On the space of free loops of an odd sphere

J. Aguadé

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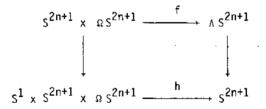
We denote by ΛX the space of maps $S^1 \rightarrow X$ and we call ΛX the space of free loops on X. There is a fibration $\Omega X \rightarrow \Lambda X \rightarrow X$, where ΩX denotes the space of(based) loops on X. The cohomology of ΛX has been studied by L. Smith ([1],[2]). Let us consider the case $X = S^{2n+1}$, an odd-dimensional sphere. In this case, the fibration $\Omega S^{2n+1} \rightarrow \Lambda S^{2n+1} \rightarrow S^{2n+1}$ is totally non-homologous to zero ([1]). Hence, ΛS^{2n+1} has the same cohomology as the product $S^{2n+1}x \ \Omega S^{2n+1}$. We consider the problem of deciding if both spaces are homotopically equivalent, i.e. if there is a splitting $\Lambda S^{2n+1} = S^{2n+1}x \ \Omega S^{2n+1}$. We consider also the related question of the triviallity of the fibration $\Omega S^{2n+1} \rightarrow \Lambda S^{2n+1} \rightarrow S^{2n+1}$. The answer to these questions is given by the following result:

Theorem The following conditions are equivalent:

i) $\Lambda S^{2n+1} \approx S^{2n+1} x \Omega S^{2n+1}$; ii) the fibration $\Omega S^{2n+1} \rightarrow \Lambda S^{2n+1} \rightarrow S^{2n+1}$ is homotopically trivial; iii) n = 0,1,3.

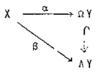
Proof: Clearly, ii \Rightarrow i. If n = 0,1,3, then S²ⁿ⁺¹ is an H-space and so the fibration $\Omega S^{2n+1} + \Lambda S^{2n+1} \rightarrow S^{2n+1}$ is trivial because we can map $S^{2n+1}x \ \Omega S^{2n+1} \rightarrow \Lambda S^{2n+1}$. We have to prove i \Rightarrow iii. Let us assume that we have a homotopy equivalence f:S²ⁿ⁺¹x Ω S²ⁿ⁺¹ + Λ S²ⁿ⁺¹ and let us consider the induced map h:S¹ x S²ⁿ⁺¹x Ω S²ⁿ⁺¹ + S²ⁿ⁺¹. Let us denote by u₁, u_{2n+1}, y, generators of H¹(S¹), H²ⁿ⁺¹(S²ⁿ⁺¹) and H²ⁿ(Ω S²ⁿ⁺¹), respectively. We will later show that in this situation we have h*(u_{2n+1}) = ±u_{2n+1} ± u₁ x y.

Let us consider now the map $g:S^{2n+1} \times (S^1 \times S^{2n}) \neq S^{2n+1}$ induced by h and let us perform the Hopf construction on g. We obtain a map $\tilde{g}:S^{2n+1} \star (S^1 \times S^{2n}) \Rightarrow S^{2n+2}$. Since the space on the left is a wedge of spheres, $S^{2n+3} \vee S^{4n+2} \vee S^{4n+3}$, we can consider a map $\hat{g}:S^{4n+3} \Rightarrow S^{2n+2}$ induced by g. The proof is complete if we show that g has Hopf invariant one, but this follows from $h^*(u_{2n+1}) = \pm u_{2n+1} \pm u_1 \times y$, using the results of [3]. Hence, we have only to show that $h^*(u_{2n+1}) = \pm u_{2n+1} \pm u_1 \times y$. Let us set $h^*(u_{2n+1}) = \lambda u_{2n+1} \pm \mu u_1 \times y$. We have a commutative diagram

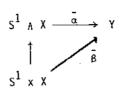


Since the right vertical map induces an isomorphism in cohomology in dimension 2n+1, we get $\lambda = \pm 1$.

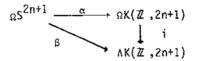
Notice that any homotopy commutative diagram



yields a homotopy commutative diagram



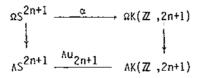
where $\bar{\alpha}$ and $\bar{\beta}$ are adjoint to α and β , respectively. In our case we can consider the diagram



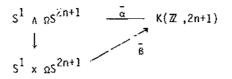
where a represents y and B is the composite

$$\Omega S^{2n+1} \xrightarrow{i} S^{2n+1} \times \Omega S^{2n+1} \xrightarrow{f} \Lambda S^{2n+1} \xrightarrow{\Lambda u_{2n+1}} \Lambda K(\mathbb{Z}, 2n+1)$$

We claim that the above diagram is homotopy commutative. Since $AK(\mathbb{Z},2n+1) = K(\mathbb{Z},2n+1) \times K(\mathbb{Z},2n)$ (because $K(\mathbb{Z},2n+1)$ is an H-space) it suffices to show that ioa and β induce the same homomorphisms in dimensions 2n+1 and 2n. This is obvious in dimension 2n+1. In dimension 2n both i* and α^* are isomorphisms. Moreover, j* is an isomorphism in dimension 2n, f* is an isomorphism in any dimension and $(Au_{2n+1})^*$ is an isomorphism in dimension 2n because the following diagram commutes



Hence, $io\alpha$ and β coincide (for some choice of the generator y) and we have a homotopy commutative diagram



Which implies $\mu = \pm 1$.

Notice that, since $S_{(p)}^{2n+1}$ is an H-space for any odd prime p we always have a splitting $\Delta S_{(p)}^{2n+1} = S_{(p)}^{2n+1} \times \Omega S_{(p)}^{2n+1}$, p odd. However, if $\Delta X = X \times \Omega X$ for some space X, X does not need to be an H-space, as we can see by considering the case of a K(G,1) where G is a non-commutative group. It is easy to see that $\Delta X(G,1) = K(G,1) \times \Omega K(G,1)$ but K(G,1) is not an H-space.

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References

- Smith,L. "On the characteristic zero cohomology of the free loop space" (preprint).
- Smith, E. "On the characteristic p cohomology of the free loop space". (in preparati n).
- Thomas, E. "On functorial cup products and the transgression theorem" Archiv d. Math 12 (1961), 435-444.

Universitat Autònoma de Barcelona. Bellaterra, Barcelona. Spain. and

Forschungsinstitut für Mathematik, ETH Zürich. Switzerland.