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q-PLURISUBHARMONICITY AND q-PSEUDOCONVEXITY IN $\mathbf{C}^n$

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Abstract

We generalize classical results for plurisubharmonic functions and hyperconvex domain to q-plurisubharmonic functions and q-hyperconvex domains. We show, among other things, that B_q-regular domains are q-hyperconvex. Moreover, some smoothing results for q-plurisubharmonic functions are also given.

According to Andreotti and Grauert (see [**AG**]), a smooth $\mathcal{C}^2$ function u on an open subset Ω of $\mathbf{C}^n$ is called q-plurisubharmonic (q-psh. for short, $0 \leq q \leq n-1$) if its complex Hessian has at least $(n-q)$ nonnegative eigenvalues at each point of Ω, or equivalently the Levi form of u at every point of Ω is positive definite when restricted to some complex linear subspace of codimension q. Later on, Hunt and Murray (see [**HM**]) found a natural extension of this notion to the class of upper semicontinuous functions.

The set of q-psh. functions has most of the properties of usual plurisubharmonic functions e.g., invariance under holomorphic maps, satisfy the maximum principle, etc. However, this class is *not* closed under addition for $q > 0$ and thus standard smoothing techniques (e.g., by convolving with an approximation of identity) available for plurisubharmonic functions do not apply, a fact which hampered early work on the subject. In fact Diederich and Fornæss [**DF**] constructed examples showing the impossibility of smoothing continuous q-psh. function by $\mathcal{C}^\infty$ smooth q-psh. ones. See also [**Sl3**, p. 154] for a related example. On the positive side, an approximation result was obtained by Slodkowski in [**Sl1**], where he shows that every q-psh. function is pointwise limit of a sequence of q-psh. functions whose second order derivatives exist almost everywhere.

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The aim of the present paper is to investigate analogues of plurisubharmonicity and pseudoconvexity in the context of q-plurisubharmonicity and q-pseudoconvexity.

Now we outline the organization of the paper. After recalling in Section 2 some background on q-psh. functions, in Section 3 we begin studying some properties of B_q-regular domains. Here we recall that a bounded domain Ω in $\mathbf{C}^n$ is said to be B_q-regular if every continuous function on $\partial\Omega$ can be extended continuously to a q-psh. function on Ω. When $q = 0$, those are precisely the B-regular domains introduced by Sibony in [**Si**]. The main result of the section is Theorem 3.6, which gives some connections between B_q-regularity of a domain and of its boundary. Here we encounter some difficulty in generalizing Theorem 2.1 of [**Si**] to the context of q-psh. functions. The main reason is, as said before, the non-additivity of the class of q-psh. functions, thus we do not know, for instance, whether a smoothly bounded B_q-regular domain should have a B_q-regular boundary. In this section we also introduce the concept of q-hyperconvexity. This is the true analogue of classical hyperconvexity. Moreover, the new class enjoys most of the properties of hyperconvexity, e.g., q-hyperconvexity is purely a local concept (Proposition 3.2).

The last section is devoted to studying smoothing results for q-psh. functions. More precisely, we show in Theorem 4.1 that on a q-pseudoconvex domains every q-psh. function is the pointwise limit of a decreasing sequence of piecewise smooth strictly q-psh. functions. This result may be considered as an analogue of a well known approximation theorem due to Fornæss and Narasimhan. In view of the above mentioned example of Diederich and Fornæss, piecewise smoothness seems to be the best possible regularity of the approximating sequence. The paper ends up with another approximation theorem (Theorem 4.3), in which we deal with approximation of *bounded from above* q-psh. functions. In particular, the theorem says that on a B_q-regular domain every bounded from above q-psh. function is the pointwise limit of a decreasing sequence of q-psh. functions which are continuous up to the boundary. This result, in the case $q = 0$ has been proved in slightly more general form in [**NW**] (see also Theorem 4.1 of [**Wi**]).

2. Preliminaries on q-plurisubharmonic functions

In this section, we will collect some known facts about q-psh. functions. For more background, the reader may consult [**HM**], [**Sl1**], [**Bu**].

Definition 2.1. Let Ω be an open set in $\mathbf{C}^n$ and $u\colon \Omega \to [-\infty, \infty)$ be an upper semicontinuous function and q be an integer, $0 \le q \le n-1$.

(i) u is said to be q-plurisubharmonic on Ω if for every complex linear subspace of dimension $q+1$ intersecting Ω, for every closed ball B (in L), and for every smooth plurisuperharmonic function g defined in a neighbourhood of B (in L) satisfying $u \leq g$ on ∂B we have $u \leq g$ on B.

(ii) If for every point $z_0 \in \Omega$ we can find a neighbourhood U and $\varepsilon > 0$ such that $u(z) - \varepsilon|z|^2$ is q-psh. on Ω then we say that u is strictly q-psh.

(iii) u is said to be q-plurisuperharmonic if $-u$ is q-psh.

(iv) If u is q-psh. and $(n-q-1)$-plurisuperharmonic then we say that u is q-Bremermann.

(v) If u is locally the maximum of a finite number of $\mathcal{C}^2$ smooth q-psh. functions, then we say that u is piecewise q-psh.

Remarks. (i) Definition 2.1 (a) is given by Hunt and Murray in [**HM**]. The definition of q-plurisubharmonicity makes sense also for $q \geq n$. However, in this case every upper semicontinuous function is q-psh. Observe that the function identically $-\infty$ is allowed to be q-psh.

(ii) According to Lemma 2.6 in [**HM**], if $u \in \mathcal{C}^2(\Omega)$ then u is q-psh. if and only if for every $z \in \Omega$ the complex Hessian $\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)_{1 \leq j,k \leq n}$ has at least $(n-q)$ nonnegative eigenvalues, or equivalently the Levi form

$$\langle \mathcal{L}(u,z)\lambda, \lambda\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\lambda_j \overline{\lambda}_k,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, is positive definite on a complex linear subspace of codimension q in $\mathbf{C}^n$. Thus, in the case of smooth functions, the concept of q-psh. functions introduced by Hunt and Murray coincides with the original one given by Andreotti and Grauert at the beginning of this paper.

(iii) The concept of q-Bremermann functions has been introduced first by Hunt and Murray in [**HM**] where they are called q-complex Monge-Ampère instead. Here we follow the terminology of Slodkowski in [**Sl1**]. If $q = 0$ then 0-Bremermann functions are precisely *maximal* plurisubharmonic function (see [**Kl**]). Likewise, piecewise smooth q-psh. functions are also called $(q+1)$-convex with corners by Diederich and Fornæss in [**DF**].

We now list basic properties of q-psh. functions that will be frequently referred to.

