^u(p_k,f^n)}$. By iv) of Lemma 1, $f^n(J) = J$. Let g be the restriction of f^n to J. Then $W^u(p_k,f^n) = W^u(p_k,g)$. By Lemma 5, either $x \in W^u(p_k,g,+)$ or $x \in W^u(p_k,g,-)$. Without loss of generality we may assume that $x \in W^u(p_k,g,+) = W^u(p_k,f^n,+)$. Then $p_i \in W^u(p_k,f^n,+)$.

Let m be the number of elements of the periodic orbit $\{p_1,\dots,p_n\}$ contained in $W^u(p_k,f^n)$. By Lemma 2, $W^u(p_i,f^n)$ contains the same number of elements of $\{p_1,\dots,p_n\}$. Then, by i) of Lemma 1, $p_k \in W^u(p_i,f^n)$ because $x \notin W^u(p_i,f^n)$. Therefore $W^u(p_k,f^n,+) \subset W^u(p_i,f^n)$. Hence $x \in W^u(p_i,f^n)$, and we get a contradiction. Q.E.D.

LEMMA 9. *(proved by Li and Yorke [8]). Let* I *be a closed interval and let* $f \in C^0(I,I)$. *Suppose there exist two closed intervals* L *and* R *such that* $L \cup R \subset f(R)$, $R \subset f(L)$ *and* $f^2(L \cap R) \cap R = \emptyset$. *Then for every* $m = 1,2,\dots$ *there exists a periodic point in* R *with period* m.

THEOREM 10. *Let* $f \in C^0(S^1,S^1)$ *and suppose* $Per(f)$ *is finite. Let* $\{p_1,\dots,p_n\}$ *be a periodic orbit of* f *with period* $n \geq 2$. *If* $W^u(p_1,f) = S^1$, *the following holds for some* $m \in \{n, n/2\}$.

i) If $(p_i,p_j) \cap \{p_1,\dots,p_n\} = \emptyset$, *then* $f^m([p_i,p_j]) = [p_i,p_j]$, *and* $f^k([p_i,p_j]) \cap (p_i,p_j) = \emptyset$, *for any* $k \in \{1,\dots,m-1\}$.

ii) $Per(f) = Per(f^m)$.

iii) $\Omega(f)=\Omega(f^m)$.

iv) By i), if $(p_i,p_j)\cap\{p_1,\dots,p_n\}=\emptyset$, *we can define* f^m_{ij} *as the restriction of* f^m *to* $[p_i,p_j]$. *Then* $Per(f^m)=\bigcup_{ij}Per(f^m_{ij})$ *and* $\Omega(f^m)=\bigcup_{ij}\Omega(f^m_{ij})$.

Proof. For any $X\subset S^1$, let Int(X) denote the interior of X. We shall show $p_k\notin \mathrm{Int}(f([p_i,p_j]))$, for $k=1,\dots,n$. If this is not the case then one of the following holds.

(1) There is a point $x\in(p_i,p_j)$ with $f(x)=p_k$ (for some $k\in\{1,\dots,n\}$) such that for every arc $[a,b]\subset(p_i,p_j)$ with $x\in(a,b)$, $p_k\in\mathrm{Int}(f([a,b]))$.

(2) There is an arc $[x,y]\subset(p_i,p_j)$ with $f([x,y])=\{p_k\}$ (for some $k\in\{1,\dots,n\}$), such that for every arc $[a,b]\subset(p_i,p_j)$ with $[x,y]\subset(a,b)$, $p_k\in\mathrm{Int}(f([a,b]))$.

Suppose (1) is true. We separate the proof into three cases.

Case I. $x\in\mathrm{Int}(W^u(p_r,f^n))$ and $\overline{W^u(p_r,f^n)}\neq S^1$, for some $r\in\{i,j\}$.

Suppose $r=i$ and let $J=W^u(p_i,f^n)$. Let g be the restriction of f^n to $\overline{J}$. Then $W^u(p_i,f^n)=W^u(p_i,g)$. By Lemma 5, $x\in\mathrm{Int}(W^u(p_i,g,+))=\mathrm{Int}(W^u(p_i,f^n,+))$.

