RELATIVE HERMITIAN MORITA THEORY
Part II: Hermitian Morita contexts

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Dedicated to the memory of Prof. Pere Menal

Abstract

We introduce the notion of a relative hermitian Morita context between torsion triples and show how these induce equivalences between suitable quotient categories of left and right modules.

Due to the lack of involutive bimodules, the induced Morita equivalences are not necessarily hermitian, however.

0. Introduction

In the first part of this paper [12], the authors essentially introduced and studied hermitian Morita theory for arbitrary involutive Grothendieck categories, generalizing previous work by Hahn [9] et al. Using the Gabriel-Popescu theorem [5], this study reduces to the study of categories of the form \((A, \sigma)\)-mod, i.e., quotient categories of the category \(A\)-mod of left modules over some ring \(A\) by a localizing subcategory \(T\), consisting of all left \(A\)-modules, which are torsion with respect to some idempotent kernel functor \(\sigma\) in \(A\)-mod.

More precisely, one starts in the aforementioned paper from the notion of a torsion triple \((A, \alpha, \sigma)\), where \((A, \alpha)\) is an algebra with involution over some fixed commutative ring \(R\) and where \(\sigma\) is an idempotent kernel functor in \(A\)-mod. The idempotent kernel functor \(\sigma\) then induces an idempotent kernel functor \(\alpha(\sigma)\) in \(\text{mod-}A\), defined by letting its Gabriel filter \(\mathcal{L}(\alpha(\sigma))\) consist of all right \(A\)-ideals \(\alpha(L)\) with \(L \in \mathcal{L}(\sigma)\). In particular, one may then localize \(A\) with respect to both \(\sigma\) and \(\alpha(\sigma)\), thus obtaining rings of quotients \(Q_{\sigma}(A)\) resp. \(Q_{\alpha(\sigma)}(A)\), which, in general,

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are not isomorphic, however. Let \((B, \beta, \tau)\) be a second torsion triple and let

\[
(P, Q, \theta : P \to Q, \mu : Q_{\tau}(P \otimes_A Q) \to Q_{\tau}, \nu : Q_{\sigma}(Q \otimes_B P) \to Q_{\sigma}(A))
\]

be a relative hermitian Morita equivalence, i.e., we assume \(P\) resp. \(Q\) to be a \((\tau, \sigma)\)-invertible \(B\)-\(A\)-bimodule resp. a \((\sigma, \tau)\)-invertible \(A\)-\(B\)-bimodule defined over some fixed commutative ring \(R\), cf. [12]; \(\mu\) resp. \(\nu\) to be isomorphisms of \(B\) resp. \(A\)-bimodules and \(\theta\) to be an additive bijection, with the property that \(\theta(bpa) = \alpha(a)\theta(p)\beta(b)\) for all \(p \in P, a \in A\) and \(b \in B\), these data satisfying some natural compatibility conditions, cf. [12]. In this case, \(P\) is a so-called involutive \(B\)-\(A\)-bimodule in the sense that there is a \(B\)-\(A\)-bimodule isomorphism \(P \cong_B [Q, Q_{\tau}(B)]^\sigma\), where for any \(A\)-\(B\)-bimodule \(M\), we define the \(B\)-\(A\)-bimodule \(\beta M^\sigma\) by \(b \cdot m \cdot a = \alpha(a)m\beta(b)\), for all \(a \in A, b \in B\) an \(m \in M\). It follows that the existence of a relative hermitian Morita equivalence between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) implies that \(Q_{\alpha(\sigma)}(A) = Q_{\sigma}(A)\) and \(Q_{\beta(\tau)}(B) = Q_{\tau}(B)\) up to canonical isomorphism.

Using this, the authors of [12] were able to derive a hermitian version of Morita I, (i.e., they showed that any relative hermitian Morita equivalence as above defines an equivalence between the categories \(S(A, \alpha, \sigma)\) and \(S(B, \beta, \tau)\) of so-called \(\sigma\)-sesquilinear left \(A\)-modules) as well as a hermitian version of Morita II.

In the present note, we introduce the notion of a relative hermitian Morita context between torsion triples. Using techniques, which are originally due to Mueller [15], we show that such Morita contexts induce equivalences between suitable quotient categories of left and right modules. However, the induced Morita equivalences are not necessarily hermitian, in general, mainly due to the lack of involutive bimodules.

We will see in this note, that relative hermitian Morita contexts exactly induce so-called relative hermitian Morita biequivalences in the sense given below, thus providing the proper framework into which Mueller's results may be generalized in the hermitian case. The behaviour of relative sesquilinear modules in the present set-up and, more generally in that of Morita contexts between so-called linked pairs will be the object of [13].

1. Generalities

Notations and terminology will be as in [12]. For the reader's convenience, let us briefly recollect here some of the main features that will be used throughout.
(1.1) In this note, \( R \) will be a commutative ring and \( A, B, \ldots \) will be \( R \)-algebras. We denote by \( \text{A-mod} \) resp. \( \text{mod-} A \) the category of left resp. right \( A \)-modules. If \( M, N \) are left resp. right \( A \)-modules, then we write \( A[M, N] \) resp. \( [M, N]_A \) for the corresponding sets of \( A \)-linear morphisms. If \( A \) and \( B \) are both \( R \)-algebras, then we denote by \( \text{A-mod-} B \) the category of \( A-B \)-bimodules defined over \( R \).