Proposition 2.2. *Let Ω be an open set of $\mathbf{C}^n$ and $0 \leq q \leq n-1$. Then*

(i) *If u is q-psh. on Ω then so are λu and $u+v$ for every $\lambda > 0$ and every 0-psh. function v.*

(ii) *For every family $\{u_\alpha\}_{\alpha \in A}$ of locally uniformly bounded from above q-psh. function on Ω, the function $(\sup\{u_\alpha : \alpha \in A\})^*$ is also q-psh. on Ω.*

(iii) *The limit of a decreasing sequence of q-psh. functions is q-psh.*

(iv) *(Maximum principle) If u is q-psh. on Ω then for every relatively compact open set U of Ω we have $\sup_U u \leq \sup_{\partial U} u$.*

(v) *If u is an upper semicontinuous function on Ω such that for every $z_0 \in \Omega$, we can find a neighbourhood U of z_0 such that $u|_U$ is q-psh., then u is q-psh. on Ω.*

(vi) *If u is q-psh. on Ω and $f\colon \Omega' \to \Omega$ is a holomorphic mapping, where Ω, Ω' are open subsets of $\mathbf{C}^n$ and $\mathbf{C}^k$, respectively, then $u \circ f$ is q-psh. on Ω'.*

(vii) *If u is q-psh. function on Ω, then the function*

$$\tilde{u} = \sum_{j=1}^{k} \chi_j(u + v_j)$$

is q-psh. on Ω, where $\chi_j\colon \mathbf{R} \to \mathbf{R}$ are convex increasing functions, v_j are 0-psh. functions on Ω. In particular, $\chi \circ u$ is q-psh. for every increasing convex function $\chi\colon \mathbf{R} \to \mathbf{R}$.

(viii) *If u and v are q- (resp. r-) psh. functions on Ω then $\max(u,v)$ is $\max(q,r)$-psh. on Ω, $u+v$ is $(q+r)$-psh. on Ω and $\min(u,v)$ is $(q+r+1)$-psh. on Ω.*

(ix) *(Gluing Lemma) If $\Omega' \subset \Omega, u$ is q-psh. on Ω and u' is q-psh. on Ω'. Assume that $\limsup_{z' \to z} u'(z') \leq u(z)$ for all $z \in \Omega \cap \partial\Omega'$, then the function*

$$v(z) = \begin{cases} u(z) & z \in \Omega \backslash \Omega' \\ \max(u(z), u'(z)) & z \in \Omega' \end{cases}$$

is q-psh. on Ω.

Here by u^* we mean the upper regularization of a function $u\colon X \to [-\infty, \infty)$, where X is a subset of $\mathbf{C}^n$ i.e.,

$$u^*(x) = \limsup_{z \to x} u(z), \quad \forall\, x \in \overline{X}.$$

For the proof, we require the following generalization of Richberg's approximation theorem for 0-psh. functions.

Bungart's Approximation Theorem (Theorem 5.3 in [**Bu**]). *Assume u is a continuous strictitly q-psh. function on an open set W of $\mathbf{C}^n$ and g a continuous function such that $u < g$ on W. Then there exists a piecewise smooth strictly q-psh. function $\tilde{u}$ on W satisfying $u < \tilde{u} < g$. In particular, there is a monotone decreasing sequence of piecewise smooth strictly q-psh. function that converges to u uniformly on W.*

Proof of Proposition 2.2: The properties (i)–(iii) follow quickly from the definition of q-plurisubharmonicity. For the more subtle ones, (iv) is proved in Lemma 2.7 in [**HM**] (see also [**Sl1**, p. 307]), the last two assertions of (viii) are deep theorems of Slodkowski (see Theorems 5.1 and 6.1 in [**Sl1**]). Note that (v) is contained in the remark following Lemma 2.7 in [**HM**]. Here is a brief proof of this fact. With no loss of generality we may assume that $q = n-1$. Assume that u is not q-psh., then we can find a ball B compactly belonging to Ω and a continuous function v on $\overline{B}$ which is 0-psh. on B such that $\max_{\overline{B}}(u+v) > \max_{\partial B}(u+v)$. Then there is $\varepsilon > 0$ so small that

$$M := \max_{\overline{B}} \tilde{u} > \max_{\partial B} \tilde{u},$$

where $\tilde{u}(z) = u(z) + v(z) + 2\varepsilon|z|^2$. Choose $z^* \in B$ such that $\tilde{u}(z^*) = M$ and set

$$f(z) = 2\varepsilon|z|^2 - M - \varepsilon|z - z^*|^2.$$

Then we have

$$(u + v + f)(z^*) = 0, \ (u + v + f)(z) \leq -\varepsilon|z - z^*|^2, \quad \forall\, z \in \overline{B}.$$

Choose a small ball B' about z^* such that $B' \subset B$ and $u|_{B'}$ is q-psh. Since $v + f$ is plurisubharmonic on $\mathbf{C}^n$, we have $0 = (u + v + f)(z^*) \leq \sup_{\partial B'}(u + v + f) < 0$, which is absurd. Now (vi) is undoubtedly well-known, but due to the lack of an explicit reference we give a proof. First we check that $u \circ f$ is q-psh. Assume that u is of class $\mathcal{C}^2$ on Ω. Then using the chain rule and the holomorphicity of f we get

$$\sum_{j.k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}(z)\lambda_j\overline{\lambda}_k = \sum_{l,m=1}^{n} \frac{\partial^2 u}{\partial w_l \partial \overline{w}_m}(f(z))\lambda'_l\overline{\lambda'}_m, \quad \forall\, z \in \Omega,$$

where $\lambda'_l = \sum_{j=1}^{n} \lambda_j \frac{\partial f_l}{\partial z_j}(z)$, $f = (f_1, \ldots, f_k)$. Since the Levi form of u is positive definite on a complex linear subspace E of codimension q, we infer, from the last expression, that the Levi form of v is positive definite on some complex linear subspace E' of codimension no larger than q.

Thus v is q-psh. This implies that v is also q-psh. if u is piecewise smooth q-psh. Now suppose that u is continuous on Ω. Using Bungart's Approximation Theorem, we deduce that u can be locally uniformly approximated by piecewise smooth q-psh. functions. Thus v is again q-psh. The general case now follows from the preceding facts and the fact that u can be approximated locally from above by a decreasing sequence of continuous q-psh. functions (Theorem 2.9 in [**Sl1**]). Finally we deal with (vii). Using the same reasonings as above, we may reduce to the case u, χ_j, v_j are $\mathcal{C}^2$ smooth functions. Now a direct computation using convexity of χ and plurisubharmonicity of v_j gives

$$\langle \mathcal{L}(\tilde{u}, z)\lambda, \lambda\rangle \geq \left(\sum_{j=1}^{k} \chi_j'(u(z) + v_j(z))\right) \langle \mathcal{L}(u, z)\lambda, \lambda\rangle.$$

It follows that $\tilde{u}$ is q-psh. Finally, for (ix) we define for each $k \geq 1$ the function

$$v_k(z) = \begin{cases} u(z) + 1/k & z \in \Omega\backslash\Omega' \\ \max(u(z) + 1/k, u'(z)) & z \in \Omega'. \end{cases}$$

Since $\limsup\limits_{z'\to z} u'(z') \leq u(z)$ for all $z \in \Omega \cap \partial\Omega'$, we deduce that $v_k \equiv u + 1/k$ on a neighbourhood of $\Omega \cap \partial\Omega'$. Thus, by (v) we infer that v_k is q-psh. on Ω. Observe that $v_k \downarrow v$ on Ω. Therefore v is q-psh. on Ω. The proof is thereby completed. $\square$

As q-psh. functions do not have the additive property, the following operator introduced by Slodkowski, seems to be a good substitute for the usual convolution.

