Let $R$ be a ring. We say that $R$ is l.a.r.i. if every left ideal is a right ideal. A ring is l.a.r.i. if every left annihilator is a right ideal. Our notation follows that of [2].

The main results are

**Theorem 1.** Let $K$ be a field and let $G$ be a nonabelian locally finite group. Then if $K[G]$ is l.a.r.i. one of the following occurs

(i) $\text{Char } K = 0$ and $G$ is a Hamilton group such that for each odd exponent, $n$, of $G$ the quaternion algebra over the field $K(\xi_n)$, where $\xi_n$ is a primitive $n$-root of the unity, is a division ring.

(ii) $\text{Char } K = 2$ and $K$ does not contain any primitive 3-root of the unity. Moreover $G \cong Q \times A$, where $Q$ is the quaternion group of order 8 and $A$ is abelian in which each element has odd order and if $n$ is an exponent for $A$, then the least integer $m \geq 1$ satisfying $2^m \equiv 1 \pmod{n}$ is odd.
Conversely if \( K[G] \) satisfies (i) or (ii), then \( K[G] \) is l.i.r.i. and, in particular, it is l.a.r.i.

Observe that if \( \text{char } K > 2 \) and \( G \) is locally finite, then \( K[G] \) is l.a.r.i. if and only if \( G \) is abelian.

**THEOREM II.** Let \( K[G] \) denote the group ring over a nonabelian group. Then the following are equivalent:

1. \( K[G] \) is l.i.r.i.
2. \( G \) is locally finite and if \( \alpha, \beta \in K[G] \) with \( \alpha \beta = 0 \), then \( \beta \alpha = 0 \).
3. \( G \) is locally finite and \( K[G] \) is l.a.r.i.

If we combine the above theorems we get necessary and sufficient conditions for \( K[G] \) to be l.i.r.i.

By using the antiautomorphism of \( K[G] \) given by

\[
\sum_{x \in G} a_x x \mapsto \sum_{x \in G} a_x x^{-1}
\]

we see that \( K[G] \) is l.i.r.i. (l.a.r.i.) if and only if \( K[G] \) is r.i.i. (r.a.i. i.e. normal.

**LEMMA I.** (i) \( K[G] \) is l.i.r.i. if and only if for every finitely generated subgroup \( H \leq G \), \( K[H] \) is l.i.r.i.

(ii) If \( K[G] \) is l.i.r.i., then all subgroups of \( G \) are normal.

(iii) Suppose that \( G \) is locally finite. If \( K[G] \) is l.a.r.i., then all subgroups of \( G \) are normal.

**PROOF.** (i) First we suppose that for every finitely generated subgroup \( H \leq G \), \( K[H] \) is l.i.r.i. Let \( I \leq K[G] \) a left ideal. Let \( \omega \in I \), \( g \in G \). We set \( H = \langle g, \sigma \omega \rangle \). Then
In $K[I]$ is a left ideal of $K[H]$ and hence $I \cap X[H]$ is an ideal of $K[H]$, since $H$ is finitely generated. Now $g \in H$ and $\alpha \in I \cap X[H]$ so $\alpha g \in I \cap X[H] \in I$. Therefore we have shown that $I_g \triangleleft J$ for any $g \in G$ and so $I$ is a right ideal. Conversely let $H$ be a subgroup of $G$ and suppose that $I \triangleleft K[H]$ is a left ideal of $K[H]$. Let $\{x_1\}$ be a set of left coset representatives for $H$ in $G$. Then $\mathbb{Z}[G]$ is a free right $K[H]$-module with basis $\{x_i\}$. Thus we have $K[G] = \sum x_i K[H]$. Denote $\sum x_i I$ by $J$. Clearly $J$ is a left ideal of $K[G]$. If we suppose that $K[G]$ is l.i.r.i., then we have that $J$ is a right ideal of $K[G]$. Let $h \in H$. Then

$$I_h \triangleleft J_h \cap K[H] \leq J \cap K[H] = I$$

and so $I$ is a right ideal.