(1.2) An \( R \)-algebra with involution is a couple \((A, \alpha)\), where \( A \) is an \( R \)-algebra and \( \alpha : A \to A \) an \( R \)-involution, i.e., an \( R \)-linear map with the property that \( \alpha^2 = \text{id}_A \) and \( \alpha(ab') = \alpha(a')\alpha(a) \) for all \( a, a' \in A \). If we only wish to consider rings with involution, it suffices to let \( R = \mathbb{Z} \).

Let \((A, \alpha)\) be an \( R \)-algebra with involution. If \( M \) is a left \( A \)-module, then by definition, the right \( A \)-module \( M^\alpha \) has the same underlying additive group as \( M \) and its right \( A \)-action is given by \( m \cdot a = \alpha(a)m \), for any \( m \in M \) and \( a \in A \). Since any left \( A \)-linear map \( u : M \to N \) induces a right \( A \)-linear map \( u^\alpha : M^\alpha \to N^\alpha \), this actually yields an exact covariant functor \((-)^\alpha : \text{A-mod} \to \text{mod-} A \). For \( M \in \text{mod-} A \), we define \( {}^\alpha M \in \text{A-mod} \) similarly, thus yielding an exact covariant functor \( {}^\alpha (-) : \text{mod-} A \to \text{A-mod} \).

If \((B, \beta)\) is another \( R \)-algebra with involution, then for any \( M \in \text{A-mod-} B \), we define \( {}^B M^\alpha \in \text{B-mod-} A \) by putting \( b \cdot m \cdot a = \alpha(a)m\beta(b) \) for any \( a \in A, b \in B \) and \( m \in M \). If \((B, \beta) = (A, \alpha)\), then we write \( M^\alpha \) for \( {}^B M^\alpha \).

In particular, if \( M \in \text{A-mod-} B \) and \( N \in \text{B-mod-} A \), then morphisms of \( A-B \)-bimodules \( M \to {}^\alpha N^\beta \) correspond bijectively to additive morphisms \( \theta : M \to N \) with the property that \( \theta(ab) = \beta(b)\theta(m)\alpha(a) \) for any \( a \in A, b \in B \) and \( m \in M \). We call such maps \( A-B \)-antimorphisms. If \( \theta \) is invertible, then we speak of an \( A-B \)-anti-isomorphism. If \((A, \alpha) = (B, \beta)\), then we will also speak of antimorphisms and anti-isomorphisms of \( A \)-bimodules.

(1.3) If \( \sigma \) is an idempotent kernel functor in \( R \)-mod with associated localization functor \( Q_\sigma \), then any \( R \)-algebra with involution \((A, \alpha)\) induces an \( R \)-algebra with involution \((Q_\sigma(A), \hat{\alpha})\), where \( \hat{\alpha} \) is the extension \( Q_\sigma(\alpha) \) of \( \alpha \) to \( Q_\sigma(A) \).

More generally, let \((A, \alpha)\) be an \( R \)-algebra with involution and let \( \sigma \) be an arbitrary idempotent kernel functor in \( A \)-mod, (we then call \((A, \alpha, \sigma)\) a (left) torsion triple - right torsion triples and related concepts are defined similarly), then \( \sigma \) induces an idempotent kernel functor \( \alpha(\sigma) \) in \( \text{mod-} A \), defined by its Gabriel filter \( L(\alpha(\sigma)) \), which consists of all right ideals \( \alpha(L) \) of \( A \) with \( L \in L(\sigma) \). In particular, for any right \( A \)-module \( M \), it follows that \( m \in M \) belongs to \( \alpha(\sigma)M \) if and only if \( m\alpha(L) = 0 \) for some \( L \in L(\sigma) \). Let \( Q^L_\sigma \) resp. \( Q^L_{\alpha(\sigma)} \) denote the localization functor
at $\sigma$ resp. $\alpha(\sigma)$ in $A$-mod resp. $\text{mod-}A$. It then has been proved in [12] that the involution $\alpha : A \to A$ extends to an anti-isomorphism $\hat{\alpha} : Q_{\alpha}(A) \to Q_{\alpha}(A)$, i.e., an additive bijection with the property that $\hat{\alpha}(q q') = \hat{\alpha}(q') \hat{\alpha}(q)$, for any $q, q' \in Q_{\alpha}(A)$. However, in general (in the absence of involutive bimodules, cf. [12, (3.7.)]), the $R$-algebras $Q_{\alpha}(A)$ and $Q_{\alpha}(A)$ are not isomorphic and $\alpha$ does not extend to an involution $\hat{\alpha} : Q_{\alpha}(A) \to Q_{\alpha}(A)$.