Definition 2.3 (see [**Sl1**, p. 309])**.** Let u and g be two functions defined on $\mathbf{C}^n$ with values in $[-\infty, \infty)$. The supremum-convolution of u and g, denoted by $u *_s g$ is defined by

$$(u *_s g)(z) := \sup\{u(x)g(z - x) : x \in \mathbf{C}^n\}, \quad \forall\, z \in \mathbf{C}^n.$$

If u is defined only on a subset U of $\mathbf{C}^n$, $u *_s g$ is understood as $\tilde{u} *_s g$, where $\tilde{u} = u$ on U and 0 on $\mathbf{C}^n\backslash U$.

Here by $B(z, r)$ we mean the open ball with center z and radius r.

The most useful properties of the supremum-convolution are summarized in the following

Proposition 2.4. *Let u be a q-psh. function on an open set Ω of $\mathbf{C}^n$ and g be a continuous function on $\mathbf{C}^n$, $0 \le g \le 1$, $g(0) = 1$, $\operatorname{supp} g \subset B(0, r)$, $r > 0$. Assume that the set $\{z \in \Omega,\ u(z) = -\infty\}$ has empty interior. Then we have*

(i) *$u *_s g$ is continuous q-psh. on $\Omega_r := \{z \in \Omega : B(z, r) \subset \Omega\}$.*

(ii) *$u *_s g_r$ converges pointwise to u on Ω as r tends to 0.*

Proof: (i) This part is essentially contained in [**Sl1**]. However, for the reader's convenience we sketch some details.

First we check the continuity of $u *_s g$ on Ω_r. As g is continuous and $\{z : u(z) = -\infty\}$ has empty interior we infer $u *_s g$ is real valued and lower semicontinuous. Now $\operatorname{supp} g$ is contained in $B(0, r)$ so

$$(u *_s g)(z) = \sup\{u(z + x) + g(x) : x \in \mathbf{B}(0, r)\}, \quad \forall\, z \in \Omega_r.$$

Now assume that $u *_s g$ is not upper semicontinuous on Ω_r. Then there are $z^* \in \Omega_r$, $\varepsilon > 0$ and a sequence $\{z_j\}$ tending to z^* such that

$$(u *_s g)(z^*) + \varepsilon < (u *_s g)(z_j), \quad \forall\, j \ge 1.$$

Choose a sequence $\{x_{j\varepsilon}\} \subset \Omega_r$ so that

$$(u *_s g)(z_j) \le u(z_j + x_{j\varepsilon}) g(x_{j\varepsilon}) + \varepsilon/3, \quad \forall\, j \ge 1.$$

Passing to a subsequence we may assume that $\{x_{j\varepsilon}\}$ converges to $x_\varepsilon \in B(0, r)$. Observe that u is upper semicontinuous so there is $j_0 \ge 1$ satisfying $u(z_j + x_{j\varepsilon}) \le u(z^* + x_\varepsilon) + \varepsilon/3$ for $j \ge j_0$. It implies that

$$\begin{aligned} u(z^* + x_\varepsilon) g(x_\varepsilon) + \varepsilon \le (u *_s g)(z^*) + \varepsilon &\le (u *_s g)(z_j) \\ &\le u(z_j + x_{j\varepsilon}) g(x_{j\varepsilon}) + \varepsilon/3 \\ &\le u(z^* + x_\varepsilon) g(x_{j\varepsilon}) + 2\varepsilon/3, \qquad \forall\, j \ge j_0. \end{aligned}$$

Letting j tend to ∞, we obtain a contradiction to the continuity of g. Thus $u *_s g$ is continuous on U_r. So it follows from Proposition 2.2 (ii) that $u *_s g$ is q-psh. on Ω_r.

(ii) We have

$$\begin{aligned} (u *_s g_r)(z) &= \sup\{u(z + x) g_r(x) : x \in B(0, r)\} \\ &= \sup\{u(z + x) g(x/r) : x \in B(0, r)\} \ge u(z) g(0) = u(z). \end{aligned}$$

Since $0 \le g \le 1$ we also have $(u *_s g_r)(z) \le \sup\{u(z + x) : x \in B(0, r)\}$. Now the upper semicontinuity of u implies that $\lim_{r \to 0} (u *_s g_r)(z) = u(z)$.

For the ease of exposition, we will say that a subset E of a domain Ω is q-pluripolar (in Ω) if there is a q-psh. function u on Ω such that $u \not\equiv -\infty$ and $u \equiv -\infty$ on E. It should be pointed out that the structure of q-pluripolar set may be very "wild" when $q > 0$. This can be seen by considering the singular locus of upper semicontinuous functions depending only on q variables. More interesting examples are provided by the next result which is a consequence of Theorem 2.5 and Proposition 5.2 in [**Sl2**]. $\square$

Proposition 2.5. *Let Ω be an open set in $\mathbf{C}^n$ and X be a complex analytic subset of codimension q in Ω. Then the function identically 0 on X and $-\infty$ elsewhere is q-psh. on Ω. In particular, $\Omega\backslash X$ is q-pluripolar.*

It follows from the above result that for $q > 0$ the union of two q-pluripolar set is in general not q-pluripolar. The next result is an analogue of the removable singularities for bounded plurisubharmonic function

Proposition 2.6. *Let Ω, Ω' be open subsets of $\mathbf{C}^n$, $\Omega' \Subset \Omega$. Let v be a q-psh. function on Ω such that $v \not\equiv -\infty$ and $E = \{z \in \Omega' : v(z) = -\infty\}$ is closed in Ω'. Then every q-psh. function u on $\Omega'\backslash E$, which is locally bounded from above near every point of E can be extended through E to a $(q+r)$-psh. function on Ω.*

Proof: Since Ω' is bounded, by subtracting a positive constant, we may assume $v < 0$ on Ω'. For $\varepsilon > 0$ we set

$$u_\varepsilon = \begin{cases} u + \varepsilon v & \text{on } \Omega\backslash E \\ -\infty & \text{on } E. \end{cases}$$

By Proposition 2.2 (ix) we have u_ε is $(q+r)$-psh. on Ω. Set $\tilde{u} = (\sup\{u_\varepsilon : \varepsilon > 0\})^*$. Then $\tilde{u}$ is $(q+r)$-psh. on Ω, in view of Proposition 2.2 (viii). Since $u_\varepsilon \leq u$ on $\Omega\backslash E$, we deduce that $\tilde{u} \leq u$ on $\Omega\backslash E$. Observe that $\lim_{\varepsilon\to 0} u_\varepsilon = u$ on $\Omega\backslash E$. Therefore $\tilde{u} = u$ on $\Omega\backslash E$. $\square$

Proposition 2.7. *Let Ω be a bounded domain in $\mathbf{C}^n$. Assume that u and v are two continuous functions on $\overline{\Omega}$ which are q-Bremermann function on Ω. Then*

$$\sup_{\Omega} |u - v| \leq \sup_{\partial\Omega} |u - v|.$$

Proof: Let $\alpha = \sup_{\partial\Omega} |u - v|$, we only need to show that $u - v \leq \alpha$ on Ω. As u and $-v$ are q- and $(n-q-1)$-psh. respectively, we have $u - v$ is $(n-1)$-psh. by Proposition 2.2 (ix). Now the desired conclusion follows from the maximum principle (Proposition 2.2 (v)). $\square$

We also need Choquet's Topological Lemma (see Lemma 2.3.4 in [**Kl**]).