(ii) In order to prove that all subgroups of $G$ are normal it suffices to see that all cyclic subgroups are normal. Let $g \in G$. Consider the left ideal $I = K[G](1 - a)$. Then $I$ is an ideal, since $K[G]$ is l.i.r.i. Thus $g^{-1}(1-a)g \subseteq I$ and $1 - g^{-1}ag = \alpha (1-a)$ for a suitable element $\alpha \in K[G]$. Now we use the $K[\langle a \rangle]$-homomorphism $\theta : K[G] \rightarrow K[\langle a \rangle]$ in which

$$\sum x \alpha x \rightarrow \sum x \alpha x$$

and we obtain $1 - \theta(g^{-1}ag) = \theta(\alpha)(1-a)$. Since $1-a$ is not invertible we have that $\theta(g^{-1}ag) \neq 0$. Hence $g^{-1}ag \in \langle a \rangle$.

(iii) Suppose that $G$ is locally finite and $K[G]$ is l.a.r.i. Let $H$ be a finite subgroup of $G$. Then Lemma 1.2 [2, Chap.3] yields
that $L(\mathcal{H}) = K[G]\omega(K[H])$. In other hand we have that

$H = \{ x \in G : x - 1 \in K[G]\omega(K[H]) \}$. By hypothesis $L(\mathcal{H})$ is

and ideal, then it is easy to see that $H$ is normal in $G$.

We recall that a nonabelian group $G$ such that all subgroups
are normal is a Hamilton group, that is [see 1, Th. 12.5.4]

$$G \cong Q \times A \times B$$

where $Q$ is the quaternion group of 3 elements, $A$ is an abelian

group such that every element has odd order, and $B$ is an abelian
group of exponent 2. For the rest of this paper we fix this
notation.

**Lemma 2.** Suppose that $G$ is locally finite and $K[G]$ is l.a.r.i. Let $\alpha, \beta \in K[G]$ such that $\alpha \beta = 0$. Then $\beta \alpha = 0$.

**Proof.** If $G$ is abelian the result is trivial. If $G$ is not

abelian, Lemma 1 (iii) yields that $G$ is a Hamilton group. Put

$G = Q \times A \times B$. If $Q$ is generated by $a, b$ with the relations

$a^4 = 1$, $aba = b$, $a^2 = b^2$, put $H = \langle a^2 \rangle \times A \times B$. $H$ is the center of $G$. By using the map $\Theta : K[G] \rightarrow K[H]$ in which

$$\sum_{x \in G} a \times x \mapsto \sum_{x \in H} a \times x$$

we can write any element $\alpha \in K[G]$ as

$$\alpha = \Theta(\nu) + \Theta(a^{-1} \alpha) a + \Theta(b^{-1} \alpha) b + \Theta(b^{-1} a^{-1} \alpha) ab.$$  

Suppose now that $\alpha \beta = 0$. A computation proves that $\Theta(\alpha \beta) = \delta(\beta \alpha)$

Therefore $\Theta(b \alpha) = 0$. Since $\alpha \in L(\beta)$ and, by hypothesis, $L(\beta)$

is an ideal we have $\alpha \times \beta = 0$ for any $x \in G$. Thus
\[ \Theta(x \beta \alpha) = 0. \] By considering (x) for \( \beta \alpha \) we conclude that 
\( \beta \alpha = 0. \)

In characteristic 2 we need the following

**Lemma 4.** Let \( K \) be a field of characteristic 2. Suppose that 
\( K \) does not contain any primitive 3-root of the unity. Put 
\( Q = \langle a, b \rangle. \) Then if \( \alpha = \sum a \in K[Q] \) such that 
\( |\alpha| = 1 \) \((\text{where } |\alpha| = \sum a \in K[Q])\) we have

\[ 1 + (\alpha b)^2 = (1 + a^2)u \]

where \( u \in K[Q] \) is a unit.

**Proof.** Let \( \alpha = a_1 + a_2a + a_3a^2 + a_4a^3 \in K[Q] \) with \( \sum a_i = 1. \)

Then a calculation proves that

\[ 1 + (\alpha b)^2 = (1 + a^2)(1 + (a_1 + a_3)(a_2 + a_4)c). \]

Since \( Q \) is a 2-group and char \( K = 2 \) we know that \( K[Q] \) is a local ring whose maximal ideal is \( \{ \alpha \in K[Q] : |\alpha| = 0 \}. \) Suppose by way of contradiction that \( 1 + (a_1 + a_3)(a_2 + a_4)c \) is not a unit. Then 
\( (a_1 + a_3)(a_2 + a_4) = 1, \) and since \( \sum a_i = 1 \) we see that \( a_1 + a_3 \)

is a primitive 3-root of the unity. Since \( K \) does not contain 
any primitive 3-root of the unity we have a contradiction.