Let $[c,d]$ be any closed arc contained in $\mathrm{Int}(W^u(p_i,f^n,+))\cap(p_i,p_j)$ with $x\in(c,d)$. We shall prove that $f^m([c,d])\supset[c,d]$, for some $m>0$. Since $p_k\in\mathrm{Int}(f([c,d]))$ and $W^u(p_k,f)=S^1$, $c\in f^r([c,d])$ for some $r>0$. If $f^r([c,d])\supset[c,d]$, we take $m=r$. Otherwise, $f^r([c,d])\supset[p_i,c]$ because $\{p_1,\dots,p_n\}\cap f^r([c,d])\neq\emptyset$ and $f^r([c,d])$ is connected. Since $d\in W^u(p_i,f^n,+)$, $d\in f^{ns}([p_i,c])$ for some $s>0$. One has $f^{ns}([p_i,c])\supset[p_i,d]$. We conclude that $f^m([c,d])\supset[c,d]$, for $m=r+ns$.

In short, for any arc $[c,d]$ with $x \in (c,d)$ and $[c,d] \subset \mathrm{Int}(W^u(p_i,f^n,+)) \cap (p_i,p_j)$, there exists an integer $m>0$ such that $f^m([c,d]) \supset [c,d]$. Since $S^1 = \mathbf{R}/\mathbf{Z}$, we may assume $[p_i,p_j] \subset (0,1)$. If the points c,d are sufficiently close to x we claim that either $f^m([c,d]) \neq S^1$ or $f^m([c,d]) = S^1$ and $\overline{f}^m([c,d]) \supset [c,d]$, for some integer $m>0$. To prove this, suppose $\overline{f}^m([c,d]) \not\supset [c,d]$ for any integer m such that $f^m([c,d]) = S^1$. Then $\overline{f}([x,x+1]) = [x,x+1]$, and this is a contradiction with $x \in \mathrm{Int}(W^u(p_i,f^n))$ and $W^u(p_i,f^n) \neq S^1$. Hence the claim is true. By Lemma 4, f has a periodic point in $[c,d]$ if c,d are sufficiently close to x. Since the arc $[c,d]$ is arbitrary with $x \in (c,d)$, $[c,d] \subset \mathrm{Int}(W^u(p_i,f^n,+)) \cap (p_i,p_j)$ and c,d sufficiently close to x, f has infinitely many periodic points, a contradiction.

Case 2. $x \in \mathrm{Int}(W^u(p_r,f^n))$ and $\overline{W^u(p_r,f^n)} = S^1$, for some $r \in \{i,j\}$.

Suppose $r=i$ and $x \in \mathrm{Int}(W^u(p_i,f^n,+))$. Let $[y,z]$ be any closed arc contained in $\mathrm{Int}(W^u(p_i,f^n,+)) \cap (p_i,p_j)$ with $x \in (y,z)$. We claim that $x \in \mathrm{Int}(f^{ns}([p_i,y]))$ for some $s>0$. To prove this, suppose $x \notin \mathrm{Int}(f^{ns}([p_i,y]))$ for all $s>0$. Since $z \in W^u(p_i,f^n,+)$, $z \in f^{nt}([p_i,y])$ for some $t>0$. Then, because $x \notin \mathrm{Int}(f^{nt}([p_i,y]))$, $f^{nt}([p_i,y]) \supset [z,p_i]$. Therefore $W^u(p_i,f^n,+) \cup W^u(p_i,f^n,-)$. That is $W^u(p_i,f^n,+) = S^1$. Let $(a_k,b_k) = S^1 - \bigcup_{0 \leq r \leq k} f^{nr}([p_i,y])$. Then, it is clear that $[a_k,b_k] \supset [a_{k+1},b_{k+1}]$, $f^n([a_k,b_k]) \supset [a_{k+1},b_{k+1}]$ and $\bigcup_{0 \leq k < +\infty} [a_k,b_k] = \{x\}$. By continuity, $\bigcup_{0 \leq k < +\infty} f^n([a_k,b_k]) = \{f^n(x)\}$. Since $\bigcup_{0 \leq k < +\infty} f^n([a_k,b_k]) \supset \bigcup_{0 \leq k < +\infty} [a_k,b_k]$, $f^n(x) = x$, a contradiction. This establishes the claim that $x \in \mathrm{Int}(f^{ns}([p_i,y]))$ for some $s>0$.

Let $[c,d]$ be any closed arc contained in $\mathrm{Int}(f^{ns}([p_i,y]))$ with $c \in (y,x)$ and $x \in (c,d)$. We shall prove that $f^m([c,d]) \supset [c,d]$

for some $m>0$. Since $p_k \in \mathrm{Int}(f([c,d])$ and $W^u(p_k,f)=S^1$, $c \in f^r([c,d])$ for some $r>0$. If $f^r([c,d])\supset[c,d]$, we take $m=r$. Otherwise $f^r([c,d])\supset[p_i,c]$ because $\{p_1,\dots,p_n\}\cap f^r([c,d])\neq\emptyset$ and $f^r([c,d])$ is connected. Since $[p_i,c]\supset[p_i,y]$, we have that $f^m([c,d])\supset[c,d]$ for $m=r+ns$.

