Publicacions Matemàtiques, Vol 36 (1992), 1075–1084.

ON THE STRUCTURE OF THE CANONICAL MODULE OF THE REES ALGEBRA AND THE ASSOCIATED GRADED RING OF AN IDEAL

S. Zarzuela

Dedicated to the memory of Pere Menal

Abstract

In this note we give the description of a morphism related with the structure of the canonocal module of the Rees algebra $R(I)$ of an ideal I in a local ring. As an application we obtain Ikeda's criteria for the Gorensteinness of $R(I)$ and a result of Herzog-Simis-Vasconcelos characterizing when the canonical module of $R(I)$ has the expected form.

Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. In [4] Ikeda characterized the Gorensteinness of the Rees algebra $R(I) = \bigoplus_{n\geq 0} I^n t^n$ by means of the canonical module of A and the canonical module of the associated graded ring $gr_A(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, under the assumptions that $R(I)$ is Cohen-Macaulay and $grade(I) \geq 2$. If in particular A is Cohen-Macaulay this characterization means that if $ht(I) \geq 2$, the Rees algebra $R(I)$ is Gorenstein if and only if the ground ring A is Gorenstein and the associated graded ring $gr_A(I)$ is Gorenstein with a-invariant $a(gr_A(I)) = -2$.

On the other hand in [3] Herzog-Simis-Vasconcelos investigated the canonical modules of $R(I)$ and $gr_A(I)$, where I is an ideal in a local Cohen-Macaulay ring $(A, \mathfrak{m})$. They were specially interested in characterizing when the canonical module of the Rees algebra $R(I)$ is isomorphic to the $R(I)$-submodule of the polynomial ring $A[t]$ generated by $1, t, \ldots, t^m$, where $m \geq 0$, or to the ideal $IR(I)$. Then it is said that the canonical module of $R(I)$ has the expected form, that occurs if in particular $R(I)$ is Gorenstein.

These results have been recently used to study the Gorensteinness of the Ress algebras and associated graded rings of powers of ideals, see [**2**].

Our deal in this paper is to give a common point of view for both results. For this, and mainly inspired in [**3**], we first give the description of a morphism of graded $R(I)$-modules

$$\Gamma : \bigoplus_{n\geq 2} \left(K_{R(I)}\right)_n \to K_{R(I)},$$

where $K_{R(I)}$ is the canonical module of the Rees algebra $R(I)$. By means of this morphism Γ some information about the structure of $K_{R(I)}$ can be transfered to the canonical module of the associated graded ring $gr_A(I)$, and conversely. This is done is section 1, proposition (1.1). Then, as a main application, we obtain in section 2 the above mentioned results of Ikeda and Herzog-Simis-Vasconcelos.

The existence of a morphism with similar properties as Γ for multi-graded Rees algebras has been obtained by H. Hiri (Helsinki).

1. The main result

We shall use the book [**1**] as a reference for unexplained results and terminology. Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring defined over a local ring R_0. If L is a finitely generated graded R-module then the Krull dimension of L, $\dim(L)$, satisfies

$$\dim(L) = \sup\left\{i \middle| \underline{H}^i_N(L) \neq 0\right\},$$

where $\underline{H}^i_N(L)$ are the $i-th$ graded local cohomology modules of L with respect to N, the maximal homogeneous ideal of R. The a-invariant of L is then defined by

$$a(L) := \sup\left\{n \middle| \left(\underline{H}^{\dim(L)}_N(L)\right)_n \neq 0\right\}.$$

Since $\underline{H}^{\dim(L)}_N(L)$ is an artinian graded R-module, $a(L)$ is a well defined integer.

Assume that R has a canonical module, K_R. Passing to the completion if necessary and by local duality one has that

$$a(R) = -\inf\left\{n \middle| (K_R)_n \neq 0\right\},$$

and R is Gorenstein if and only if R is Cohen-Macaulay and $K_R \simeq R(a(R))$.