**The Proof of Theorem 1.** Suppose that \( G \) is a nonabelian locally 
finite group and \( K[G] \) is l.a.r.i. Then Lemma 1(iii) yields 
that \( G = Q \times A \times B. \) First we observe that the case char \( K > 2 \)
is not possible. Since \( K[G] \) is l.a.r.i. clearly \( K[Q] \) so. But in 
char \( > 2 \) we have

\[ K[Q] \cong K \times K \times K \times K \times M(2, K). \]
and this is a contradiction, since \(K(\mathbb{Z}/2)\) is not l.a.r.i.

Suppose \(\text{char} K = 0\). Let \(n\) be an exponent for \(A\) and let \(x \in A\) such that \(o(x) = n\). Then \(K[<x>]\) is a product of fields

\[K[<x>] \cong K(\xi_n) \times L_1 \times \ldots \times L_m\]

where \(o(\xi_n) = n\). In other hand we have

\[K[Q] \cong K \times K \times K \times K \times \left(\frac{-1,-1}{K}\right)\]

where the last factor is the quaternion algebra over \(K\). Since

\[K[Q \times <x>] \cong K[Q] \otimes K[<x>]\]

we get that \(\left(\frac{-1,-1}{K}\right) \otimes K(\xi_n) = \left(\frac{-1,-1}{K(\xi_n)}\right)\)

is a direct factor of \(K[Q \times <x>]\) and so \(\left(\frac{-1,-1}{K(\xi_n)}\right)\) is l.a.r.i.

Therefore the quaternion algebra over \(K(\xi_n)\) is a division ring.

Conversely suppose that \(K[G]\) satisfies (i). Then we will prove that \(K[G]\) is l.i.r.i. . It follows from Lemma 1(i) that it satisfies to consider \(G\) finite. Then

\[G \cong Q \times A \times (\mathbb{Z}/2)^{2m}\]

and we get

\[K[G] = K[Q \times A] \times \ldots \times K[Q \times A]\]

Clearly we can suppose that \(G = Q \times A\). Then it is easy to see that

\[K[G] = K[A] \times K[A] \times K[A] \times K[A] \times \prod_{i} \left(\frac{-1,-1}{K(\xi_i)}\right)\]

where \(o(\xi_i)\) are exponents for \(A\). Hence we see that \(K[G]\) is
a product of l.i.r.i. rings. Therefore \( K[G] \) is l.i.r.i.

Char \( K = 2 \). First we observe that if \( K \) contains a primitive 3-root of the unity, then \( K[G] \) is not l.a.r.i. From Lemma 2 it suffices to exhibit elements \( \alpha, \beta \in K[G] \) such that
\[
\alpha \beta = 0 \text{ but } \beta \alpha \neq 0.
\]
If \( \xi \) is a primitive 3-root of the unity we set
\[
\alpha = (1 + \xi)(1 + \xi a)b \quad \beta = (1 + \xi)(1 + \xi a)b(1 + a)b.
\]
A calculation proves that \( \alpha \beta = 0 \) but \( \beta \alpha \neq 0 \). We now prove that \( G = Q \times A \). If this is not the case there exists an element \( x \in G - Q \) of order 2 which centralizes \( G \). Again there exist elements
\[
\alpha = 1 + (a + b + ab)x \quad \beta = (a + b + ab)(1 + a) + (1 + a)x
\]
such that \( \alpha \beta = 0 \) but \( \beta \alpha \neq 0 \) and so \( K[G] \) is not l.a.r.i.

Let \( n \) be an exponent for \( A \) and \( x \in A \) such that \( o(x) = n \). Since char \( K = 2 \) we have that \( K[x] \) is semisimple, and so
\[
K[x] = K(\xi) x \ldots x L_m \text{ where } o(\xi) = n.
\]
Then \( K[Q] \otimes K(\xi) \cong K(\xi)[Q] \) is a direct factor of \( K[Q \times < x>] \). By hypothesis \( K(\xi)[Q] \) is l.a.r.i. By above \( K(\xi) \) does not contain any primitive 3-root of the unity. Therefore \( 2 \nmid m \), where \( m \) is the degree of the extension \( (\mathbb{Z}/2\mathbb{Z})(\xi)/\mathbb{Z}/2\mathbb{Z}) \). But \( m \) is precisely the least integer satisfying \( 2^m = 1 \mod n \).