In short, for any arc $[c,d]$ with $c \in (y,x)$, $x \in (c,d)$ and $[c,d]\subset \mathrm{Int}(f^{ns}([p_i,y]))$ there exists an integer $m>0$ such that $f^m([c,d])\supset[c,d]$. Since $S^1=\mathbf{R}/\mathbf{Z}$, we may assume $[p_i,p_j]\subset(0,1)$. If the points c,d are sufficiently close to x we have either $f^m([c,d])\neq S^1$ or $f^m([c,d])=S^1$ and $\overline{f}^m([c,d])\supset[c,d]$, for some integer $m>0$. To prove this suppose $\overline{f}^m([c,d])\not\supset[c,d]$ for any integer m such that $f^m([c,d])=S^1$. Then $\overline{f}([x,x+1])=[x,x+1]$. Since $x \in \mathrm{Int}(W^u(p_i,f^n,+))$, we have $W^u(p_i,f^n,+)=S^1$. Let z be the closest point to p_i such that $z \in (p_i,x)$, $f^n(z)=x$ and $f^n(V)\subset(p_i,x]$ for any neighborhood V of z sufficiently small. Let g be the restriction of f^n to $[p_i,x]$, and let $L=[p_i,z]$ and $R=[z,x]$. Then, by Lemma 9, g has infinitely many periodic points, a contradiction. Hence the claim is true. By Lemma 4, f has a periodic point in $[c,d]$ if c,d are sufficiently close to x.

Since the arc $[c,d]$ is arbitrary with $c \in (y,x)$, $x \in (c,d)$, $[c,d]\subset \mathrm{Int}(f^{ns}([p_i,y]))$ and c,d sufficiently close to x, f has infinitely many periodic points, a contradiction. Hence $x \notin \mathrm{Int}(W^u(p_i,f^n,+))$.

The proof is similar if $x \in \mathrm{Int}(W^u(p_i,f^n,-))$. Otherwise $x \notin \mathrm{Int}(W^u(p_i,f^n,+))$ and $x \notin \mathrm{Int}(W^u(p_i,f^n,-))$. From the definition of the one-sided unstable manifold we have that

$f^n(W^u(p_i,f^n,+)) \subset W^u(p_i,f^n,+)$ and $f^n(W^u(p_i,f^n,-)) \subset W^u(p_i,f^n,-)$. Then, by ii) and iv) of Lemma 1 and since $x \in Int(W^u(p_i,f^n))$, we have that $\overline{W^u(p_i,f^n,+)} = [p_i,x]$, $\overline{W^u(p_i,f^n,-)} = [x,p_i]$, $f^n([p_i,x]) = [p_i,x]$ and $f^n([x,p_i]) = [x,p_i]$. Therefore, $f^n(x) \in \{x,p_i\}$. Since $f(x) = p_k$, $f^n(x) = p_i$. Because $f^n([p_i,x]) = [p_i,x]$, there is a point $y \in (p_i,x)$ with $f^n(y) = x$.

Let g be the restriction of f^n to $[p_i,x]$, and let $L = [p_i,y]$ and $R = [y,x]$. Then, by Lemma 9, g has infinitely many periodic points, a contradiction.

Case 3. $x \notin Int(W^u(p_i,f^n))$ and $x \notin Int(W^u(p_j,f^n))$.

Since $W^u(p_k,f) = S^1$, by Lemma 8, either $x \in W^u(p_i,f^n)$ or $x \in W^u(p_j,f^n)$. Without loss of generality we may assume that $x \in W^u(p_i,f^n)$. Because $x \notin Int(W^u(p_i,f^n))$, x is a boundary point of $W^u(p_i,f^n)$ and $\overline{W^u(p_i,f^n)}$ is a closed-arc. Let $I = \overline{W^u(p_i,f^n)}$ and let h be the restriction of f^n to I. By Lemma 5, $W^u(p_i,h,+) = [p_i,x]$. Since $h(W^u(p_i,h,+)) \subset W^u(p_i,h,+)$, $h(x) \in [p_i,x]$. By Lemma 6, $h(x) \in (p_i,x]$. That is, $f^n(x) \in (p_i,x]$. This is a contradiction because $f(x) = p_k$ and $f^n(x) \in \{p_1,\dots,p_n\}$.

