Pub. Mat. UAB
N° 25, Juny 1981

On the space of free loops of an odd sphere

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Rebut el 13 de Març del 1981

We denote by ΛX the space of maps $S^1 \to X$ and we call ΛX the space of free loops on X. There is a fibration $\Omega X \to \Lambda X \to X$, where ΩX denotes the space of(based) loops on X. The cohomology of ΛX has been studied by L. Smith ([1],[2]). Let us consider the case $X = S^{2n+1}$, an odd-dimensional sphere. In this case, the fibration $\Omega S^{2n+1} \to \Lambda S^{2n+1} \to S^{2n+1}$ is totally non-homologous to zero ([1]). Hence, ΛS^{2n+1} has the same cohomology as the product $S^{2n+1} \times \Omega S^{2n+1}$. We consider the problem of deciding if both spaces are homotopically equivalent, i.e. if there is a splitting $\Lambda S^{2n+1} \simeq S^{2n+1} \times \Omega S^{2n+1}$. We consider also the related question of the triviality of the fibration $\Omega S^{2n+1} \to \Lambda S^{2n+1} \to S^{2n+1}$. The answer to these questions is given by the following result:

<u>Theorem</u> The following conditions are equivalent:

i) $\Lambda S^{2n+1} \simeq S^{2n+1} \times \Omega S^{2n+1}$;

ii) the fibration $\Omega S^{2n+1} \to \Lambda S^{2n+1} \to S^{2n+1}$ is homotopically trivial;

iii) $n = 0,1,3$.

Proof: Clearly, ii $\Rightarrow$ i. If $n = 0,1,3$, then S^{2n+1} is an H-space and so the fibration $\Omega S^{2n+1} \to \Lambda S^{2n+1} \to S^{2n+1}$ is trivial because we can map $S^{2n+1} \times \Omega S^{2n+1} \to \Lambda S^{2n+1}$. We have to prove i $\Rightarrow$ iii. Let us assume that we have a homotopy equivalence

$f:S^{2n+1} \times \Omega S^{2n+1} \to \Lambda S^{2n+1}$ and let us consider the induced map $h:S^1 \times S^{2n+1} \times \Omega S^{2n+1} \to S^{2n+1}$. Let us denote by u_1, u_{2n+1}, y, generators of $H^1(S^1)$, $H^{2n+1}(S^{2n+1})$ and $H^{2n}(\Omega S^{2n+1})$, respectively. We will later show that in this situation we have $h^*(u_{2n+1}) = \pm u_{2n+1} \pm u_1 \times y$.

Let us consider now the map $g:S^{2n+1} \times (S^1 \times S^{2n}) \to S^{2n+1}$ induced by h and let us perform the Hopf construction on g. We obtain a map $\tilde{g}:S^{2n+1} * (S^1 \times S^{2n}) \to S^{2n+2}$. Since the space on the left is a wedge of spheres, $S^{2n+3} \vee S^{4n+2} \vee S^{4n+3}$, we can consider a map $\hat{g}:S^{4n+3} \to S^{2n+2}$ induced by g. The proof is complete if we show that g has Hopf invariant one, but this follows from $h^*(u_{2n+1}) = \pm u_{2n+1} \pm u_1 \times y$, using the results of [3]. Hence, we have only to show that $h^*(u_{2n+1}) = \pm u_{2n+1} \pm u_1 \times y$. Let us set $h^*(u_{2n+1}) = \lambda u_{2n+1} + \mu u_1 \times y$. We have a commutative diagram

$$\begin{tikzcd} S^{2n+1} \times \Omega S^{2n+1} \arrow[r, "f"] \arrow[d] & \Lambda S^{2n+1} \arrow[d] \\ S^1 \times S^{2n+1} \times \Omega S^{2n+1} \arrow[r, "h"] & S^{2n+1} \end{tikzcd}$$

Since the right vertical map induces an isomorphism in cohomology in dimension 2n+1, we get $\lambda = \pm 1$.

Notice that any homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \Omega Y \arrow[d, dashed, "\Gamma"] \\ & \Lambda Y \end{tikzcd}$$

yields a homotopy commutative diagram

$$\begin{tikzcd} S^1 \wedge X \arrow[r, "\bar{\alpha}"] & Y \\ S^1 \times X \arrow[u] \arrow[ur, "\bar{\beta}"'] & \end{tikzcd}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are adjoint to α and β, respectively. In our case we can consider the diagram

$$\begin{tikzcd} \Omega S^{2n+1} \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \Omega K(\mathbb{Z},2n+1) \arrow[d, "i"] \\ & \Lambda K(\mathbb{Z},2n+1) \end{tikzcd}$$

where α represents y and β is the composite

$$\Omega S^{2n+1} \xrightarrow{\ i\ } S^{2n+1} \times \Omega S^{2n+1} \xrightarrow{\ f\ } \Lambda S^{2n+1} \xrightarrow{\Lambda u_{2n+1}} \Lambda K(\mathbb{Z},2n+1)$$

We claim that the above diagram is homotopy commutative. Since $\Lambda K(\mathbb{Z},2n+1) = K(\mathbb{Z},2n+1) \times K(\mathbb{Z},2n)$ (because $K(\mathbb{Z},2n+1)$ is an H-space) it suffices to show that $i \circ \alpha$ and β induce the same homomorphisms in dimensions $2n+1$ and $2n$. This is obvious in dimension $2n+1$. In dimension $2n$ both i^* and α^* are isomorphisms. Moreover, j^* is an isomorphism in dimension $2n$, f^* is an isomorphism in any dimension and $(\Lambda u_{2n+1})^*$ is an isomorphism in dimension $2n$ because the following diagram commutes

$$\begin{tikzcd} \Omega S^{2n+1} \arrow[r, "\alpha"] \arrow[d] & \Omega K(\mathbb{Z},2n+1) \arrow[d] \\ \Lambda S^{2n+1} \arrow[r, "\Lambda u_{2n+1}"] & \Lambda K(\mathbb{Z},2n+1) \end{tikzcd}$$

Hence, $i \circ \alpha$ and β coincide (for some choice of the generator y) and we have a homotopy commutative diagram

$$
\begin{tikzcd}
S^1 \wedge \Omega S^{2n+1} \arrow[r, "\bar{\alpha}"] \arrow[d] & K(\mathbb{Z}, 2n+1) \\
S^1 \times \Omega S^{2n+1} \arrow[ur, dashed, "\bar{\beta}"'] &
\end{tikzcd}
$$

Which implies $\mu = \pm 1$.

Notice that, since $S^{2n+1}_{(p)}$ is an H-space for any odd prime p we always have a splitting $\Lambda S^{2n+1}_{(p)} \simeq S^{2n+1}_{(p)} \times \Omega S^{2n+1}_{(p)}$, p odd. However, if $\Lambda X \simeq X \times \Omega X$ for some space X, X does not need to be an H-space, as we can see by considering the case of a $K(G,1)$ where G is a non-commutative group. It is easy to see that $\Lambda K(G,1) \simeq K(G,1) \times \Omega K(G,1)$ but $K(G,1)$ is not an H-space.

I am grateful to A. Zabrodsky for some helpful conversations on the subject of this paper.

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