Choquet's Lemma. *Let $\{u_\alpha\}_{\alpha\in A}$ be a family of functions on an open set $\Omega \subset \mathbf{C}^n$, which are locally bounded from above. Then there exists a countable subfamily $\{\alpha_j\} \subset A$ such that*

$$(\sup\{u_\alpha : \alpha \in A\})^* = (\sup\{u_{\alpha_j} : j \geq 1\})^*.$$

Moreover, if u_α is lower continuous for every $\alpha \in A$, then we can choose $\{\alpha_j\}$ such that

$$\sup\{u_\alpha : \alpha \in A\} = \sup\{u_{\alpha_j} : j \geq 1\}.$$

3. q-pseudoconvex domains and Dirichlet problem

We first recall, according to Slodkowski (see [**Sl2**, p. 121]), that a domain Ω is said to be q-pseudoconvex if there is a neighbourhood U of $\partial\Omega$ so that the function $-\log d(z)$ is q-psh. on $U \cap \Omega$, where $d(z) = \operatorname{dist}(z, \partial\Omega)$. It follows from Theorem 4.3 in [**Sl2**] that Ω is q-psh. if and only if there exists a neighbourhood U of $\partial\Omega$ and a q-psh. function u on $U \cap \Omega$ such that $\lim_{z\to\partial\Omega} u(z) = \infty$. As in the proof of Theorem 2.6.10 in [**Hö**], if Ω is bounded, by gluing the function $-\log d(z)$ with a suitable convex increasing function of $|z|^2$ we get a continuous q-psh. exhaustion function for Ω. Adding $|z|^2$ to this function we obtain a continuous strictly q-psh. exhaustion function φ for Ω. By Bungart's Approximation Theorem, we can even assume that this function is piecewise smooth strictly q-psh.

Definition 3.1. A bounded domain Ω is said to be q-hyperconvex if it admits a negative continuous q-psh. exhaustion function.

Remarks. (a) As in the 0-hyperconvex case, it is easy to check that every q-hyperconvex domain is q-pseudoconvex. On the other hand, not every bounded q-pseudoconvex domain is q-hyperconvex. Indeed, consider $\Omega = D\backslash E$, where D is a q-pseudoconvex domain in $\mathbf{C}^n$ and E is the zero set of a holomorphic function f on D, $f \not\equiv 0$. Let u be a q-psh. exhaustion function for Ω, it is clear that $u - \log|f|$ is a q-psh. exhaustion function for $D\backslash E$. Thus Ω is q-pseudoconvex. On the other hand, Ω is not q-hyperconvex in view of Proposition 2.6 and the maximum principle (Proposition 2.2 (iv)).

(b) We do not know if a bounded domain Ω is q-hyperconvex if it admits a negative q-psh. exhaustion (not necessarily continuous) function. This is true when $q = 0$ (see Theorem 1.6 in [**Bl**]).

Concerning q-hyperconvexity we have the following results which are analogous to the well known facts for hyperconvexity.

Proposition 3.2. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then Ω is q-hyperconvex if one of the following conditions holds.*

(a) *Ω is locally q-hyperconvex i.e., for every $p \in \partial\Omega$ there is a neighbourhood U of p such that $\Omega \cap U$ is q-hyperconvex.*

(b) *Ω is q-pseudoconvex and $\partial\Omega$ is $\mathcal{C}^1$ smooth.*

Proof: The proof is almost the same as the ones given by Kerzman and Rosay in [**KR**] (see also [**De**] and [**CM**]). For convenience of the reader, we sketch a proof of q-hyperconvexity of Ω under the assumption (a). From the compactness of $\partial\Omega$ and the local hyperconvexity of Ω, we infer that there are open subsets $U_i'' \Subset U_i' \Subset U_i$ of $\mathbf{C}^n$, $1 \le i \le k$ such that:

(i) $\partial\Omega \subset \cup_i U_i''$.

(ii) For any $1 \le i \le k$, there is a negative q-psh. exhaustion function v_i for U_i.

For every $1 \le i < j \le k$ such that $U_i' \cap U_j' \cap \Omega \neq \emptyset$ we define

$$E_{ij}(x) = \inf\{v_j(z) : z \in U_i' \cap U_j' \cap \Omega : v_i(z) \ge x\}.$$

It follows from (ii) that E_{ij} are increasing and $\lim_{x\to 0} E_{ij}(x) = 0$. Now using Lemma 2 in [**CM**] we find a continuous increasing function $\tau\colon (-\infty, 0) \to \mathbf{R}$ such that:

(iii) $\lim_{x\to 0} \tau(x) = \infty$.

(iv) $|\tau - \tau \circ E_{ij}| \le 1/2$ for every $1 \le i < j \le k$ with $U_i' \cap U_j' \cap \Omega \neq \emptyset$.

It follows from (iv) that if $z \in U_i' \cap U_j' \cap \Omega$ then $|\tau \circ v_i - \tau \circ v_j| \le 1$. Moreover, as τ is increasing and convex, we also get for all $\varepsilon > 0$ sufficiently small

$$|\tau(v_i(z) - \varepsilon) - \tau(v_j(z) - \varepsilon)| \le 1, \quad \forall\, z \in U_i' \cap U_j' \cap \Omega.$$

Choose $\mathcal{C}^\infty$ smooth functions φ_j satisfying $0 \le \varphi_j \le 1$, $\operatorname{supp}\varphi_j \subset U_j'$ and $\varphi_j = 1$ on a neighbourhood of $\overline{U_j''}$. Let $\lambda > 0$ so large that $|z|^2 - \lambda < 0$ on Ω and that $\varphi_j(z) + \lambda|z|^2$ is plurisubharmonic for every j. Next we set

$$v_j^\varepsilon = \tau(u_j(z) - \varepsilon) + \varphi_j(z) - 1 + \lambda(|z|^2 - \lambda),$$

and

$$v^\varepsilon(z) = \max\{v_j^\varepsilon(z) : z \in U_j\}.$$

As $v_j^\varepsilon \le v_k^\varepsilon$ on $\partial U_j' \cap U_k'' \cap \Omega$, we deduce that v^ε is q-psh. on $\Omega \cap (\cup U_j'')$. Define

$$v(z) = \sup_{\varepsilon>0} \left(\frac{v^\varepsilon(z)}{\tau(-\varepsilon)} - 1 \right)^* .$$

Then v is a negative q-psh. function on $\Omega \cap (\cup U_j'')$. Let K be a compact subset of Ω such that $(\Omega \backslash (\cup U_j'')) \subset K$ and $\partial K \subset \cup U_j''$. Let $\theta = \max_{\partial K} v < 0$ and $\varphi = \max(v, \theta)$ on $\Omega \backslash K$ and $\varphi = \theta$ on K. It is easy to check that θ is a negative q-psh. exhaustion function for Ω. The desired conclusion now follows. □

The next result should be compared to Corollary 4.9 in [**Sl2**].