Thus (2) must be true. Let X denote the quotient space of S^1 obtained by identifying all points of $[x,y]$ to a single point, and let $g\colon X \longrightarrow X$ be the quotient map of f obtained by this identification. Then, g verifies (1) and the hypotheses of this theorem. Hence, we have a contradiction.

In short, the interior of $f([p_i,p_j])$ and $\{p_1,\dots,p_n\}$ do not intersect. Since $f(S^1) = S^1$ (because $W^u(p_1,f) = S^1$), i) is easy to verify.

ii) follows immediately from i).

iii) Let $y \in \Omega(f) - \{p_1, \ldots, p_n\}$ and let V be a neighborhood of y contained in $S^1 - \{p_1, \ldots, p_n\}$. Then, if $f^r(V) \cap V \neq \emptyset$, m is a divisor of r. Therefore $\Omega(f) \subset \Omega(f^m)$. Because $\Omega(f^m)$ is always contained in $\Omega(f)$, we have $\Omega(f) = \Omega(f^m)$.

iv) follows readily from definitions. Q.E.D.

§4. <u>Proof of Theorem A</u>

LEMMA 11. *Let $f \in C^0(S^1, S^1)$ and suppose $Per(f)$ is finite. If $p \in Fix(f)$ and $W^u(p, f) = S^1$, then $p \notin Int(f([a,b]))$ for any arc $[a,b] \subset S^1 - \{p\}$ with $f^{-1}(p) \cap [a,b]$ connected.*

Proof. We shall show that there is not an arc $[a,b] \subset S^1 - \{p\}$ with $f^{-1}(p) \cap [a,b]$ connected such that $p \in Int(f([a,b]))$. Otherwise, one of the following holds.

(1) There is a point $x \in S^1 - \{p\}$ with $f(x) = p$ such that for every arc $[a,b] \subset S^1 - \{p\}$ with $x \in (a,b)$, $p \in Int(f([a,b]))$.

(2) There is an arc $[x,y] \subset S^1 - \{p\}$ with $f([x,y]) = \{p\}$ such that for every arc $[a,b] \subset S^1 - \{p\}$ with $[x,y] \subset (a,b)$, $p \in Int(f([a,b]))$.

Suppose (1) is true. If $x \in Int(W^u(p,f,+))$, let $[c,d]$ be any arc contained in $Int(W^u(p,f,+)) \cap (S^1 - \{p\})$ with $x \in (c,d)$. By the same argument used in the proof of case 2 of statement i) of Theorem 10, we should show that $f^m([c,d]) \supset [c,d]$ for some $m > 0$, and that f has infinitely many periodic points, a contradiction. Similarly, if $x \in Int(W^u(p,f,-))$.

Assume $x \notin Int(W^u(p,f,+))$ and $x \notin Int(W^u(p,f,-))$. Again, by the argument used in the proof of case 2 of statement i) of

Theorem 10, we have a contradiction.

Thus (2) must be true. Let X denote the quotient space of S^1 obtained by identifying all points of $[x,y]$ to a single point, and let g: $X \longrightarrow X$ be the quotient map of f obtained by this identification. Therefore g verifies (1) and the hypotheses of this lemma. Hence, we have a contradiction. Q.E.D.

LEMMA 12. *Let* $f \in C^0(S^1,S^1)$. *Suppose* $Per(f)$ *is finite,* $Fix(f) = \{p_1,\ldots,p_r\}$ *with* $r > 1$, *and* $W^u(p_k,f) \neq S^1$ *for any* $p_k \in Fix(f)$. *If* $f([p_i,p_j]) \supset [p_i,p_j]$ *and* $(p_i,p_j) \cap Fix(f) = \emptyset$, *then either* $W^u(p_i,f,+) \supset [p_i,p_j)$ *or* $W^u(p_j,f,-) \supset (p_i,p_j]$.

Proof. We claim that either $f([p_i,x]) \subset [p_i,p_j]$ for some x sufficiently close to p_i, or $f([y,p_j]) \subset [p_i,p_j]$ for some y sufficiently close to p_j. Otherwise, there is an arc $[x,y] \subset (p_i,p_j)$ such that $f([x,y]) \supset [x,y]$. By Lemma 4, f has a fixed point in $[x,y]$, a contradiction.