From now on $(A, \mathfrak{m})$ will be a local ring and I an ideal of A. We shall use the following notations:

$S = R(I) = \bigoplus_{n\geq 0} I^n t^n \subset A[t]$, the Rees algebra of I,

$S_+ = \bigoplus_{n\geq 1} I^n t^n = t(IR(I)) = (tI)\, R(I)$,

$M = \mathfrak{m} \oplus S_+$, the maximal homogeneous ideal of $R(I)$,

$G = gr_A(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, the associated graded ring of I,

$K_S = K_{R(I)}$, $K_G = K_{gr_A(I)}$, K_A the canonical modules of $R(I)$, $gr_A(I)$ and A.

Since $gr_A(I) = R(I)/IR(I)$ and $A = R(I)/S_+$, the canonical modules $K_{gr_A(I)}$ and K_A exist if the canonical module $K_{R(I)}$ exists. Assume moreover that $R(I)$ is Cohen-Macaulay and $ht(I) > 0$. By [4, (2.1)] the a-invariant $a(gr_A(I))$ is negative, and from the same proof it can be deduced that $a(R(I)) = -1$.

Proposition (1.1). *Let $(A, \mathfrak{m})$ be a local ring and $I \subset A$ be an ideal of A with $ht(I) > 0$. Assume that the Rees algebra $R(I)$ is Cohen-Macaulay and has a canonical module $K_{R(I)}$. Put*

$$\left(K_{R(I)}\right)_+ := \bigoplus_{n\geq 2} \left(K_{R(I)}\right)_n .$$

Then there exists a morphism of graded $R(I)$-modules

$$\Gamma : (K_{R(I)})_+ \to K_{R(I)}$$

such that:

(i) *Γ is injective of degree -1.*

(ii) *For any $n < -a\big(gr_A(I)\big)-1$, $\Gamma_{|(K_{R(I)})_{n+1}} : (K_{R(I)})_{n+1} \to (K_{R(I)})_n$ is bijective.*

(iii) *For any element $\beta \in (K_{R(I)})_+$, $\Gamma(\beta)$ is the only element in $K_{R(I)}$ such that $s\beta = (t\, s)\,\Gamma(\beta)$ for any $s \in IR(I)$.*

(iv) *For any $r \geq 1$ and any $\alpha \in K_{R(I)}$, $\Gamma\big((t^r y)\alpha\big) = \big(t^{r-1} y\big)\alpha$ for any $y \in I^r$.*

(v) *There exists an isomorphism $\bigoplus_{n\geq 2}\big(K_{gr_A(I)}\big)_n \simeq \Big(K_{R(I)}/\Gamma\big((K_{R(I)})_+\big)\Big)(-1)$.*

Proof. Consider the exact sequences of graded S-modules

$$0 \to S_+ \to S \to A \to 0$$
$$0 \to IS \to S \to G \to 0.$$

Dualizing with K_S we get the exact sequences

$$0 \to \underline{\mathrm{Hom}}_S(A, K_S) \to \underline{\mathrm{Hom}}_S(S, K_S) \to \underline{\mathrm{Hom}}_S(S_+, K_S) \to \underline{\mathrm{Ext}}^1_S(A, K_S) \to \underline{\mathrm{Ext}}^1_S(S, K_S) = 0$$

and

$$0 \to \underline{\mathrm{Hom}}_S(G, K_S) \to \underline{\mathrm{Hom}}_S(S, K_S) \to \underline{\mathrm{Hom}}_S(IS, K_S) \to \underline{\mathrm{Ext}}^1_S(G, K_S) \to \underline{\mathrm{Ext}}^1_S(S, K_S) = 0.$$