Corollary 3.3. *Let Ω_1, Ω_2 be bounded q_1 (resp. q_2-) hyperconvex domains in $\mathbf{C}^n$. Assume that Ω_1 (resp. Ω_2) has a local q_1 (resp. q_2-) psh. defining function near every point of $a \in \partial\Omega_1 \cap \partial\Omega$. Then $\Omega_1 \cup \Omega_2$ is (q_1+q_2+1)-hyperconvex.*

Here we say that a domain Ω has a local q-psh. defining function at $a \in \partial\Omega$ if there is a neighbourhood U of a and a q-psh. function ρ on U such that $U \cap \Omega = \{z : \rho(z) < 0\}$.

Proof: Let $q = q_1 + q_2 + 1$. In view of Proposition 3.2 (a), it is enough to show that $\Omega_1 \cup \Omega_2$ is locally q-hyperconvex at every point $p \in \partial(\Omega_1 \cup \Omega_2)$. This is true for any point $p \in (\partial(\Omega_1) \backslash \Omega_2) \cup (\partial(\Omega_2) \backslash \Omega_1)$, as Ω_1 (resp. Ω_2) is q_1 (resp. q_2-) hyperconvex respectively. Now let $p \in \partial\Omega_1 \cap \partial\Omega_2$. Then we can find a small open ball B around p with radius r and functions u_1, u_2 which are q_1-psh. and q_2-psh. on B such that

$$B \cap \Omega_i = \{z \in B : u_i(z) < 0\}, \quad i = 1; 2.$$

Then

$$\begin{aligned} B \cap (\Omega_1 \cup \Omega_2) = \{z \in B : \min(&\max(u_1(z), |z-p|^2 - r^2), \\ &\max(u_2(z), |z-p|^2 - r^2)) < 0\}. \end{aligned}$$

By Proposition 2.2 (viii), the function $\min(u_1, u_2)$ is q-plurisubharmonic, so $\Omega_1 \cup \Omega_2$ is locally q-hyperconvex at p. The desired conclusion follows. □

The concepts described below are inspired from the seminal work of Sibony [**Si**], where a complete characterization of the domains for which the Dirichlet problem with respect to psh. function admits a solution is given.

Definition 3.4. (a) A bounded domain Ω in $\mathbf{C}^n$ is called B_q-regular $(0 \le q \le n-1)$ if for every real valued continuous function φ on $\partial\Omega$ there is a continuous function u on $\overline{\Omega}$ such that $u \equiv \varphi$ on $\partial\Omega$ and u is q-Bremermann on Ω.

(b) A compact set K of $\mathbf{C}^n$ is called B_q-regular if $P_q(K) = \mathcal{C}(K)$, where $\mathcal{C}(K)$ is the algebra of real-valued continuous function on K and $P_q(K)$ is the closure in $\mathcal{C}(K)$ of continuous q-psh. functions defined on neighbourhoods of K.

Regarding the Dirichlet problem for q-psh. functions, we have the following result due to Bungart (Theorem 3.7 in [**Bu**]). Before formulating it, we recall that a bounded domain Ω is said to have an q-psh. barrier at a point $p \in \partial\Omega$ if there is a continuous function u on $\overline{\Omega}$ which is q-psh. on Ω and satisfies $u(p) = 1$, $u < 1$ elsewhere.

Theorem 3.5. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then the following assertions are equivalent*

(i) *Ω is B_q-regular.*

(ii) *Ω has a q-psh. barrier at every boundary point of Ω.*

It should be said that the plurisubharmonic analogue of the above theorem is contained in Theorem 2.1 of [**Si**]. Notice also that weaker versions of Theorem 3.5 have appeared in [**HM**], [**Sl1**], etc. The next theorem is motivated by Theorem 2.1 in [**Si**].

Theorem 3.6. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then*

(i) *If Ω is q-hyperconvex $(0 \le q \le n-1)$ and $\partial\Omega$ is B_r-regular $(0 \le r \le n-q)$ then Ω is B_{q+r}-regular.*

(ii) *Assume Ω is B_q-regular, then Ω is q-hyperconvex. Moreover, $\partial\Omega$ is B_q-regular if at every point $z_0 \in \partial\Omega$ one of the following conditions holds:*

(a) *There is a continuous q-psh. function u on a neighbourhood of z_0 such that $u(z_0) = 1$ but $u < 1$ on $\partial\Omega\backslash\{z_0\}$.*

(b) *There are a neighbourhood U of z_0 such that $U \cap \partial\Omega$ is $\mathcal{C}^1$ smooth, a continuous function u on $\overline{U \cap \Omega}$ such that u is $\mathcal{C}^2$ smooth, $u(z_0) = 1$ while $u < 1$ elsewhere and at every point of $U \cap \Omega$ the Levi form of u is positive definite on a complex linear subspace L of codimension q.*

Proof: (i) We first claim that there is a negative continuous strictly q-psh. exhaustion function for Ω. To see this, we will borrow an argument from [**KR**, p. 178]. More precisely, pick a negative continuous q-psh. exhaustion function φ for Ω. Choose $R > 0$ so large that $|z| < R$ on Ω. Set

$$\psi(z) = \sup_{j\geq 1}\left(j\varphi(z) - \frac{1}{j} + \frac{1}{jR^2}|z|^2\right).$$

Then ψ is continuous strictly q-psh. on Ω since locally it is defined by maximum of a *finite* number of such functions. Since φ tends to 0 on $\partial\Omega$, we infer that so does ψ. Thus the claim is proved. Now let f be a real valued, continuous function on $\partial\Omega$ we must show that there is a $(q+r)$-Bremermann function on Ω which is continuous on $\overline{\Omega}$ and extends f. As $\partial\Omega$ is B_r-regular, according to Theorem 2.9 in [**Sl1**], we can find a sequence u_j of continuous r-psh. functions on neighbourhoods of $\partial\Omega$ that converges uniformly to f on $\partial\Omega$ and that $u_j(z) + L_j|z|^2$ is locally convex in neighbourhoods of $\partial\Omega$, where L_j is a positive constant. Fix $j \geq 1$, by the claim above, we can choose a negative continuous strictly q-psh. exhaustion function ψ for Ω. Pick a $\mathcal{C}^\infty$ function θ with support in ω and $\theta = 1$ on a neighbourhood of $\partial\Omega$. For $C_j > 0$ large enough, the function $C_j\psi + \theta u_j$ is $(q+r)$-psh. on Ω and extends u_j. It follows that there is a $(q+r)$-Bremermann function $\tilde{u}_j$ which is continuous on $\overline{\Omega}$ such that $\tilde{u}_j = u_j$ on $\partial\Omega$. Applying Proposition 2.7, we get $\{\tilde{u}_j\}$ converges uniformly on $\overline{\Omega}$ to a $(q+r)$-Bremermann function $\tilde{u}$. Of course we have $\tilde{u} \equiv \varphi$ on $\partial\Omega$. Thus Ω is B_{q+r}-regular.