(1.4) We leave it as an exercise to the reader to verify that for any $M \in A$-mod resp. $N \in \text{mod-}A$ and any idempotent kernel functor $\sigma$ in $A$-mod we have $\alpha(\sigma)M^\sigma = (\sigma M)^\sigma$ resp. $\alpha(\sigma N) = \alpha(\sigma N)$. It is then fairly easy to deduce, for example, that for any $A$-bimodule $M$ the $A$-bimodules $Q_{\alpha}(M)$ and $Q_{\alpha}(M)$ are isomorphic i.e., that there exists an anti-isomorphism $Q_{\alpha}(M) \cong Q_{\alpha}(M)$ of $A$-bimodules.

From this, it easily follows, for example, that the ring $A$ is $\sigma$-closed if and only if $A$ is $\alpha(\sigma)$-closed. Indeed, the involution $\alpha$ induces an isomorphism of $A$-bimodules $A \cong A$. But then, if $A$ is $\sigma$-closed, the $A$-bimodule isomorphism $A \cong Q_{\alpha}(A)$ induces an isomorphism

$$A \cong A \cong Q_{\alpha}(A) \cong Q_{\alpha}(A),$$

which proves that $A$ is $\alpha(\sigma)$-closed. The converse may be proved in a similar way.

2. Hermitian Biequivalences

(2.1) Let us fix a pair of $R$-algebras $A$ and $B$ and idempotent kernel functors $\sigma$ resp. $\tau$ in $A$-mod resp. $B$-mod. Recall from [18], [19] that an $A$-$B$-bimodule $Q$ is said to be $(\sigma, \tau)$-flat, if it possesses the following two properties:

(2.1.1) if $M'' \in B$-mod is torsion with respect to $\tau$, then $Q \otimes_{B} M'' \in A$-mod is torsion with respect to $\sigma$;

(2.1.2) for any monomorphism $i : M' \to M$ in $B$-mod, the kernel $\text{Ker}(Q \otimes_{B} i) \in A$-mod is $\sigma$-torsion.

In particular, if $Q$ is a $(\sigma, \tau)$-flat $A$-$B$-bimodule, then it follows, that for any left $B$-module $M$ there is a canonical isomorphism of left $A$-modules $Q_{\alpha}(Q \otimes_{B} M) \cong Q_{\alpha}(Q \otimes_{B} Q_{\tau}(M))$. Applying this to $M = B$, it follows that any $\sigma$-closed $(\sigma, \tau)$-flat $A$-$B$-bimodule $Q$ is automatically $Q_{\alpha}(A) Q_{\tau}(B)$-bimodule.

Finally, recall that a $\sigma$-closed $A$-$B$-bimodule $Q$ is said to be $(\sigma, \tau)$-invertible, if it is $(\sigma, \tau)$-flat and if there exists another $\tau$-closed $(\tau, \sigma)$-flat
$B$-$A$-bimodule $P$ together with bimodule isomorphisms $\mu: Q^l_\tau(P \otimes_A Q) \cong Q^l_\tau(B)$ resp. $\nu: Q^l_\sigma(Q \otimes_B P) \cong Q^l_\sigma(A)$.

The right-handed version of these notions is defined similarly.

(2.2) As in [18], one defines a relative (left) Morita equivalence between $(A,\sigma)$ and $(B,\tau)$ to be a tuple

$$(P,Q, \mu: Q^l_\tau(P \otimes_A Q) \to Q^l_\tau(B), \nu: Q^l_\sigma(Q \otimes_B P) \to Q^l_\sigma(A)),$$

where $P$ and $Q$ are $(\tau,\sigma)$ resp. $(\sigma,\tau)$-invertible bimodules and where $\mu$ resp. $\nu$ are bimodule isomorphisms, all of these data being defined over $R$. Relative right Morita equivalences are defined similarly.

It has been proved in [18], that one may always modify the isomorphisms $\mu$ and $\nu$ in such a way that they satisfy for any $p, p' \in P$, $q, q' \in Q$, $a \in A$ and $b \in B$:

$$\mu(p \perp q)p' = p\nu(q \perp p') \text{ resp. } \nu(q \perp p)q' = q\mu(p \perp q'),$$

where, for example, $p \perp q$ denotes the image of $p \otimes q$ under the canonical map $P \otimes_A Q \to Q^l_\tau(P \otimes_A Q)$. So, without restriction of generality, we will assume this condition to be satisfied for any left resp. right Morita equivalence used in this text.

(2.3) With notations as in the previous paragraph, it has been proved in [18] that any relative Morita equivalence $(P,Q,\mu,\nu)$ determines an equivalence between the quotient categories $(A,\sigma)$-$\text{mod}$ an $(B,\tau)$-$\text{mod}$, given by the functor

$$F = Q^l_\tau(P \otimes_A -): (A,\sigma)$-$\text{mod}$ $\to$ $(B,\tau)$-$\text{mod}$$

and its adjoint

$$G = Q^l_\sigma(Q \otimes_B -): (B,\tau)$-$\text{mod}$ $\to$ $(A,\sigma)$-$\text{mod}$$.