By [**1**, (36.8)] K_S is a Cohen-Macaulay S-module with $depth(K_S) = \dim(S) = \dim(A) + 1$, hence given that $\dim(A) = \dim(G)$ we obtain $\underline{\mathrm{Hom}}_S(A, K_S) = \underline{\mathrm{Hom}}_S(G, K_S) = 0$. On the other hand by [**1**, (36.14)] $K_A \simeq \underline{\mathrm{Ext}}^1_S(A, K_S)$, $K_G \simeq \underline{\mathrm{Ext}}^1_S(G, K_S)$, thus we have exact sequences

$$0 \to \underline{\mathrm{Hom}}_S(S, K_S) \xrightarrow{\pi} \underline{\mathrm{Hom}}_S(S_+, K_S) \xrightarrow{\psi} \underline{K}_A \to 0$$
$$0 \to \underline{\mathrm{Hom}}_S(S, K_S) \xrightarrow{\sigma} \underline{\mathrm{Hom}}_S(IS, K_S) \xrightarrow{\varphi} \underline{K}_G \to 0$$

where π and σ are the obvious restriction maps.

Furthermore $\underline{\mathrm{Hom}}_S(S, K_S)$ may be identified with K_S by the map that in degree n is given by

$$\begin{aligned} \underline{\mathrm{Hom}}_S(S, K_S)_n &\to (K_S)_n, \\ f &\to f(1) \end{aligned}$$

thus we may finally write exact sequences

$$0 \to W_S \xrightarrow{\pi} \underline{\mathrm{Hom}}_S(S_+, K_S) \xrightarrow{\psi} K_A \to 0$$
$$0 \to K_S \xrightarrow{\sigma} \underline{\mathrm{Hom}}_S(IS, K_S) \xrightarrow{\varphi} K_G \to 0.$$

Consider the morphism of graded S-modules

$$\begin{aligned} \tau : \underline{\mathrm{Hom}}_S(S_+, K_S) &\to \underline{\mathrm{Hom}}_S(IS, K_S) \\ f &\to \tau(f) : IS \to K_S \\ &\qquad s \to \tau(f)(s) = f(ts). \end{aligned}$$

Since $S_+ = t(IS)$, τ is an isomorphism of degree 1 and for any n there exists a diagram with exact sequences

$$\begin{array}{c} 0 \to (K_S)_n \xrightarrow{\pi} \underline{\mathrm{Hom}}_S(S_+, K_S)_n \xrightarrow{\psi} (K_G)_n \to 0 \\ \tau \big\downarrow \\ 0 \to (K_S)_{n+1} \xrightarrow{\sigma} \underline{\mathrm{Hom}}_S(IS, K_S)_{n+1} \xrightarrow{\varphi} (K_A)_{n+1} \to 0. \end{array}$$

Define $\omega := \tau\pi$. K_A is a graded S-module reduced to degree 0 while $(K_S)_n = 0$ for any $n \le 0$, hence

$$\pi : K_S \to \bigoplus_{n\ge 1} \underline{\mathrm{Hom}}_S(S_+, K_S)_n$$

is an isomorphism and

$$\omega : K_S \to \bigoplus_{n\ge 2} \underline{\mathrm{Hom}}_S(IS, K_S)_n$$

is also an isomorphism of degree 1. Hence we may define

$$\Gamma := \omega^{-1}\sigma : (K_S)_+ \to K_S,$$

a morphism of graded S-modules of degree -1. Since σ is injective Γ is injective too, proving (i).

Set $a := -a\big(gr_A(I)\big)$. It is clear that $\sigma_{|(K_S)_n}$ is an isomorphism for any $n < a$, hence

$$\Gamma_{|(K_S)_{n+1}} : (K_S)_{n+1} \to (K_S)_n$$

is an isomorphism for any $n < a-1$, that is (ii).

To show (iii) take an element $\beta \in (K_S)_+$: By definition of the restriction map σ,

$$\begin{aligned} \sigma(\beta) : IS &\to K_S \\ s &\to s\beta. \end{aligned}$$

Moreover, by definition of π

$$\begin{aligned} \pi\,\Gamma(\beta) : S_+ &\to K_S \\ ts &\to (ts)\,\Gamma(\beta). \end{aligned}$$

Therefore $s\beta = (ts)\Gamma(\beta)$ for any $s \in IS$. Now assume that $(ts)\alpha = (ts)\,\Gamma(\beta)$ for any $s \in IS$, where $\alpha \in W_S$. Then $(ts)(\alpha - \Gamma(\beta)) = 0$ for any $s \in IS$ and in particular $\big(0 : (\Gamma(\beta) - \alpha)\big) \supset S_+$, with $ht(S_+) = 1$. This implies that $\alpha = \Gamma(\beta)$ since K_S is a Cohen-Macaulay S-module with $depth(K_S) = \dim(S)$.