Without loss of generality we can assume that $f([p_i,x]) \subset [p_i,p_j]$ for x sufficiently close to p_i. Then, either $x \in (p_i,f(x))$ or $f(x) \in (p_i,x)$, for some x sufficiently close to p_i if it is necessary. By continuity and Lemma 5, $W^u(p_i,f,+) \supset [p_i,p_j)$ if $x \in (p_i,f(x))$.

Now, suppose $f(x) \in (p_i,x)$ for some $x \in (p_i,p_j)$. Then, $f([y,p_j]) \subset [p_i,p_j]$ for y sufficiently close to p_j. Otherwise, there exists an arc $[x,y] \subset (p_i,p_j)$ such that $f([x,y]) \supset [x,y]$, a contradiction. Therefore, either $y \in (f(y),p_j)$ or $f(y) \in (y,p_j)$, for some y sufficiently close to p_j it it is necessary. By continuity and Lemma 5, $W^u(p_j,f,-) \supset (p_i,p_j]$ if $y \in (f(y),p_j)$. But if $f(y) \in (y,p_j)$ the arc $[x,y] \subset (p_i,p_j)$ is such that

$f([x,y]) \supset [x,y]$, a contradiction. Q.E.D.

THEOREM 13. *Let* $f \in C^0(S^1,S^1)$. *Suppose* $Per(f) = Fix(f) = \{p_1,\ldots,p_r\}$ *and* $f(S^1) = S^1$. *Then* $\bigcup_{1 \le k \le r} W^u(p_k,f) = S^1$.

Proof. We define $W = \bigcup_{1 \le k \le r} W^u(p_k,f)$. Suppose $r > 1$ and $W \ne S^1$. We claim that $S^1 - W$ has more than one connected component. To prove this, suppose $S^1 - W$ has only one connected component. By v) of Lemma 1, $S^1 - W = (p_i,p_j)$ with $(p_i,p_j) \cap Fix(f) = \emptyset$. From iv) of Lemma 1 it follows that $f(W) = W$. Then, since $f(S^1) = S^1$, $f([p_i,p_j]) \supset [p_i,p_j]$. By Lemma 12, $(p_i,p_j) \subset W^u(p_i,f,+) \cup W^u(p_j,f,-) \subset W$, a contradiction. This establishes the claim.

Let (p_i,p_j) and (p_l,p_k) be two distinct connected components of $S^1 - W$. It is clear that $(p_i,p_j) \cap Fix(f) = \emptyset$ and $(p_l,p_k) \cap Fix(f) = \emptyset$. From Lemma 12 it follows that $f([p_i,p_j]) \not\supset [p_i,p_j]$ and $f([p_l,p_k]) \not\supset [p_l,p_k]$. Then $f([p_i,p_j]) \supset [p_j,p_i] \supset [p_l,p_k]$ and similarly $f([p_l,p_k]) \supset [p_i,p_j]$. Hence $f^2([p_i,p_j]) \supset [p_i,p_j]$. By Lemma 12, $(p_i,p_j) \subset W^u(p_i,f^2,+) \cup W^u(p_j,f^2,-) \subset W$, a contradiction.

Now, suppose $r = 1$ and $W \ne S^1$. We may assume that there exists a neighborhood V of $p = p_1$ such that $f^{-1}(p) \cap V = \{p\}$. Otherwise, there is an arc $[x,y]$ such that $p \in [x,y]$, $f([x,y]) = \{p\}$ and $f([a,b]) \ne \{p\}$ for every arc $[a,b]$ with $[x,y] \subset (a,b)$. Let X denote the quotient space of S^1 obtained by identifying all points of $[x,y]$ to the single point p, and let $g\colon X \longrightarrow X$ be the quotient map of f obtained by this identification. Then g verifies the hypotheses of the theorem and there exists a neighborhood V of p such that $g^{-1}(p) \cap V = \{p\}$. We separete the proof into five cases.

Case 1. Suppose $f([p,x]) \supset [p,x]$, for some x sufficiently close to p.

This implies that there exists y sufficiently close to p such that $y \in (p,f(y))$. Therefore $W^u(p,f,+) = S^1$, a contradiction.

Case 2. Suppose $f([x,p]) \supset [x,p]$, for some x sufficiently close to p.

Similarly, $W^u(p,f,-) = S^1$, a contradiction.

Case 3. Suppose $f([p,x]) \subset [p,x]$, for some x sufficiently close to p.

Then $f([x,p]) \supset [x,p]$. By case 2, we have a contradiction.

Case 4. Suppose $f([x,p]) \subset [x,p]$, for some x sufficiently close to p.