Conversely, if $(F,G): (A,\sigma)$-$\text{mod}$ $\cong$ $(B,\tau)$-$\text{mod}$ is an equivalence of categories, then $(F,G)$ determines a relative Morita equivalence $(P,Q,\mu,\nu)$. Indeed, put $P = F(Q^l_\sigma(A))$ resp. $Q = G(Q^l_\tau(B))$ both endowed with their canonical bimodule structure. One may then show that (up to isomorphism) $F = Q^l_\tau(P \otimes_A -)$ resp. $G = Q^l_\sigma(Q \otimes_B -)$. We then let $\mu$ resp. $\nu$ denote the morphisms obtained by applying the natural isomorphisms $FG \cong \text{id}_{(B,\tau)$-$\text{mod}}$ resp. $GF \cong \text{id}_{(A,\sigma)$-$\text{mod}}$ to $Q^l_\tau(B)$ resp. $Q^l_\sigma(A)$. It may finally be shown that these data determine each other up to isomorphism.
(2.4) Let us define a relative hermitian Morita biequivalence between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) to consist of a relative left Morita equivalence

\[(P_l, Q_l, \nu_l : Q^l_\alpha(P_l \otimes_A Q_l) \to Q^l_\beta(B), \nu_r : Q^l_\alpha(Q_l \otimes_B P_l) \to Q^l_\beta(A))\]

between \((A, \sigma)\) and \((B, \tau)\) and a relative right Morita equivalence

\[(P_r, Q_r, \nu_r : Q^r_\beta(P_r \otimes_A Q_r) \to Q^r_\alpha(B), \nu_r : Q^r_\alpha(Q_r \otimes_B P_r) \to Q^r_\beta(A))\]

between \((A, \alpha(\sigma))\) and \((B, \beta(\tau))\), linked by a \(B-A\)-anti-isomorphism

\[\theta' : P_l \cong Q_r.\]

Of course, one then easily verifies that the map \(\theta'\) also induces a \(B-A\)-anti-isomorphism \(\theta'' : P_r \cong Q_l\).

Note also that \(\theta'\) induces bimodule isomorphisms

\[P_l \cong \beta^\alpha_A[P_r, Q^l_\alpha(A)]^\sigma\text{ resp. } Q_l \cong \beta^\alpha_B[Q_r, Q^l_\beta(B)]^\beta\]

and

\[P_r \cong \beta^\alpha_B[P_l, Q^l_\beta(B)]^\sigma\text{ resp. } Q_r \cong \beta^\alpha_A[Q_l, Q^l_\alpha(A)]^\beta.\]

(2.5) On the other hand, define a hermitian biequivalence between torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) to be a couple of equivalences

\[(F^l, G^l) : (A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}\]

resp.

\[(F^r, G^r) : \text{mod-}(A, \alpha(\sigma)) \approx \text{mod-}(B, \beta(\tau)),\]

linked through an isomorphism

\[\Theta' : F^l((-)^\beta) \cong F^r((-)^\sigma).\]

Since

\[G^lF^l \cong id_{(A, \sigma)\text{-mod}}\text{ resp. } G^rF^r \cong id_{\text{mod-}(A, \alpha(\sigma))},\]

it then also follows that there exists an isomorphism

\[\Theta'' : G^l((-)^\alpha) \cong G^r((-)^\beta).\]
Indeed, if \( N \in (B, \tau)\)-mod, say with \( N = F^iM \) for \( M \in (A, \sigma)\)-mod, then, up to natural isomorphism
\[
G^i(N)^{\sigma} = (G^iF^iM)^{\sigma} = M^{\sigma}
\]
\[
= G^rF^r(M^{\sigma}) = G^r((F^iM)^{\beta})
\]
\[
= G^r(N^{\beta}),
\]
which proves the assertion.

(2.6) Consider a relative hermitian Morita biequivalence
\[
((F^i, G^i), (F^r, G^r), \Theta') : F^i((-)^{\beta} \cong F^r((-)^{\alpha}))
\]
between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) and let \((C, \gamma)\) be a third algebra with involution. Let \( M \) be a \( \sigma \)-closed \( A-C \)-bimodule. Clearly, right multiplication by elements of \( C \) is \( \sigma \)-linear, hence this action canonically endows \( F^iM \) with a \( C-B \)-bimodule structure. From this, it easily follows that the isomorphism \( \Theta' \) extends to an isomorphism
\[
\Theta' : \gamma F^i((-)^{\beta} \cong F^r((-)^{\alpha}),
\]
when restricted to \( A-C \)-bimodules in \((A, \sigma)\)-mod.

(2.7) Lemma. Let \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) be torsion triples, let \( P \) be a \( B-A \)-bimodule and \( Q \) an \( A-B \)-bimodule and let \( \theta : P \to Q \) be an \( A-B \)-anti-isomorphism. Then for any left \( A \)-module \( M \), there is a natural isomorphism
\[
Q^l_r(P \otimes_A M)^{\beta} \cong Q^r_{\beta(\tau)}(M^{\alpha} \otimes_A Q)
\]
of right \( B \)-modules.