Furthermore for any element $\alpha \in K_S$ and any integer $r \ge 1$ we have that $s\big(t^r y\big)\,\alpha = (ts)(t^{r-1}y)\,\alpha$ for any $s \in IS$ and $y \in I^r$. By (iii) $\Gamma\big((t^r y)\alpha\big) = (t^{r-1}y)\alpha$, showing (iv).

Finally it is clear that $\bigoplus_{n\ge 2}(K_C)_n \simeq \Big(\bigoplus_{n\ge 2} \underline{\mathrm{Hom}}_S(IS, K_S)_n\Big)/\sigma\big((K_S)_+\big) \simeq \Big(K_S/\Gamma\big((K_S)_+\big)\Big)(-1)$, hence (v) . ∎

Remark (1.2). If in proposition (1.1) we assume in particular that $a(gr_A(I)) \leq -2$, then $(K_S)_1 \underset{\sigma}{\simeq} \underline{\mathrm{Hom}}_S(IS, K_S)_1 \underset{\tau}{\simeq} \underline{\mathrm{Hom}}_S(S_+, K_S)_0 \underset{\psi}{\simeq} K_A$.

Remark (1.3). Next we list some of the cases for which the a-invariant $a(gr_A(I))$ can be explicitly computed.

(i) Let I be an $\mathfrak{m}$-primary ideal in a local ring $(A, \mathfrak{m})$ with infinite residue field. Assume that $gr_A(I)$ is Cohen-Macaulay. Then

$$a(gr_A(I)) = \delta(I) - \dim(A),$$

where $\delta(I)$ is the reduction exponent of I, see [**2**, (2.4)].

(ii) Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and I a strongly Cohen-Macaulay ideal in A (that is, an ideal such that for any $r \geq 0$ the Koszul homology $H_r(K(\underline{a}))$ is zero or a maximal Cohen-Macaulay A/I-module, where $\underline{a}$ is any system of generators of I). Assume that the minimal number of generators $\mu(I_\mathfrak{p}) \leq ht(\mathfrak{p})$ for all prime ideals $\mathfrak{p} \supseteq I$. Then

$$a(gr_A(I)) = -ht(I),$$

see [**2**, (2.5)].

(iii) Let I be an almost complete intersection ideal in a Cohen-Macaulay ring $(A, \mathfrak{m})$ (we say that I is almost complete intersection if $\mu(I) = ht(I) + 1$ and $\mu(I_\mathfrak{p}) = ht(\mathfrak{p})$ for all prime ideals $\mathfrak{p} \in Min(A/I)$). Then

$$a(gr_A(I)) = -ht(I),$$

see [**5**, (7.3)].

As we have already commented in the introduction, Ikeda proved in [**4**] that if $grade(I) \geq 2$ and $R(I)$ is Gorenstein the a-invariant $a(gr_A(I)) = -2$. This is a particular case of the following result that we may obtain from the proof of proposition (1.1).