(ii) Let $f(z) = -|z-z_0|^2$ where z_0 is some point in $\partial\Omega$. As Ω is B_q-regular, there is u continuous on $\overline{\Omega}$ which is B_q-regular on Ω such that $u \equiv f$ on $\partial\Omega$. Set $\varphi(z) = u(z) + |z-z_0|^2$, applying the maximum principle we see that either φ is a negative q-psh. exhaustion function for Ω or $\varphi \equiv 0$. The latter one is ruled out as $-|z-z_0|^2$ is not q-psh. Thus Ω is hyperconvex. For the B_q-regularity of $\partial\Omega$, we divide the proof into two steps.

Step 1. We will show for every $z_0 \in \partial\Omega$ there is $h \in P_q(\partial\Omega)$ satisfying $h(z_0) = 0$ but $h < 0$ on $\partial\Omega\backslash\{z_0\}$. To see this, consider first the case there is an u satisfying (a) of (ii). Then the claim follows by gluing u with a suitable negative constant. Now assume that there are a small neighbourhood U of z_0 and u satisfying (b) of (ii). Since $(\partial\Omega)\cap U$ is $\mathcal{C}^1$ smooth, we can find $r_0 > 0$, $\varepsilon_0 > 0$ such that the closed ball $B(z_0, r_0)$ is contained in U and for all $\varepsilon \in (0, \varepsilon_0)$ the set $(B(z_0, r_0) \cap \partial\Omega) + \varepsilon\mathbf{n}$ is relatively compact in $U \cap \Omega$, where $\mathbf{n}$ is the unit outward normal at p.

Set $u_\varepsilon(z) = u(z - \varepsilon\mathbf{n})$. Then u_ε is q-psh. and continuous on a neighbourhood of $B(z_0, r_0) \cap \partial\Omega$. Let $\{r_j\}_{j\geq 1}$ be a sequence, $r_j \downarrow 0$, $r_j < r_0$. Choose $a_j > 0$ such that

$$\sup\{u(z) : z \in \partial\Omega,\ r_j < |z - z_0| < r_0\} < a_j < 1, \quad \forall\, j \geq 1.$$

Since u is continuous on $\overline{U \cap \Omega}$, we can find a sequence $\varepsilon_j \downarrow 0$ satisfying

$$u_{\varepsilon_j} < a_j \text{ on } B(z_0, r_0)\backslash B(z_0, r_j),\ \frac{1 + a_j}{2} < u_{\varepsilon_j}(z_0) := b_j.$$

Set

$$v_j = c_j(u_{\varepsilon_j} - b_j) - 1,$$

where c_j satisfies $\frac{1}{b_j - a_j} < c_j < \frac{1}{1 - b_j}$. Then v_j is continuous q-psh. on $B(z_0, r_0) \cap \partial\Omega$ and satisfy $v_j < 0$ on $B(z_0, r_0) \cap \partial\Omega$, $v_j(z_0) = -1$, $v_j < -2$ on $(B(z_0, r_0) \cap \partial\Omega)\backslash B(z_0, r_j)$. Since the Levi form of u at every point of $U \cap \Omega$ is definite positive on L we infer that for every $a_1, \ldots, a_k > 0$ and every $n_1, \ldots, n_k \geq 1$ the Levi form of the function $a_1 v_{n_1} + \cdots + a_k v_{n_k}$ on some neighbourhood of $B(z_0, r_0) \cap \partial\Omega$ is definite positive on L as well. Thus, this function is continuous and q-psh. on a neighbourhood of $B(z_0, r_0) \cap \partial\Omega$. By applying the proof of Lemma 3.2 in [**Po**], we obtain a function $v \in P_q(\partial\Omega \cap B(z_0, r_0))$ satisfying $v(z_0) = 0, v(z) < 0,\ \forall\, z \in (\partial\Omega \cap B(z_0, r_0))\backslash\{z_0\}$. Let

$$0 > -\varepsilon = \sup_{\partial\Omega \cap \partial B(z_0, r_0)} v.$$

Reasoning as in the proof of Lemma 1.5 in [**Si**], we conclude that

$$h = \begin{cases} \max(u, -\varepsilon) & \text{on } \partial\Omega \cap B(z_0, r_0) \\ -\varepsilon & \text{on } \partial\Omega\backslash B(z_0, r_0) \end{cases}$$

defines a function in $P_q(\partial\Omega)$ satisfying $h(z_0) = 0, h < 0$ on $\partial\Omega\backslash\{z_0\}$.

Step 2. Fix a continuous real valued function f on $\partial\Omega$. Set

$$F(z) = \sup\{u(z) : u \in P_q(\partial\Omega),\ u \leq f \text{ on } \partial\Omega\}, \quad \forall\, z \in \partial\Omega.$$

Fix $z_0 \in \partial\Omega$ and choose a function h as in Step 1. Then given $\varepsilon > 0$ there is $t > 0$ so large that $f(z_0) + t(h(z) - 1) - \varepsilon \leq f(z)$ on $\partial\Omega$. It follows that $f(z_0) + t(h(z) - 1) - \varepsilon \leq F(z)$ on $\partial\Omega$. In particular, $f(z_0) - \varepsilon \leq F(z_0)$. Let $\varepsilon \to 0$ we conclude that $F \equiv f$ on $\partial\Omega$. The Choquet Topological Lemma implies that f is the increasing limit of a sequence in $P_q(\partial\Omega)$. By Dini's Theorem, the convergence is uniform on $\partial\Omega$. This proves the theorem. $\square$

Remark. Let Ω be the "annulus" $\Omega = \{(z,w) \in \mathbf{C}^2 : 1 < |z|^2+|w|^2 < 2\}$. Then $\partial\Omega = \{(z,w) \in \mathbf{C}^2 : |z|^2+|w|^2 = 1\} \cup \{(z,w) \in \mathbf{C}^2 : |z|^2+|w|^2 = 2\}$. Thus, being the union of two strictly pseudoconvex boundaries $\partial\Omega$ is B_0-regular. On the other hand $\partial\Omega$ is $\mathcal{C}^1$ smooth (even real analytic) and Ω is 1-pseudoconvex (Proposition 4.6 in [**Sl2**]), thus applying Proposition 3.2 (b) we deduce that Ω is 1-hyperconvex. However Ω is *not* 0-pseudoconvex in view of the Hartogs extension phenomenom. This shows that the conclusion of Theorem 3.6 (i) is somehow sharp. On the other hand, we do not know if (a) and (b) are really needed for (ii) of the theorem.

The last result of the section generalizes the example in the above remark.

Corollary 3.7. *Let Ω and Ω' be bounded B_q-regular domains in $\mathbf{C}^n$ $(0 \le q \le n-1)$ such that $\Omega' \Subset \Omega$. Assume further that for every $p \in \partial\Omega'$ there is a neighbourhood U of p such that $U \backslash \Omega'$ is r-hyperconvex, and that $\partial\Omega$ and $\partial\Omega'$ are B_q-regular compact sets. Then $\Omega\backslash\Omega'$ is B_{q+s}-regular, where $s = \max(q,r)$.*

Proof: Let $\Omega'' = \Omega\backslash\Omega'$. Then $\partial\Omega'' = \partial\Omega \cup \partial\Omega'$. Since $\partial\Omega$ and $\partial\Omega'$ are disjoint B_q-regular compact sets we have $\partial\Omega''$ is also B_q-regular. Now we claim that Ω'' is s-hyperconvex. By Proposition 3.2 (a), it suffices to check Ω'' is locally s-hyperconvex. Obviously this true at every point $p \in \partial\Omega'$, for $p \in \partial\Omega$, it follows from the B_q-regularity of Ω. $\square$

4. Approximations of q-psh. functions

The next result is an analogue of the Fornæss-Narasimhan approximation theorem for 0-psh. functions on pseudoconvex domains (Theorem 5.5 in [**FN**]).