Proof: First note that there is a natural isomorphism
\[
\psi_M : (P \otimes_A M)^{\beta} \cong M^{\alpha} \otimes_A Q : p \otimes m \mapsto m \otimes \theta(p)
\]
of right \( B \)-modules. So,
\[
Q^l_r(P \otimes_A M)^{\beta} = Q^l_r((P \otimes_A M)^{\beta}) = Q^r_{\beta(\tau)}((P \otimes_A M)^{\beta})
\]
\[
\overset{(*)}{\cong} Q^r_{\beta(\tau)}(M^{\alpha} \otimes_A Q),
\]
where the isomorphism \((*)\) is just \( Q^r_{\beta(\tau)}(\psi_M) \). This proves the assertion.

We may now prove:
Theorem. Let \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) be torsion triples. Then there is a bijective correspondence between

(2.8.1) \((\text{isomorphism classes of}) \) relative hermitian Morita biequivalences between \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\);

(2.8.2) \((\text{isomorphism classes of}) \) hermitian biequivalences between \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\).

Proof: First, let us consider a relative hermitian Morita biequivalence between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\), determined by the relative Morita equivalences

\[
(P_l, Q_l, \mu_l : Q^l_l(P_l \otimes_A Q_l) \rightarrow Q^l_l(B), \nu_l : Q^l_l(Q_l \otimes_B P_l) \rightarrow Q^l_l(A))
\]
resp.

\[
(P_r, Q_r, \mu_r : Q^r_r(P_r \otimes_A Q_r) \rightarrow Q^r_r(B), \nu_r : Q^r_r(Q_r \otimes_B P_r) \rightarrow Q^r_r(A))
\]
and the \(B\)-\(A\)-anti-isomorphism \(\theta' : P_l \sim Q_r\). Then we may associate to it a hermitian biequivalence

\[
((F^l, G^l), (F^r, G^r), \Theta' : F^l(-)^\beta \sim F^r((-)^\alpha))
\]
between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\), with

\[
F^l = Q^l_l(P_l \otimes_A -), \quad G^l = Q^l_l(Q_l \otimes_B -), \quad F^r = Q^r_r(Q_r \otimes_A P_r) \quad \text{resp.}
\]
\[
G^r = Q^r_r(- \otimes_B Q_r),
\]
and with

\[
\Theta' : F^l(-)^\beta = Q^l_l(P_l \otimes_A -)^\beta \sim Q^r_r((-)^\alpha \otimes_A Q_r) = F^r((-)^\alpha)
\]
determined by the \(B\)-\(A\)-anti-isomorphism \(\theta' : P_l \sim Q_r\), using (2.7).

Conversely, consider a hermitian biequivalence between the torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) given by the equivalences

\[
(F^l, G^l) : (A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}
\]
and

\[
(F^r, G^r) : \text{mod-}(A, \alpha(\sigma)) \approx \text{mod-}(B, \beta(\tau)),
\]
linked through the isomorphism

\[
\Theta' : F^l(-)^\beta \sim F^r((-)^\alpha).
\]

By the remarks made in (2.3), we then obtain a relative left Morita equivalence

\[
(P_l = F^l(Q^l_l(A)), Q_l = G^l(Q^l_l(B)), \mu_l, \nu_l),
\]
where \( \mu_t \) and \( \nu_t \) are induced by the isomorphisms \( F^t G^t \cong \text{id}(B, r) \)-mod resp. \( G^t F^t \cong \text{id}(A, \sigma) \)-mod, and a relative right Morita equivalence

\[
(P_r = G^r(Q_{\beta(\tau)}(B)), Q_r = F^r(Q_{\alpha(\sigma)}(A)), \mu_r, \nu_r),
\]

where \( \mu_r \) and \( \nu_r \) are defined similarly.

Finally, if we restrict to \( A \)-bimodules, then the remarks made in (2.6) show that the isomorphism \( \Theta' \) induces an isomorphism

\[
\alpha F^t(-) \beta \cong F^r((-)^{\alpha}) = F^r((-)^{\alpha}),
\]

so, in particular, this yields an isomorphism

\[
\alpha F^t(Q^t_\alpha(A)) \beta \cong F^r(Q^r_\alpha(A)).
\]

With \( Q_r = F^r(Q^r_{\alpha(\sigma)}(A)) \in \text{mod}-(B, \beta(\tau)) \), we clearly obtain a second isomorphism

\[
F^r(Q^r_\alpha(A)) = F^r(Q^r_{\alpha(\sigma)}(A)) = Q^r_{\beta(\tau)}(Q^r_{\alpha(\sigma)}(A) \otimes_A Q_r)
\]

\[
= Q^r_{\beta(\tau)}(A \otimes_A Q_r) \cong Q^r_{\beta(\tau)}(Q_r) = Q_r,
\]

where the last isomorphism of \( A-B \)-bimodules is induced by

\[
A \otimes_A Q_r \rightarrow Q_r : a \otimes q \mapsto a(a)q.
\]

Composing the previous isomorphisms (with \( P_t = F^t Q^t_\alpha(A) \)), we then obtain an isomorphism of \( A-B \)-bimodules \( \alpha P_t \beta \cong Q_t \), i.e., a \( B-A \)-anti-isomorphism \( \theta' : P_t \rightarrow Q_t \) defined over \( R \).