Corollary (1.4). *Let $(A, \mathfrak{m})$ be a local ring and $I \subset A$ a non principal ideal of A such that $I^{-1} = A$. If $R(I)$ is Gorenstein then $a(gr_A(I)) = -2$.*

Proof: We use the same notation as in the proof of proposition (1.1). First we show that $a(G) \leq -2$. For this it is enough to see that $\sigma_{|(K_S)_1}$ is an isomorphism, that is, that any $f \in \underline{\mathrm{Hom}}_S(IS, K_S)_1$ is given by the product by an element $a \in (K_S)_1$. And this follows immediately from the fact that $(K_S)_1 \simeq A$ (S is Gorenstein), I generates IS and $\mathrm{Hom}_A(I, A) = I^{-1} = A$. Now by proposition (1.1) we have $(K_G)_{-2} \simeq ((K_S)_1/\Gamma((K_S)_2)) \neq 0$ since Γ is injective and $(K_S)_2 \simeq I$ is not principal. ■

2. Applications

Throughout this section we shall use the same notation as in the preceding one. Let I be an ideal of A such that the associated graded ring $gr_A(I)$ has a canonical module. First we ask when there exists a finitely generated A-module P such that $gr_P(I) := \bigoplus_{n\geq 0} I^nP/I^{n+1}P$ is isomorphic to $K_{gr_A(I)}(r)$, where r is some integer (see **[6]** for this question when I is the maximal ideal of A). Observe that in any case $r = -a\big(gr_A(I)\big)$. Suppose that $gr_A(I)$ is Cohen-Macaulay (then A is Cohen-Macaulay too). By **[1**, (36.11)**]** $K_{gr_A(I)}$ is a graded $gr_A(I)$-module of finite injective dimension with $rk_k\left(\underline{\mathrm{Ext}}^i_{gr_A(I)}\big(k, K_{gr_A(I)}\big)\right) = 1$ if $i = \dim\big(gr_A(I)\big)$, and 0 otherwise, where k is the residue field of A. By standard methods it is easy to show that if $gr_P(I) \simeq K_{gr_A(I)}(r)$ then P is an A-module of finite injective dimension with $rk_k\big(\underline{\mathrm{Ext}}^i_A(k, P)\big) = 1$ if $i = \dim(A)$, and 0 otherwise. Hence P is a canonical module of A.

Denote by $(1,t)^m$ the $R(I)$-submodule of the polynomial ring $A[t]$ generated by $1, t, \ldots, t^m$, where $m \geq 0$, and by $(1,t)^{-1}$ the ideal $IR(I)$. If P is a finitely generated A-module let $P(1,t)^m$ be the extension $P \underset{A}{\otimes} (1,t)^m$. The above question can be answered by means of these modules in the following way:

Theorem (2.1) ([3, (2.4)]). *Let $(A, \mathfrak{m})$ be a local ring and $I \subset A$ be an ideal of A with $ht(I) \geq 1$. Assume that $R(I)$ is Cohen-Macaulay and has a canonical module. Let $a := -a\big(gr_A(I)\big)$. Then the following are equivalent:*

(i) $K_{gr_A(I)} \simeq gr_{K_A}(I)(-a)$.
(ii) $K_{R(I)} \simeq K_A(1,t)^{a-2}(-1)$.

Proof: (i) $\Longrightarrow$ (ii) Assume $K_G \simeq \left(\bigoplus_{n\geq 0} I^nK_A/I^{n+1}K_A\right)(-a)$. First observe that if $a \geq 2$ then $(K_S)_1 \simeq K_A$ by remark (1.2). On the other hand if $a = 1$ we have the following diagram with exact sequences

$$\begin{array}{ccc} & \underline{\mathrm{Hom}}_S(S_+, K_S)_0 \xrightarrow{\psi} K_A \to 0 & \\ & \Big\downarrow \tau & \\ 0 \to (K_S)_1 \underset{\sigma}{\to} & \underline{\mathrm{Hom}}_S(IS, K_S)_1 & \underset{\varphi}{\to} (K_G)_1 \to 0 \end{array}$$

where τ and ψ are isomorphisms. Thus $(K_S)_1 \simeq \sigma\big((K_S)_1\big) = Ker\varphi \simeq Ker\big(\varphi\tau\psi^{-1}\big)$, and since $(K_G)_1 \simeq K_A/IK_A$ and $\varphi\tau\psi^{-1}$ is an epimorphism we have that $(K_S)_1 \simeq IK_A$.