Theorem 4.1. *Let Ω be a q-pseudoconvex domain in $\mathbf{C}^n$. Then for every r-psh. function u on Ω, there is a sequence of piecewise smooth strictly s-psh. function on Ω that decreases to u, where $s = \max(q,r)$.*

Proof: We follow the ideas given in the proof of Fornæss-Narasimhan's theorem. By the remarks made at the beginning of Section 3, there is a piecewise smooth strictly q-psh. exhaustion function φ for Ω. For $j \ge 1$, set $\Omega_j = \{z \in \Omega : \varphi(z) < j\}$. Then $\cup\Omega_j = \Omega$ and $\Omega_j \Subset \Omega_{j+1}$. By composing φ with suitable increasing convex functions, we get a sequence of continuous strictly q-psh. function $\{\varphi_j\}$ satisfying $\varphi_j < -j$ on $\overline{\Omega_j}$ and $\varphi_j > a_j = \max_{\partial\Omega_{j+1}} u$ on $\partial\Omega_{j+1}$. After replacing φ_j by $\max_{m \ge j} \varphi_m + \varepsilon_j$

for some sufficiently fast decreasing sequence $\varepsilon_j \downarrow 0$, we may achieve that $\varphi_j > \varphi_{j+1}$ on Ω. Now we consider two cases.

Case 1. u is bounded from below. Fix $j \geq 1$, then there is a sequence $\{\delta_{j,m}\}$ decreasing to 0 so fast that $u_{j,m} := u *_s \rho_{\delta j,m} + \delta_{j,m}(1 + |z|^2) > u$ on Ω_{j+2} and $u_{j,m} < \varphi_j$ on $\partial\Omega_{j+1}$. Applying Bungart's Approximation Theorem to the sequence $u_{j,m}$ we can find a sequence of piecewise smooth strictly r-psh. functions $v_{j,m}$ on Ω_{j+2} such that $v_{j,m} \downarrow u$ on Ω_{j+2} as $m \to \infty$, $v_{j,m} > u$ on Ω_{j+2} and $v_{j,m} < \varphi_j$ on $\partial\Omega_{j+1}$. The Gluing Lemma implies that

$$\tilde{u}_{j,m} = \begin{cases} \max(v_{j,m}, \varphi_j) & \text{on } \Omega_{j+1} \\ \varphi_j & \text{on } \Omega \backslash \Omega_{j+1} \end{cases}$$

defines a piecewise smooth, strictly s-psh. function on Ω. Moreover, $\tilde{u}_{j,m} \downarrow \max(u, \varphi_j)$ on Ω_{j+1} as m tends to ∞. It follows that given j there is $p(j)$ so large that $\tilde{u}_{j+1,p} < \tilde{u}_{j,m}$ on Ω_j for $p \geq p(j)$ and $\tilde{u}_{j+1,p} < \tilde{u}_{r,m}$ on Ω_r for $r \leq j$ and $p \geq p(j)$. Thus we can choose a sequence $\{m(j)\}$ tending to ∞ fast enough so that $\tilde{u}_{j+1,m(j+1)} < \tilde{u}_{j,m(j)}$ on Ω_j and $\tilde{u}_{j+1,m(j+1)} < \tilde{u}_{r,j}$ on Ω_r for $r \leq j$. Finally we define

$$u_j = \max_{p \geq j} \tilde{u}_{p,m(p)}.$$

Observe the maximum is locally taken by a *finite* number of piecewise smooth strictly s-psh. functions. This implies that u_j is piecewise smooth strictly s-psh. and $u_j \downarrow u$.

Case 2. General u. From the first case, we deduce that for each $N \geq 1$ there is a sequence $u_{N,k}$ of piecewise smooth strictly s-psh. functions that decreases to $\max(u, -N)$ as k tends to ∞. For each m, choose $p(m) > m$ so large that

$$u_{m,p(m)} < u_{j,m} + \frac{1}{m^2} \text{ on } \overline{\Omega_j}, \quad 1 \leq j \leq m.$$

This is possible because by Dini's Theorem $\max(u_{m,l}, u_{j,m})$ converges uniformly to $u_{j,m}$ on $\overline{\Omega_j}$ as l goes to ∞. Set

$$u_j = \max_{m \geq j} \left(u_{m,p(m)} + \frac{1}{m} \right).$$

Observe that for $z \in \Omega_l$ and $m \geq p(l)$ we have

$$u_{m,p(m)}(z) + \frac{1}{m} < u_{l,m}(z) + \frac{1}{m^2} + \frac{1}{m} < u_{l,p(l)}(z) + \frac{1}{l}.$$

It implies that *locally* u_j is maximum of a finite number of piecewise smooth strictly s-psh. functions. Thus u_j is piecewise smooth strictly s-psh. and $u_j \downarrow u$.

The theorem is completely proved. $\square$

Corollary 4.2. *Let Ω be a q-pseudoconvex domain in $\mathbf{C}^n$, let K be a compact subset of Ω and ω an open neighbourhood of the q-plurisubharmonic hull $\hat{K}_\Omega^P$ of K i.e.,*

$$\hat{K}_\Omega^P = \{z \in \Omega : u(z) \leq \sup_K u,\ u \text{ is } q\text{-psh. on } \Omega\}.$$

Then there is a piecewise smooth strictly q-psh. function u on Ω such that

(a) $u < 0$ *on* K *and* $u > 0$ *on* $\Omega \backslash \omega$.

(b) $\{z : u(z) < c\}$ *is relatively compact in* Ω *for every* $c \in \mathbf{R}$.

Proof: Using a compactness argument and Proposition 2.4, as in the proof of Theorem 2.6.11 in [**Hö**], we can find a continuous q-psh. function v on Ω satisfying (a) and (b). Applying Theorem 4.1 we get a piecewise smooth strictly q-psh. function u satisfying (a) and (b). We are done. $\square$

Theorem 4.3. *Let Ω be a bounded q-hyperconvex domain in $\mathbf{C}^n$. Assume that there is a compact set P in $\partial\Omega$ having the following properties.*

(a) *For every $p \in (\partial\Omega)\backslash P$, there is a q-psh. barrier at p.*

(b) *There is a negative, locally bounded q'-psh. function g on Ω such that $P = \{z \in \overline{\Omega} : g^*(z) = -\infty\}$.*

Then for every bounded from above r-psh. function u on Ω and every compact set K of $(\partial\Omega)\backslash P$ there is a sequence of bounded from above strictly s-psh. functions $\{u_j\}$ on Ω which are continuous on $\Omega \cup K$ such that $u_j \downarrow u^*$ on $\Omega \cup K$ where $s = \max\{r + q', 2q\}$.

