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GROUP RINGS IN WHICH EVERY LEFT IDEAL IS A RIGHT IDEAL

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ABSTRACT. Let K[G] denote the group ring of G over the field K. In this note we characterize those group rings in which all left ideals are right ideals.

Let R be a ring. We say that R is l.i.r.i. if every left ideal is a right ideal. A ring is l.a.r.i. if every left annihilator is a right ideal. Our notation follows that of [2] .

The main results are

THEOREM I. Let K be a field and let G be a nonabelian locally finite group. Then if K[G] is l.a.r.i. one of the following occurs

(i) Char K = 0 and G is a Hamilton group such that for each odd exponent, n , of G the quaternion algebra over the field $K(\xi_n)$, where ξ_n is a primitive n-root of the unity, is a division ring.

(ii) Char K = 2 and K does not contain any primitive 3-root of the unity. Moreover $G \cong Q \times A$, where Q is the quaternion group of order 8 and A is abelian in which each element has odd order and if n is an exponent for A, then the less integer $m \geq 1$ satisfying $2^m \equiv 1 \pmod{n}$ is odd.

Conversely if K[G] satisfies (i) or (ii), then K[G] is l.i.r.i. and, in particular, it is l.a.r.i. .

Observe that if char $K > 2$ and G is locally finite, then K[G] is l.a.r.i. if and only if G is abelian.

THEOREM II. Let K[G] denote the group ring over a nonabelian group. Then the following are equivalent

(i) K[G] is l.i.r.i. .

(ii) G is locally finite and if $\alpha, \beta \in K[G]$ with $\alpha\beta = 0$, then $\beta\alpha = 0$.

(iii) G is locally finite and K[G] is l.a.r.i. .

If we combine the above theorems we get necessary and sufficient conditions for K[G] to be l.i.r.i. .

By using the antiautomorphism of K[G] given by $\sum_{x \in G} a_x x \longmapsto \sum_{x \in G} a_x x^{-1}$ we see that K[G] is l.i.r.i.(l.a.r.i) if and only if K[G] is r.i.l.i. (r.a.l.i.).

LEMMA 1. (i) K[G] is l.i.r.i. if and only if for every finitely generated subgroup $H \leq G$, K[H] is l.i.r.i. .

(ii) If K[G] is l.i.r.i. , then all subgroups of G are normal.

(iii) Suppose that G is locally finite. If K[G] is l.a.r.i., then all subgroups of G are normal.

PROOF. (i) First we suppose that for every finitely generated subgroup $H \leq G$, K[H] is l.i.r.i. . Let $I \leq K[G]$ a left ideal. Let $\alpha \in I$, $g \in G$. We set $H = \langle g, \operatorname{supp}\alpha \rangle$. Then

$I \cap K[H]$ is a left ideal of $K[H]$ and hence $I \cap K[H]$ is an ideal of $K[H]$, since H is finitely generated. Now $g \in H$ and $\alpha \in I \cap K[H]$ so $\alpha g \in I \cap K[H] \subseteq I$. Therefore we have shown that $Ig \subseteq I$ for any $g \in G$ and so I is a right ideal. Conversely let H be a subgroup of G and suppose that $I \subseteq K[H]$ is a left ideal of $K[H]$. Let $\{x_i\}$ be a set of left coset representatives for H in G. Then $K[G]$ is a free right $K[H]$ - module with basis $\{x_i\}$. Thus we have $K[G] = \dot{\sum} x_i K[H]$. Denote $\dot{\sum} x_i I$ by J. Clearly J is a left ideal of $K[G]$. If we suppose that $K[G]$ is l.i.r.i., then we have that J is a right ideal of $K[G]$. Let $h \in H$. Then

$$Ih \subseteq Jh \cap K[H] \subseteq J \cap K[H] = I$$

and so I is a right ideal.