Then $f([p,x]) \supset [p,x]$. By case 1, we have a contradiction.

Case 5. Suppose $f([p,x]) \subset [a,p]$ and $f([y,p]) \subset [p,b]$ for x and y sufficiently close to p, and for some $a,b \in S^1 - \{p\}$.

Hence, by the above cases we have a contradiction for the map f^2. Q.E.D.

COROLLARY 14. *Let $f \in C^0(S^1,S^1)$. Suppose $Per(f) = \{p_1,\ldots,p_r\}$ and $f(S^1) = S^1$. Then $\bigcup_{1 \leq k \leq r} W^u(p_k,f) = S^1$.*

Proof. Let n be the product of the periods of all the periodic points of f. Then all the periodic points of f are fixed points of f^n. By Theorem 13, $\bigcup_{1 \leq k \leq r} W^u(p_k,f^n) = S^1$. Since $W^u(p_k,f^n) \subset W^u(p_k,f)$, $\bigcup_{1 \leq k \leq r} W^u(p_k,f) = S^1$. Q.E.D.

THEOREM 15. *Let X be an arbitrary interval of the real line, and let $f \in C^0(X,X)$. If Per(f) is finite then for some integer $n \geq 0$, $P(f) = \{1,2,4,\ldots,2^n\}$.*

This theorem is contained in a theorem of Sharkovskii (see [6], [9] and [10]) which says the following. Order the positive integers as follows: $3,5,7,\ldots,2\cdot 3,2\cdot 5,2\cdot 7,\ldots,4\cdot 3,4\cdot 5,4\cdot 7,\ldots,8\cdot 3,8\cdot 5,8\cdot 7,\ldots,8,4,2,1$. Then if m is to the right of n and f has a periodic point of period n, then f has a periodic point of period m.

THEOREM A. *Let* $f \in C^0(S^1,S^1)$ *and suppose* $Per(f)$ *is finite. Then there are integers* $m \geq 1$ *and* $n \geq 0$*, such that* $P(f) = \{m, 2m, 4m, \ldots, 2^n m\}$.

Proof. We separate the proof into three cases.

Case 1. There is a periodic point p of f with period $r \geq 2$ and $W^u(p,f) = S^1$.

By Theorem 10, $Per(f) = Per(f^m) = \bigcup_{ij} Per(f^m_{ij})$, where $m \in \{r, r/2\}$ and f^m_{ij} is the restriction of f^m to $[p_i,p_j]$ with $p_i,p_j \in orb(p)$ and $(p_i,p_j) \cap orb(p) = \emptyset$. By Theorem 15, for every f^m_{ij} there is an integer $n(ij) \geq 0$ such that $P(f^m_{ij}) = \{1,2,4,\ldots,2^{n(ij)}\}$. Let n be the greatest element of $\{n(ij)\}$. Then $P(f) = \{m,2m,4m,\ldots,2^n m\}$.

Case 2. There is a fixed point p of f with $W^u(p,f) = S^1$.

We represent S^1 as the interval $[0,1]$ identifying the points 0 and 1 to the point p. Let $g\colon [0,1] \longrightarrow S^1$ be the natural map defined by this identification. By Lemma 11, there exists a map $h\colon [0,1] \longrightarrow [0,1]$ such that $f \circ g = g \circ h$. Therefore $P(f) = P(h) = \{1,2,4,\ldots,2^n\}$ for some integer $n \geq 0$.

Case 3. For every periodic point p of f we have that $W^u(p,f) \neq S^1$.

Let $g \in C^0(S^1,S^1)$ and let X be a subset of S^1 such that $g(X) \subset X$. From now on $g|X$ will denote the restriction of g to X.

If $f(S^1) \neq S^1$, let $J = f(S^1)$. Then $P(f) = P(f|J)$. By Theorem 15, there is an integer $n \geq 0$ such that $P(f|J) = \{1,2,4,\ldots,2^n\}$. Hence, the theorem is proved. Therefore, we shall assume that $f(S^1) = S^1$.