Proof: We follow the lines of Theorem 3.2 in [**NW**]. Choose a negative continuous q-psh. exhaustion function v for Ω. Since u^* is upper semicontinuous on $\partial\Omega$, there is a sequence $\{\varphi_j\}$ of continuous functions on $\partial\Omega$ that decreases to u^* on $\partial\Omega$. Now for each j we set

$$\Phi_j = \sup\{\varphi : \varphi \text{ is } q\text{-psh. on } \Omega,\ \text{continuous on } \overline{\Omega},\ \varphi \leq \varphi_j \text{ on } \partial\Omega\}.$$

Then Φ_j is bounded and lower semicontinuous on $\overline{\Omega}$. It follows from (a) that $\Phi_j = \varphi_j$ on $(\partial\Omega)\backslash P$. It also follows from Choquet's Topological Lemma that there is a sequence $\{\varphi_{k,j}\}_{k\geq 1}$ of q-psh. function on Ω, continuous on $\overline{\Omega}$ that increases to Φ_j on $\overline{\Omega}$. It follows from the hypothesis (a) that $\varphi_{k,j} \uparrow \varphi_j$ on $(\partial\Omega)\backslash P$.

Fix a function $\rho \in \mathcal{C}(\mathbf{C}^n)$ such that $\rho(0) = 1$, $0 \le \rho \le 1$ and $\operatorname{supp} \rho \subset \mathbf{B}(0,1)$. Fix $j \ge 1$, we claim that there are $\delta_j \in (0, 1/j)$, $a_j \ge 1$ such that

$$(u *_s \rho_{\delta_j}) - \frac{1}{j} + \frac{1}{j}(g *_s \rho_{\delta_j}) \le jv + \varphi_{a_j,j}$$

on Ω_{δ_j}. To see this, we argue by contradiction. Assume otherwise, then there are sequences $\{\delta_m\} \downarrow 0$, $\{b_m\} \uparrow \infty$ and a sequence of points $\{x_m\}$, $x_m \in \Omega_{\delta_m}$ such that

$$(u *_s \rho_{\delta_m})(x_m) - \frac{1}{j} + \frac{1}{j}(g *_s \rho_{\delta_m})(x_m) \ge jv(x_m) + \varphi_{b_m,j}(x_m).$$

Passing to a subsequence we may assume that $\{x_m\}$ converges to $x^* \in \partial\Omega$. Using the upper semicontinuity of u^* and g^* on $\overline{\Omega}$ and the definition of supremum-convolution, we have

$$\limsup_{m\to\infty} \left((u *_s \rho_{\delta_m})(x_m) + \frac{1}{j}(g *_s \rho_{\delta_m})(x_m) \right) \le u^*(x^*) + \frac{1}{j} g^*(x^*).$$

Since $\lim_{z\to\partial\Omega} v(z) = 0$, we deduce that

$$\limsup_{m\to\infty} (jv(x_m) + \varphi_{b_m,j}(x_m)) \ge \limsup_{m\to\infty} \varphi_{b_m,j}(x_m).$$

Putting all this together, we obtain

$$u^*(x^*) \ge u^*(x^*) + \frac{1}{j} g^*(x^*) \ge \limsup_{m\to\infty} \varphi_{b_m,j}(x_m).$$

Combining this inequality with (b) we infer $x^* \notin P$. Thus $x^* \in (\partial\Omega)\backslash P$. But then we have $\varphi_{b_m,j}(x^*) \uparrow \varphi_j(x^*) > u^*(x^*)$. A contradiction! The claim is therefore proved. This implies, in view of the Gluing Lemma and Proposition 2.2 (viii), that the function

$$v_j = \begin{cases} \max\left\{ (u *_s \rho_{\delta_j}) - \dfrac{1}{j} + \dfrac{1}{j}(g *_s \rho_{\delta_j}), jv + \varphi_{a_j,j} \right\} & \text{on } \Omega_{\delta_j} \\ jv + \varphi_{a_j,j} & \text{on } \overline{\Omega}\backslash\Omega_{\delta_j} \end{cases}$$

is s-psh. on Ω and continuous on $\overline{\Omega}$. Moreover v_j converges pointwise to u^* on $\Omega \cup K$. Now set

$$u_j(z) = \sup_{m \ge j} \{v_m(z)\} + |z|^2/j.$$

It is clear that $u_j \downarrow u^*$ on $\Omega \cup K$. Fix $j \geq 1$, we will show that u_j is continuous on $\Omega \cup K$. Since each u_j is continuous on $\overline{\Omega}$ we have u_j is lower semicontinuous there. It remains to check that u_j is upper semicontinuous on $\Omega \cup K$. Assume otherwise, then there is $a \in \Omega \cup K$, $\varepsilon > 0$ and sequences $j_k \uparrow \infty$, $z_{j_k} \to a$ such that

$$v_{j_k}(z_{j_k}) > v_{j_k}(a) + \varepsilon,\ z_{j_k} \in \Omega \cup K, \quad \forall\, k \geq 1.$$

Consider two cases.

Case 1. $a \in \Omega$. Then we may assume that $z_{j_k} \in \Omega_{\delta_{j_k}}$, $\forall\, k$. Since g is bounded near a and $v < 0$, from the definition of v_j we infer that for all k large enough the following inequalities hold

$$v_{j_k}(a) \geq (u *_s \rho_{\delta_{j_k}})(a) - \varepsilon/2,\ v_{j_k}(z_{j_k}) \leq (u *_s \rho_{\delta_{j_k}})(z_{j_k}).$$

This is absurd, in view of the upper semicontinuity of u at a.

Case 2. $a \in K$. Since $v_j \equiv \varphi_{a_j,j}$ on K for all j, we may assume that $z_{j_k} \in \Omega$, $\forall\, k$. Since $v < 0$ on Ω we have $v_{j_k} < \varphi_{a_{j_k},j_k}$ on $\Omega \backslash \Omega_{\delta_{j_k}}$ and $v_{j_k} < \max(\varphi_{a_{j_k},j_k}, (u *_s \rho_{\delta_{j_k}}))$ on $\Omega_{\delta_{j_k}}$. Since the $\varphi_{a_{j_k},j_k}$ are continuous on $\overline{\Omega}$ and u^* is upper semicontinuous at a, we get a contradiction.

Thus u_j is continuous on $\Omega \cup K$. The desired conclusion now follows. $\square$

Remark. Let Ω be the Hartogs triangle $\{(z,w) \in \mathbf{C}^2 : 0 < |z| < |w| < 1\}$. Then Ω is pseudoconvex and 1-strictly pseudoconvex, since it can be defined as $\{(z,w) \in \mathbf{C}^2 : \max(|z|^2 - |w|^2, |w|^2 - 1) < 0\}$. In particular Ω is B_1-regular. Consider the bounded psh. function $u(z,w) = |z/w|$. By the maximum principle we can check that u can not be approximated from above by continuous functions on $\overline{\Omega}$ which are 0-psh. on Ω. (For details see Section 4 of [**Wi**]). On the other hand, by Theorem 4.3 we see that u^* is the limit on $\overline{\Omega}$ of a decreasing continuous functions on $\overline{\Omega}$ which are 1-psh. functions on Ω. This shows that the conclusion of Theorem 4.3 is somehow optimal.

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