Claim. If $a \geq 2$ then for any $n \geq a-1$.

$$(K_S)_n = \left(t^{n-a+1} I^{n-a+1}\right) (K_S)_{a-1},$$

and if $a = 1$ then for any $n \geq 1$

$$(K_S)_n = \left(t^{n-a} I^{n-a}\right) (K_S)_a.$$

The proof of the claim is by induction on n, First assume that $a \geq 2$, and let $n \geq a-1$ such that

$$(K_S)_n = \left(t^{n-a+1} I^{n-a+1}\right) (K_S)_{a-1}.$$

Consider the diagram we used in the proof of proposition (1.1)

$$\begin{tikzcd} (K_S)_n \arrow[drr, "\omega"] & & & & \\ 0 \arrow[r] & (K_S)_{n+1} \arrow[r, "\sigma"] & \underline{\mathrm{Hom}}_S(IS, K_S)_{n+1} \arrow[r, "\varphi"] & (K_G)_{n+1} \arrow[r] & 0 \end{tikzcd}$$

By definition

$$\Gamma\big((K_S)_{n+1}\big) = \omega^{-1}\sigma\big((K_S)_{n+1}\big) = \omega^{-1}\big(Ker(\varphi)\big) = Ker(\omega\varphi),$$

while by assumption $(K_G)_{n+1} = I^{n+1-a}K_A/I^{n+2-a}K_A$ with $K_A \simeq (K_S)_{a-1}$. Hence $(K_G)_{n+1} \simeq I^{n+1-a}\big(K_S\big)_{a-1}/I^{n+2-a}(K_S)_{a-1} = (K_S)_n/I(K_S)_n$, and since $\omega\varphi$ is an epimorphism this implies that $Ker(\omega\varphi) = I(K_S)_n$. Therefore $\Gamma\big((K_S)_{n+1}\big) = I(K_S)_n = \Gamma\big((tI)(K_S)_n\big)$ by proposition (1.1), (iv) and since Γ is injective we get that

$$(K_S)_{n+1} = (tI)(K_S)_n = \left(t^{n-a+2} I^{n-a+2}\right) (K_S)_{a-1}.$$

With similar arguments we can also proof the claim when $a = 1$.

Therefore we can write

$$\begin{aligned} K_S = (K_S)_1 \oplus \cdots \oplus (K_S)_{a-1} \oplus (tI)(K_S)_{a-1} \\ \oplus (t^2I^2)(K_S)_{a-1} \oplus \cdots, \end{aligned}$$

with $(K_S)_1 \underset{g}{\simeq} K_A$ if $a \geq 2$, and

$$K_S = (K_S)_1 \oplus (tI)(K_S)_1 \oplus (t^2t^2)(K_S)_1 \oplus \cdots$$

with $(K_S)_1 \underset{h}{\simeq} IK_A$ if $a = 1$.

Suppose $a \geq 2$. By definition

$$K_A(1,t)^{a-2} = K_A \oplus t\,K_A \oplus \cdots \oplus t^{a-2}K_A \oplus t^{a-1}\,I\,K_A \oplus t^a\,I^2\,K_A \oplus \cdots .$$

Consider the map

$$\xi : K_S \to K_A(1,t)^{a-2}$$

defined by $\xi_{|(K_S)_1} = g$ and $\xi_{|(K_S)_n} = t\left(\xi_{|(K_S)_{n-1}}\Gamma\right)$ if $n \geq 2$. ξ is a morphism of graded S-modules, for if $\alpha \in (K_S)_n$ and $a \in I^r$, $\xi\big((t^r\,a)\alpha\big) = t\,\xi\Big(\Gamma\big((t^r\,a)\alpha\big)\Big) = t\,\xi\big((t^{r-1}\,a)\,\alpha\big)$ by proposition (1.1), (iv) and applying this repeatdly we get $\xi\big((t^r\,a)\,\alpha\big) = t^r\,\xi(a\alpha) = (t^r a)\,\xi(\alpha)$.