(ii) In order to prove that all subgroups of G are normal it suffices to see that all cyclic subgroups are normal. Let $a, g \in G$. Consider the left ideal $I = K[G](1-a)$. Then I is an ideal, since $K[G]$ is l.i.r.i. . Thus $g^{-1}(1-a)g \in I$ and $1 - g^{-1}ag = \alpha(1-a)$ for a suitable element $\alpha \in K[G]$. Now we use the $K[\langle a \rangle]$ - homomorphism $\theta : K[G] \longrightarrow K[G]$ in which $\sum_{x \in G} a_x x \longmapsto \sum_{x \in \langle a \rangle} a_x x$ and we obtain $1 - \theta(g^{-1}ag) = \theta(\alpha)(1-a)$. Since $1-a$ is not invertible we have that $\theta(g^{-1}ag) \neq 0$. Hence $g^{-1}ag \in \langle a \rangle$.

(iii) Suppose that G is locally finite and $K[G]$ is l.a.r.i. . Let H be a finite subgroup of G. Then Lemma 1.2 [2, Chap.3] yields that $\mathcal{l}(\hat{H}) = K[G]\omega(K[H])$. In other hand we have that $H = \{ x \in G: x - 1 \in K[G]\omega(K[H]) \}$. By hypothesis $\mathcal{l}(\hat{H})$ is and ideal , then it is easy to see that H is normal in G.

We recall that a nonabelian group G such that all subgroups are normal is a Hamilton group, that is [see 1, Th. 12.5.4]

$$G \cong Q \times A \times B$$

where Q is the quaternion group of 8 elements, A is an abelian group such that every element has odd order , and B is an abelian group of exponent 2. For the rest of this paper we fix this notation.

LEMMA 2. Suppose that G is locally finite and K[G] is l.a.r.i. . Let $\alpha, \beta \in K[G]$ such that $\alpha\beta = 0$. Then $\beta\alpha = 0$.

PROOF. If G is abelian the result is trivial. If G is not abelian, Lemma 1 (iii) yields that G is a Hamilton group. Put $G = Q \times A \times B$. If Q is generated by a,b with the relations $a^4 = 1$, $aba = b$, $a^2 = b^2$, put $H = \langle a^2 \rangle \times A \times B$. H is the center of G. By using the map $\theta : K[G] \longrightarrow K[H]$ in which $\sum_{x \in G} a_x x \longmapsto \sum_{x \in H} a_x x$ we can write any element $\alpha \in K[G]$ as

$$(*) \quad \alpha = \theta(\alpha) + \theta(a^{-1}\alpha)a + \theta(b^{-1}\alpha)b + \theta(b^{-1}a^{-1}\alpha)ab.$$

Suppose now that $\alpha\beta = 0$. A computation proves that $\theta(\alpha\beta) = \theta(\beta\alpha)$ Therefore $\theta(\beta\alpha) = 0$. Since $\alpha \in \mathcal{l}(\beta)$ and, by hypothesis, $\mathcal{l}(\beta)$ is an ideal we have $\alpha x \beta = 0$ for any $x \in G$. Thus

$\theta(x\beta\alpha) = 0$. By considering (*) for $\beta\alpha$ we conclude that $\beta\alpha = 0$.

In characteristic 2 we need the following

LEMMA 4. Let K be a field of characteristic 2. Suppose that K does not contain any primitive 3-root of the unity. Put $Q = \langle a,b\rangle$. Then if $\alpha = \sum a_x x \in K[\langle a\rangle]$ such that $|\alpha| = 1$ (where $|\alpha| = \sum a_x$) we have

$$1 + (\alpha b)^2 = (1+a^2)u$$

where $u \in K[\langle a\rangle]$ is a unit.

PROOF. Let $\alpha = a_1 + a_2 a + a_3 a^2 + a_4 a^3 \in K[\langle a\rangle]$ with $\sum a_i = 1$. Then a calculation proves that

$$1 + (\alpha b)^2 = (1+a^2)(1 + (a_1 + a_3)(a_2 + a_4)a).$$

Since Q is a 2-group and char K = 2 we know that $K[Q]$ is a local ring whose maximal ideal is $\{\alpha \in K[Q] : |\alpha| = 0\}$. Suppose by way of contradiction that $1 + (a_1 + a_3)(a_2 + a_4)a$ is not a unit. Then $(a_1 + a_3)(a_2 + a_4) = 1$, and since $\sum a_i = 1$ we see that $a_1 + a_3$ is a primitive 3-root of the unity. Since K does not contain any primitive 3-root of the unity we have a contradiction.