Conversely suppose that \( K[G] \) satisfies (ii). We will prove that...
$K[Q]$ is l.i.r.i. Again from Lemma 1(i) we can consider that $G$ is finite. Then

$$K[A] \cong K(\xi_1) \times \cdots \times K(\xi_m)$$

and so

$$K[Q \times A] \cong K(\xi_1)[Q] \times \cdots \times K(\xi_m)[Q].$$

By hypothesis the field $K(\xi_i)$ does not contain any primitive 3-root of the unity. Since a product of l.i.r.i. rings is a l.i.r.i., we have only to prove that if a field $K$ does not contain any primitive 3-root of the unity, then $K[Q]$ is l.i.r.i.

Let $I \subseteq K[Q]$ a left ideal. Suppose that $\alpha \in I$. We can write $\alpha$ in the form $\alpha = \alpha_1 + \alpha_2 b$, where $\alpha_1 \in K[\langle a \rangle]$. The first task is to show that $\alpha_1(1 + a^2) \in I$. Note that if $\alpha_1(1 + a^2) \in I$, then, since $1 + a^2$ is central, $\alpha_2 b(1 + a^2) \in I$. Again $\alpha_2(1 + a^2)$ is central and therefore $b \alpha_2(1 + a^2) \in I$. Since $I$ is a left ideal $\alpha_2(1 + a^2) \in I$. Thus we need only to prove that $\alpha_1(1 + a^2) \in I$.

If $\alpha$ is a unit, then $I = K[Q]$. Thus we may suppose that $\alpha$ is not a unit. Then we have $|\alpha_1 + \alpha_2| = 0$. Suppose that $\alpha_1$ is a unit. Then $1 + \alpha_1^{-1} \alpha_2 b \in I$. Clearly $1 + (\alpha_1^{-1} \alpha_2 b)^2 \in I$, so Lemma 4 yields that $1 + a^2 \in I$. Hence $\alpha_1(1 + a^2) \in I$. If $\alpha_1$ is not a unit, then we have $|\alpha_1| = 0$ and hence $|\alpha_2| = 0$. Therefore $\alpha_1 = \beta_1(1 + a)$ and $\alpha_2 = \beta_2(1 + a)$ for suitable elements $\beta_1 \in K[\langle a \rangle]$

Thus $\alpha = (\beta_1 + \beta_2 ab)(1 + a)$. If $\beta_1 + \beta_2 ab$ is a unit we obtain that $1 + a \in I$ and so $\alpha_1(1 + a^2) = \alpha_1(1 + a)^2 \in I$. Hence we may consider that $|\beta_1 + \beta_2| = 0$. If $\beta_1$ is a unit, then $(1 + \beta_1^{-1} \beta_2 ab)(1 + a) \in I$. Again we use Lemma 4 and we get that $(1 + a^2)(1 + a) \in I$. Thus
\( \omega_1(1+a^2) = \beta_1(1+a)(1+a^2) \in I. \) Finally if \( \beta_1 \) is not a unit we have \( \beta_1 = \gamma_1(1+a) \) for certain \( \gamma_1 \in K[<a>] \). Therefore \( \omega_1(1+a^2) = \gamma_1(1+a^2)(1+a^2) = 0 \) and, certainly, \( \omega_1(1+a^2) \in I. \)

Now we will prove that \( \omega x \in I \) for any \( x \in Q \). Since \( Q = \langle a, b \rangle \) it suffices to see that \( \omega a, \omega b \in I. \) By using the automorphism of \( Q \) given by \( a \rightarrow b, b \rightarrow a \) we see that we have only to prove that \( \omega a \in I. \) But

\[ \omega a = \omega_1 a + \omega_2 ba = \omega_1 a + ab\omega_2(1+a^2). \]

Since \( ax \in I \) and by above \( \omega_2(1+a^2) \in I \), the result follows.

**THE PROOF OF THEOREM II. (i) \( \rightarrow \) (ii).** It follows from Lemma 1 (ii) that all subgroups of \( G \) are normal. Since \( G \) is not abelian, it is a Hamilton group and, clearly, locally finite. If a ring is l.i.r.i., then it is l.a.r.i. Lemma 2 completes the proof. Trivially (ii) implies (iii). It follows from Th. 1 that (iii) implies (i). The result follows.

**REFERENCES**