It is clear that ξ is injective (Γ it is), hence to show that ξ is an isomorphism we must see that $\xi\big((K_S)_n\big) = \big(K_A(1,t)^{a-2}\big)_{n-1}$ for any n. This is trivial for $n < a$, thus assume $n \geq a$. By the claim $(K_S)_n = (tI)(K_S)_{n-1}$, hence $\Gamma\big((K_S)_n\big) = I(K_S)_{n-1}$ by proposition (1.1), (iv). By induction we then get that ξ is an isomorphism. Since it is of degree -1 we finally obtain

$$K_S \simeq K_A(1,t)^{a-2}(-1).$$

Similarly we could proof the case $a = 1$.

(ii) $\Longrightarrow$ (i) If $a \geq 2$ this is a direct consequence of proposition (1.1), (v). Hence assume $a = 1$ and $K_S \simeq K_A(1,t)^{-1}(-1)$. Consider the exact sequence

$$0 \to K_S \xrightarrow{\sigma} \underline{\mathrm{Hom}}_S(IS,\, K_S) \xrightarrow{\varphi} K_G \to 0.$$

Passing to the completion if necessary and by local duality [**1**, (36.8)] we obtain the exact sequence

$$0 \to K_A IS(-1) \to K_A S(-1) \to K_G \to 0,$$

thus $K_G \simeq gr_{K_A}(I)(-1)$ as we wanted to show.

Now we can also obtain the following version of Ikeda's result characterizing when the Rees algebra is Gorenstein:

Theorem (2.2) ([4, (3.1)]). *Let $(A, \mathfrak{m})$ be a local ring and I be an ideal of A. Suppose that $R(I)$ is Cohen-Macaulay and $I^{-1} = A$. The following conditions are equivalent:*

(i) *$R(I)$ is Gorenstein.*

(ii) *$K_A \simeq A$ and $K_{gr_A(I)} \simeq gr_A(I)\,(-2)$.*

Proof: (i) $\Longrightarrow$ (ii) By corollary (1.4) $a(G) = -2$, and by remark (1.2) $K_A \simeq A$. Hence by theorem (2.1) $K_G \simeq G(-2)$.

(ii) $\Longrightarrow$ (i) By theorem (2.1) we get $K_S \simeq (1, t)(-1) = S(-1)$, and S is Gorenstein since by hypothesis S is Cohen-Macaulay. ∎

Remark (2.3). Assume under the hypothesis of theorem (2.1) that $gr_A(I)$ is Gorenstein and $a(gr_A(I)) = -1$. Then $K_{R(I)} \simeq IS(-1)$, thus in particular $I \simeq (K_{R(I)})_1$ and $R(I)$ cannot be Gorenstein if I is not principal.

Acknowledgements.

I must thank M. Herrmann, J. Ribbe and H. Hiri for the discussions we had about this problem during my stay in the Mathematical Institute of the University of Cologne (Germany). I was partially supported by DGICYT-grant PB 88-0224 (Spain).

References

1. M. HERRMANN, S. IKEDA AND U. ORBANZ, "*Equimultiplicity and Blowing Up*," Springer–Verlag, Berlin Heidelberg, 1988.
2. M. HERRMANN, J. RIBBE AND S. ZARZUELA, On the Gorenstein property of Rees and form rings of powers of ideals, to appear in *Transactions of the American Mathematical Society*.
3. J. HERZOG, A. SIMIS AND W. V. VASCONCELOS, On the canonical module of the Rees algebra and the associated graded ring of an ideal, *Journal of Algebra* **105** (1987), 285–302.
4. S. IKEDA, On the Gorensteinness of Rees algebras over local rings, *Nagoya Mathematical Journal* **102** (1986), 135–154.
5. J. RIBBE, Dissertation, Universität zu Köln (1991).
6. A. OOISHI, On the associated graded modules of canonical modules, *Journal of Algebra* **141, no. 1** (1991), 143–157.

Departament d'Àlgebra i Geometria
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
E-08007 Barcelona
SPAIN

Rebut el 18 de Desembre de 1991