THE PROOF OF THEOREM I. Suppose that G is a nonabelian locally finite group and $K[G]$ is l.a.r.i. . Then Lemma 1(iii) yields that $G = Q \times A \times B$. First we observe that the case char $K > 2$ is not posible. Since $K[G]$ is l.a.r.i. clearly $K[Q]$ so. But in char > 2 we have

$$K[Q] \cong K \times K \times K \times K \times M(2, K)$$

and this is a contradiction, since $M(2,K)$ is not l.a.r.i. . Suppose char$K = 0$. Let n be an exponent for A and let $x \in A$ such that $o(x) = n$. Then $K[<x>]$ is a product of fields

$$K[<x>] \cong K(\xi_n) \times L_1 \times \dots \times L_m$$

where $o(\xi_n) = n$. In other hand we have

$$K[Q] \cong K \times K \times K \times K \times \left(\frac{-1,-1}{K}\right)$$

where the last factor is the quaternion algebra over K. Since $K[Q \times <x>] \cong K[Q] \otimes_K K[<x>]$ we get that $\left(\frac{-1,-1}{K}\right) \otimes K(\xi_n) = \left(\frac{-1,-1}{K(\xi_n)}\right)$

is a direct factor of $K[Q \times <x>]$ and so $\left(\frac{-1,-1}{K(\xi_n)}\right)$ is l.a.r.i..

Therefore the quaternion algebra over $K(\xi_n)$ is a division ring. Conversely suppose that $K[G]$ satisfies (i). Then we will prove that $K[G]$ is l.i.r.i. . It follows from Lemma 1(i) that it suffices to consider G finite. Then

$$G \cong Q \times A \times (Z/2Z) \times \overset{m)}{\dots} \times (Z/2Z)$$

and we get

$$K[G] = K[Q \times A] \times \overset{2m)}{\dots} \times K[Q \times A]$$

Clearly we can suppose that $G = Q \times A$. Then it is easy to see that

$$K[G] = K[A] \times K[A] \times K[A] \times K[A] \times \prod_i \left(\frac{-1,-1}{K(\xi_i)}\right)$$

where $o(\xi_i)$ are exponents for A. Hence we see that $K[G]$ is a product of l.i.r.i. rings. Therefore $K[G]$ is l.i.r.i. .

Char $K = 2$. First we observe that if K contains a primitive 3-root of the unity, then $K[G]$ is not l.a.r.i. . From Lemma 2 it suffices to exhibit elements $\alpha, \beta \in K[G]$ such that $\alpha\beta = 0$ but $\beta\alpha \neq 0$. If ξ is a primitive 3-root of the unity we set

$$\alpha = (1 + \xi(1 + \xi a)b)$$
$$\beta = (1 + \xi(1 + \xi a)b)(1 + a)b.$$

A calculation proves that $\alpha\beta = 0$ but $\beta\alpha \neq 0$. We now prove that $G = Q \times A$. If this is not the case there exists an element $x \in G - Q$ of order 2 which centralizes G. Again there exist elements

$$\alpha = 1 + (a + b + ab)x$$
$$\beta = (a + b + ab)(1 + a) + (1 + a)x$$

such that $\alpha\beta = 0$ but $\beta\alpha \neq 0$ and so $K[G]$ is not l.a.r.i..
Let n be an exponent for A and $x \in A$ such that $o(x) = n$. Since char $K = 2$ we have that $K[\langle x \rangle]$ is semisimple, and so

$$K[\langle x \rangle] = K(\xi_1) \times \dots \times L_m \quad \text{where } o(\xi_1) = n.$$

Then $K[Q] \otimes K(\xi_1) \cong K(\xi_1)[Q]$ is a direct factor of $K[Q \times \langle x \rangle]$. By hypothesis $K(\xi_1)[Q]$ is l.a.r.i. . By above $K(\xi_1)$ does not contain any primitive 3-root of the unity. Therefore $2 \nmid m$, where m is the degree of the extension $(Z/2Z(\xi_1))/(Z/2Z)$. But m is precisely the less integer satisfying $2^m \equiv 1 \pmod n$.
Conversely suppose that $K[G]$ satisfies (ii). We will prove that