It is now easy to verify that the isomorphism

\[
F^t(-) \beta = Q^t_{\tau}(P_t \otimes_A -) \beta \cong Q^r_{\beta(\tau)}((-)^{\alpha} \otimes_A Q_r) = F^r((-)^{\alpha})
\]

is actually induced by \( \theta' : P_t \rightarrow Q_t \), exactly as in (2.7), proving that the previous constructions are mutually inverse to each other. \( \blacksquare \)

(2.9) Note. With notations as before, let us call a relative hermitian Morita biequivalence involutive, if \( P_t = P_r \) (up to isomorphism). The anti-isomorphisms \( \theta' : P_t \rightarrow Q_r \) and \( \theta'' : P_r \rightarrow Q_t \) may then be used to show that \( Q_t = Q_r \) as well. But then, it follows that \( \theta' \) and \( \theta'' \) induce anti-isomorphisms \( \theta' : P_t \rightarrow Q_t \) resp. \( \theta'' : P_r \rightarrow Q_r \). In particular, it follows that the relative Morita equivalences determined by \( (P_t, Q_t) \) resp. \( (P_r, Q_r) \) are actually relative hermitian Morita equivalences, in the sense of [12]. The theory developed in [12] then shows that the involution \( \alpha \) resp. \( \beta \) on \( A \) resp. \( B \) uniquely extends to an involution \( \hat{\alpha} \) resp. \( \hat{\beta} \) on \( Q^t_\alpha(A) = Q^r_{\alpha(\sigma)}(A) \) resp. \( Q^r_\beta(B) = Q^r_{\beta(\tau)}(B) \). Moreover, \( P_t = P_r \) resp. \( Q_t = Q_r \) is then canonically endowed with a \( Q^t_\tau(B)-Q^r_\tau(A) \)-bimodule structure.
3. Hermitian Morita contexts

(3.1) For any pair of rings \((A, B)\), we define a Morita context between \(A\) and \(B\) to be a tuple

\[ \mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A), \]

where \(P\) is a \(B\)-\(A\)-bimodule, \(Q\) an \(A\)-\(B\)-bimodule and where \(\mu\) resp. \(\nu\) are morphisms of \(B\)- resp. \(A\)-bimodules, satisfying for any \(p, p' \in P\), \(q, q' \in Q\),

\[ \mu(p \otimes q)p' = p\nu(q \otimes p') \text{ resp. } \nu(q \otimes p)q' = q\mu(p \otimes q'). \]

In other words, we want the following diagrams to commute:

\[
\begin{array}{ccc}
P \otimes A A & \xrightarrow{\mu \otimes P} & B \otimes B P \\
\downarrow & & \downarrow \\
Q \otimes A A & \xrightarrow{\nu \otimes Q} & A \otimes A Q \\
\downarrow & & \downarrow \\
Q \otimes B B & \xrightarrow{\nu \otimes Q} & B \otimes B P
\end{array}
\]

The images of \(\nu\) resp. \(\mu\) are two-sided ideals of \(A\) resp. \(B\) and will be denoted by \(J_A\) resp. \(J_B\); they are called the trace ideals of the Morita context \(\mathcal{M}\).

(3.2) Let us now assume \((A, \alpha)\) and \((B, \theta)\) to be \(R\)-algebras with involution. We then define a hermitian Morita context between \((A, \alpha)\) and \((B, \beta)\) to be a tuple

\[ \mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A) \]

where \(P\) is a \(B\)-\(A\)-bimodule, \(Q\) an \(A\)-\(B\)-bimodule (both defined over \(R\)) \(\theta\) an additive bijection and where \(\mu\) resp. \(\nu\) are morphisms of \(B\)- resp. \(A\)-bimodules.

Moreover, these data should satisfy the following requirements for any \(p, p' \in P\), \(q, q' \in Q\), \(a \in A\) and \(b \in B\):

(3.2.1) \(\theta(bpa) = \alpha(\alpha)\theta(p)\beta(b)\);

(3.2.2) \(\mu(p \otimes q)p' = p\nu(q \otimes p') \text{ resp. } \nu(q \otimes p)q' = q\mu(p \otimes q')\);

(3.2.3) \(\mu(p \otimes \theta(p')) = \beta(\mu(p \otimes \theta(p))) \text{ resp. } \nu(\theta(p) \otimes p') = \alpha(\nu(\theta(p') \otimes p))\).
(Of course, the first condition just says that the bijection $\theta : P \to Q$ is a $B$-$A$-anti-isomorphism).

**3.3 Examples.** It is easy to see that a hermitian Morita equivalence in the sense of [3], [9] determines a hermitian Morita context, in the obvious way.