Let p be a periodic point of f with period r and let J be a connected component of $W^u(p,f)$. Since $W^u(p,f) \neq S^1$, $J \neq S^1$. By iii) and iv) of Lemma 1, $f^r(J) = J$. From Theorem 15 it follows that $P(f^r|J) = \{1,2,4,\ldots,2^s\}$ for some integer $s \geq 0$. Because $f^r(\bar{J}) = \bar{J}$, $P(f^r|\bar{J}) = \{1,2,4,\ldots,2^t\}$ where $t = s$ if $s \geq 1$, and $t \in \{0,1\}$ if $s = 0$. For each connected component of $W^u(p,f)$ we have an integer $t \geq 0$. Let $t(p)$ be the greatest integer associated to some connected component of $W^u(p,f)$. Then $P(f^r|\overline{W^u(p,f)}) = \{1,2,4,\ldots,2^{t(p)}\}$. Hence $P(f|\overline{W^u(p,f)}) = \{r,2r,4r,\ldots,2^{t(p)}r\}$.

Let m be the smallest element of P(f) and let p be a periodic point of f with period m. We claim that $P(f|\overline{W^u(p,f)} \cup \overline{W^u(q,f)}) = \{m,2m,4m,\ldots,2^t m\}$ for any periodic point q of f such that $\overline{W^u(p,f)} \cap \overline{W^u(q,f)} \neq \emptyset$, and for some integer $t = t(p,q)$. We shall prove this claim. By Corollary 14, there are periodic points q such that $\overline{W^u(p,f)} \cap \overline{W^u(q,f)} \neq \emptyset$. Let q be such a periodic point with period k. By v) of Lemma 1, the sets $P(f|\overline{W^u(p,f)}) = \{m,2m,4m,\ldots,2^{t(p)}m\}$ and $P(f|\overline{W^u(q,f)}) = \{k,2k,4k,\ldots,2^{t(q)}k\}$ intersect. Then, since $k \geq m$, we obtain that $k = 2^a m$, for some integer $a \geq 0$. Therefore, if $t(p,q)$ is the greatest element of $\{t(p), a + t(q)\}$, the claim is proved. By the same argument and by Corollary 14, the theorem follows. Q.E.D.

§5. Proof of Theorem B

LEMMA 16. *Let $f \in C^0(S^1,S^1)$. Suppose $\Omega(f)$ is finite and $W^u(p,f) \neq S^1$ for all $p \in Per(f)$. Then $\Omega(f) = Per(f)$.*

Proof. Suppose $x \in \Omega(f)$ and $x \notin \mathrm{Per}(f)$. By vi) of Lemma 1, for some periodic point p_1, there exists $z \in W^u(p_1,f)$ such that $f(z) = p_1$ and z is not periodic. Let n be the period of p_1 and let $\mathrm{orb}(p_1) = \{p_1,\ldots,p_n\}$. By iii) of Lemma 1, $z \in W^u(p_k,f^n)$ for some $k \in \{1,\ldots,n\}$. Note that $f^n(z) \in \{p_1,\ldots,p_n\}$ and (by iv) of Lemma 1) $f^n(z) \in W^u(p_k,f^n)$. We separate the proof into two cases.

Case 1. p_1 is a periodic point with period $n \geqslant 2$.

Then, by Theorem 7, $f^n(z) = p_k$. Let $J = W^u(p_k,f^n)$. By iv) of Lemma 1, $f^n(J) = J$. Let g be the restriction of f^n to J. Then $z \in W^u(p_k,f^n) = W^u(p_k,g)$, and $g(z) = p_k$. By Lemma 6, $z = p_k$. This is a contradiction, because z is not periodic.

Case 2. p_1 is a fixed point.

Then $n = 1$, and $f(z) = p_1$. The proof is identic to the above case. Q.E.D.

THEOREM 17 *(proved by Block in [6]). Let I be an arbitrary interval of the real line. Let $f \in C^0(I,I)$ and suppose $\Omega(f)$ is finite. Then $\Omega(f) = \mathrm{Per}(f)$.*

LEMMA 18. *Let $f \in C^0(S^1,S^1)$. Suppose $\Omega(f)$ is finite and $W^u(p_1,f) = S^1$ for some periodic orbit $\{p_1,\ldots,p_n\}$ with $n \geqslant 2$. Then $\Omega(f) = \mathrm{Per}(f)$.*

Proof. By Theorem 10, $\mathrm{Per}(f) = \mathrm{Per}(f^m) = \bigcup_{ij} \mathrm{Per}(f^m_{ij})$ and $\Omega(f) = \Omega(f^m) = \bigcup_{ij} \Omega(f^m_{ij})$, where $m \in \{n,n/2\}$ and f^m_{ij} is the restriction of f^m to $[p_i,p_j]$, if $(p_i,p_j) \cap \{p_1,\ldots,p_n\} = \emptyset$. By Theorem 17, $\Omega(f^m_{ij}) = \mathrm{Per}(f^m_{ij})$. Hence $\Omega(f) = \mathrm{Per}(f)$. Q.E.D.