$K[G]$ is l.i.r.i.. Again from Lemma 1(i) we can consider that G is finite. Then

$$K[A] \cong K(\xi_1) \times \dots \times K(\xi_m)$$

and so

$$K[Q \times A] \cong K(\xi_1)[Q] \times \dots K(\xi_m)[Q] .$$

By hypothesis the field $K(\xi_i)$ does not contain any primitive 3-root of the unity. Since a product of l.i.r.i. rings is a l.i.r.i., we have only to prove that if a field K does not contain any primitive 3-root of the unity, then $K[Q]$ is l.i.r.i.. Let $I \subseteq K[Q]$ a left ideal. Suppose that $\alpha \in I$. We can write α in the form $\alpha = \alpha_1 + \alpha_2 b$, where $\alpha_i \in K[\langle a \rangle]$. The first task is to show that $\alpha_i(1+a^2) \in I$. Note that if $\alpha_1(1+a^2) \in I$, then, since $1+a^2$ is central, $\alpha_2 b(1+a^2) \in I$. Again $\alpha_2(1+a^2)$ is central and therefore $b\alpha_2(1+a^2)$ I. Since I is a left ideal $\alpha_2(1+a^2) \in I$. Thus we need only to prove that $\alpha_1(1+a^2) \in I$. If α is a unit, then $I = K[Q]$. Thus we may suppose that α is not a unit. Then we have $|\alpha_1| + |\alpha_2| = 0$. Suppose that α_1 is a unit. Then $1 + \alpha_1^{-1}\alpha_2 b \in I$. Clearly $1 + (\alpha_1^{-1}\alpha_2 b)^2 \in I$, so Lemma 4 yields that $1+a^2 \in I$. Hence $\alpha_1(1+a^2) \in I$. If α_1 is not a unit, then we have $|\alpha_1| = 0$ and hence $|\alpha_2| = 0$. Therefore $\alpha_1 = \beta_1(1+a)$ and $\alpha_2 = \beta_2(1+a)$ for suitable elements $\beta_i \in K[\langle a \rangle]$. Thus $\alpha = (\beta_1 + \beta_2 ab)(1+a)$. If $\beta_1 + \beta_2$ ab is a unit we obtain that $1+a \in I$ and so $\alpha_1(1+a^2) = \alpha_1(1+a)^2 \in I$. Hence we may consider that $|\beta_1| + |\beta_2| = 0$. If β_1 is a unit, then $(1 + \beta_1^{-1}\beta_2 ab)(1+a) \in I$. Again we use Lemma 4 and we get that $(1+a^2)(1+a) \in I$. Thus

$\alpha_1(1+a^2) = \beta_1(1+a)(1+a^2) \in I$. Finally if β_1 is not a unit we have $\beta_1 = \gamma_1(1+a)$ for certain $\gamma_1 \in K[\langle a \rangle]$. Therefore $\alpha_1(1+a^2) = \gamma_1(1+a^2)(1+a^2) = 0$ and, certainly, $\alpha_1(1+a^2) \in I$. Now we will prove that $\alpha x \in I$ for any $x \in Q$. Since $Q = \langle a, b \rangle$ it suffices to see that $\alpha a, \alpha b \in I$. By using the automorphism of Q given by $a \longrightarrow b$ $b \longrightarrow a$ we see that we have only to prove that $\alpha a \in I$. But

$$\alpha a = \alpha_1 a + \alpha_2 ba = a\alpha + ab\alpha_2(1+a^2).$$

Since $a\alpha \in I$ and by above $\alpha_2(1+a^2) \in I$, the result follows.

THE PROOF OF THEOREM II. (i) $\Longrightarrow$ (ii). It follows from Lemma 1 (ii) that all subgroups of G are normal. Since G is not abelian, it is a Hamilton group and, clearly, locally finite. If a ring is l.i.r.i., then it is l.a.r.i. Lemma 2 completes the proof. Trivially (ii) implies (iii). It follows from Th. I that (iii) implies (i). The result follows.

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