More generally, recall that a relative hermitian Morita equivalence between torsion triples $(A, \alpha, \sigma)$ and $(B, \beta, \tau)$ is defined in [12] to be a tuple $\mathcal{H}_0$ of the form

$$(P, Q, \theta : P \to Q, \mu : Q^t_\sigma(P \otimes_A Q) \to Q^t_\tau(B), \nu : Q^t_\sigma(Q \otimes_B P) \to Q^t_\sigma(A)),$$

where $P$ resp. $Q$ is a $(\tau, \sigma)$-invertible $B$-$A$-bimodule resp. a $(\sigma, \tau)$-invertible $A$-$B$-bimodule defined over $R$, where $\theta$ is a $B$-$A$-anti-isomorphism and where $\mu$ resp. $\nu$ is an isomorphism of $B$- resp. $A$-bimodules, these data satisfying relations as in (3.2), but replacing tensor products with relative tensor products, i.e., considering for example $Q^t_\sigma(P \otimes_A Q)$ instead of $P \otimes_A Q$. It may be shown that the existence of a relative hermitian Morita equivalence between $(A, \alpha, \sigma)$ and $(B, \beta, \tau)$ implies the involutions $\alpha$ resp. $\beta$ to extend to involutions $\bar{\alpha}$ resp. $\bar{\beta}$ on $Q^t_\sigma(A)$ resp. $Q^t_\tau(B)$, cf. [12, (3.7)]. Writing $j_{P \otimes Q} : P \otimes_A Q \to Q^t_\tau(P \otimes_A Q)$ resp. $j_{Q \otimes P} : Q \otimes_B P \to Q^t_\sigma(Q \otimes_B P)$ for the canonical morphisms, it then follows that $\mathcal{H}_0$ induces a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q, \bar{\mu} : P \otimes_A Q \to Q^t_\tau(B), \bar{\nu} : Q \otimes_B P \to Q^t_\sigma(A))$$

between $(Q^t_\sigma(A), \bar{\alpha})$ and $(Q^t_\tau(B), \bar{\beta})$, with $\bar{\mu} = \mu \circ j_{P \otimes Q}$ resp. $\bar{\nu} = \nu \circ j_{Q \otimes P}$.

**3.4** Consider a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A)$$

between the $R$-algebras with involution $(A, \alpha)$ and $(B, \beta)$ and let $J_A$ and $J_B$ be the trace ideals of the associated Morita context

$$\mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A).$$

We will then also call $J_A$ and $J_B$ the trace ideals of the hermitian Morita context $\mathcal{H}$. It is then easy to see that $\alpha(J_A) = J_A$ and $\beta(J_B) = J_B$. Indeed, if $j \in J_A$, then $j = \nu(\sum_i q_i \otimes p_i)$, for some $p_i \in P$ and some $q_i \in Q$. For any index $i$, pick $p'_i \in P$ with $q_i = \theta(p'_i)$, then

$$\alpha(j) = \alpha \left( \nu \left( \sum_i q_i \otimes p_i \right) \right) = \sum_i \alpha(\nu(\theta(p'_i) \otimes p_i)) = \sum_i \nu(\theta(p_i) \otimes p'_i) \in J_A.$$
So, $\alpha(J_A) \subseteq J_A$, whence equality as $J_A = \alpha(\alpha(J_A)) \subseteq \alpha(J_A)$ as well. A similar argument shows that $\beta(J_B) = J_B$.

(3.5) For any twosided ideal $I$ of a ring $A$ and any left resp. right $A$-module $M$, let us denote by $\text{Ann}^I_M(I)$ resp. $\text{Ann}^r_M(I)$ the set of all $m \in M$ with the property that $Im = 0$ resp. $mI = 0$. Then, as in [15], for any twosided ideal $I$ of a ring $A$, the set of all $M \in A$-mod with the property that $\text{Ann}^I_M(I) = 0$ is the torsionfree class of an idempotent kernel functor in $A$-mod, denoted by $\sigma^I_I$. The corresponding Gabriel filter $\mathcal{L}(\sigma^I_I)$ consists of all left ideals $L$ of $A$ with the property that $\text{Ann}^I_M(I) = 0$ implies that $\text{Ann}^I_M(L) = 0$ for all $M \in A$-mod. The quotient category $(A, \sigma^I_I)$-mod of $A$-mod associated to $\sigma^I_I$ consists of all $M \in A$-mod with the property that the canonical map $M \to \alpha[A, M] \to A[I, M]$ is bijective.

Note also that if $A$ is left noetherian, then $\sigma^I_I$ is just the usual symmetric idempotent kernel functor $\sigma_I$ in $A$-mod associated to $I$, with Gabriel filter generated by the positive powers $I^n$ of $I$. Indeed, clearly $\sigma_I \geq \sigma^I_I$. On the other hand, if $\sigma^I_I M = 0$ and if $m \in \sigma_I M$, then $I^nm = 0$ for some positive integer $n$. Since $\text{Ann}^I_M(I) = 0$, it follows that $m = 0$, hence $\sigma_I M = 0$ as well. So, $\sigma^I_I = \sigma_I$, indeed.

The idempotent kernel functor $\sigma^I_I$ associated to $I$ in $\text{mod-}A$ is defined similarly, using $\text{Ann}^I_M(I)$ instead of $\text{Ann}^I_M(I)$.