LEMMA 19. *Let $f \in C^0(S^1,S^1)$. Suppose $\Omega(f)$ is finite and $W^u(q,f) \neq S^1$ for any $q \in Per(f)$ with period greater than 1. If $W^u(q,f) = S^1$ for some $q \in Fix(f)$, then $\Omega(f) = Per(f)$.*

Proof. Suppose $y \in \Omega(f)$ and $y \notin Per(f)$. By vi) of Lemma 1, for some periodic point p, there exists $z \in W^u(p,f)$ such that $f(z) = p$ and z is not periodic. Let n be the period of p. We separate the proof into three cases.

Case 1. p is a periodic point with period $n \geq 2$.

Since $W^u(p,f) \neq S^1$, by the same argument used in the proof of case 1 of Lemma 16, we would have a contradiction.

Case 2. p is a fixed point with $W^u(p,f) \neq S^1$.

Now, we should have a contradiction by the same argument used in the proof of case 2 of Lemma 16.

Case 3. p is a fixed point with $W^u(p,f) = S^1$.

By the proof of case 2 of Theorem A, there are two continuous maps $g\colon [0,1] \longrightarrow S^1$ and $h\colon [0,1] \longrightarrow [0,1]$ such that $f \circ g = g \circ h$. By Theorem 17, $\Omega(h) = Per(h)$. Then $\Omega(f) = Per(f)$. Q.E.D.

THEOREM B. *Let $f \in C^0(S^1,S^1)$ and suppose $\Omega(f)$ is finite. Then $\Omega(f) = Per(f)$.*

Theorem B follows immediately from Lemmas 16, 18 and 19.

§6. Proofs of Theorems C and D

THEOREM 20 *(proved by Block in [5]). Let X denote an arbitrary interval of the real line, and let $f \in C^0(X,X)$. Suppose $Per(f) = Fix(f)$ is finite. Then $\Omega(f) = Fix(f)$.*

THEOREM D. *Let* $f \in C^0(S^1, S^1)$. *Suppose* $Per(f) = Fix(f) = \{p_1, \ldots, p_r\}$. *Then* $\Omega(f) = Fix(f)$.

Proof. We separate the proof into two cases.

Case 1. There is a fixed point p of f with $W^u(p,f) = S^1$.

By the same argument used in the proof of case 3 of Lemma 19 and by Theorem 20, we have that $\Omega(f) = \mathrm{Fix}(f)$.

Case 2. For every fixed point p, $W^u(p,f) \neq S^1$.

If $f(S^1) \neq S^1$, let $J = f(S^1)$. Then, by Theorem 20, $\Omega(f) = \Omega(f|J) = \mathrm{Fix}(f|J) = \mathrm{Fix}(f)$. Hence, the theorem is proved. Therefore, we shall assume that $f(S^1) = S^1$.

From Theorem 13 it follows that $r > 1$. Let p be a fixed point of f. By i) of Lemma 1, $W^u(p,f)$ is connected. By iv) of Lemma 1, $f(W^u(p,f)) = W^u(p,f)$. From Theorem 20 we have that $\Omega(f|W^u(p,f)) = \mathrm{Fix}(f|W^u(p,f))$. Then, by Theorem 13, $\Omega(f) = \mathrm{Fix}(f)$. Q.E.D.

LEMMA 21 *(proved by Adler, Konheim and McAndrew in [1]). Let f be a continuous map of a compact topological space and let n be a positive integer. Then* $ent(f^n) = n \cdot ent(f)$.

LEMMA 22 *(proved by Bowen [7]). Let f be a continuous map of a compact metric space and suppose* $\Omega(f)$ *is finite. Then* $ent(f) = 0$.

Now, the proof of Theorem C is identical to the proof of Theorem A of [5]. We include it here by its brevity.

THEOREM C. *Let* $f \in C^0(S^1, S^1)$ *and suppose* $Per(f)$ *is finite. Then* $ent(f) = 0$.

Proof. Let n be the product of the periods of all the periodic points of f. Then $\mathrm{Per}(f^n) = \mathrm{Fix}(f^n)$. By Theorem D, $\Omega(f^n) = \mathrm{Per}(f^n)$.

In particular, $\Omega(f^n)$ is finite. Hence $\mathrm{ent}(f^n) = 0$, by Lemma 22. Thus, by Lemma 1, $\mathrm{ent}(f) = 0$. Q.E.D.

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