(3.6) Lemma. Let $(A, \alpha)$ be an $R$-algebra with involution and let $I$ be a twosided ideal of $A$, then $\alpha(\sigma^I_I) = \sigma^\alpha(I)$.

Proof: The idempotent kernel functor $\alpha(\sigma^I_I)$ is defined through its Gabriel filter $\mathcal{L}(\alpha(\sigma^I_I))$, which consists of all right $A$-ideals $\alpha(L)$ with $L \in \mathcal{L}(\sigma^I_I)$. So, let $L \in \mathcal{L}(\sigma^I_I)$ and let $N \in \text{mod-}A$ be such that $\text{Ann}^\alpha_N(\alpha(L)) = 0$. If $M = \alpha N$, then $\text{Ann}^I_M(I) = 0$, as one easily checks. So, by definition, $\text{Ann}^I_M(L) = 0$ as well. But then, it also follows that $\text{Ann}^\alpha_N(\alpha(L)) = 0$, so $\alpha(L) \in \mathcal{L}(\sigma^\alpha(I))$. We thus find that $\alpha(\sigma^I_I) \leq \sigma^\alpha_I$, hence $\alpha(\sigma^I_I) = \sigma^\alpha_I$, by symmetry.

(3.7) To the trace ideals $J_A$ and $J_B$ of a hermitian Morita context

$$H = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A),$$

we may associate, by the foregoing, idempotent kernel functors $\sigma^I_A = \sigma^I_{J_A}$ resp. $\sigma^r_B = \sigma^r_{J_B}$ in $A$-mod resp. $B$-mod and $\sigma^r_A = \sigma^r_{J_A}$ resp. $\sigma^r_B = \sigma^r_{J_B}$ in $\text{mod-}A$ resp. $\text{mod-}B$. Moreover, as $\alpha(J_A) = J_A$ and $\beta(J_B) = J_B$, it follows that $\alpha(\sigma^I_A) = \sigma^\alpha_A$ and $\beta(\sigma^r_B) = \sigma^r_B$. The associated localization functors will be denoted by $Q^I_A$ resp. $Q^r_B$ and $Q^r_A$ resp. $Q^r_B$. 

Recall from [15], [18] that any Morita context

\[ \mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A) \]

induces an equivalence between the quotient categories \((A, \sigma_A^l)\)-mod and \((B, \sigma_B^l)\)-mod through the functor

\[ F^l = A[Q, -] : (A, \sigma_A^l)\text{-mod} \to (B, \sigma_B^l)\text{-mod} \]

and its adjoint

\[ G^l = B[P, -] : (B, \sigma_B^l)\text{-mod} \to (A, \sigma_A^l)\text{-mod}. \]

From [15], we also retain that, up to canonical isomorphism, we have

\[ B[P, Q_B^l(B)] = Q_A^l(Q) \text{ resp. } A[Q, Q_A^l(A)] = Q_B^l(P). \]

Indeed, for any \( M \in (A, \sigma_A^l)\)-mod, we have


so \( B[P, Q_B^l(B)] = Q_A^l(Q) \). The other isomorphism may be derived in a similar way.

In view of the general theory developed in [18], it then follows that

\[ F^l = A[Q, -] = Q_B^l(Q_B^l(P) \otimes_A -) = Q_B^l(P \otimes_A -) \]

and similarly

\[ G^l = B[P, -] = Q_A^l(Q_A^l(Q) \otimes_B -) = Q_A^l(Q \otimes_B -). \]

It also follows that the maps \( \mu \) and \( \nu \) extend to bimodule isomorphisms

\[ \mu_l : Q_B^l(P \otimes_A Q) \to Q_B^l(B), \]

and

\[ \nu_l : Q_A^l(Q \otimes_B P) \to Q_A^l(A). \]

In particular, we thus obtain that the Morita context \( \mathcal{M} \) induces a relative left Morita equivalence

\[ (Q_B^l(P), Q_A^l(Q), \mu_l : Q_B^l(P \otimes_A Q) \to Q_B^l(B), \nu_l : Q_A^l(Q \otimes_B P) \to Q_A^l(A)) \]
between \((A, \sigma)\) and \((B, \tau)\), which we will usually denote by \(M^t\).

(3.9) Of course, in a completely similar way, one may derive that \(M\) induces an equivalence between the categories \(\text{mod-}(A, \sigma^*_A)\) and \(\text{mod-}(B, \sigma^*_B)\) through the functor

\[
F^r = [P, -]_A : \text{mod-}(A, \sigma^*_A) \to \text{mod-}(B, \sigma^*_B)
\]

and its adjoint

\[
G^r = [Q, -]_B : \text{mod-}(B, \sigma^*_B) \to \text{mod-}(A, \sigma^*_A).
\]

Moreover, we then also have that

\[
F^r = [P, -]_A = Q_B^r(- \otimes_A Q_B^r(Q)) = Q_B^r(- \otimes_A Q)
\]

and that

\[
G^r = [Q, -]_B = Q_A^r(- \otimes_B Q_A^r(P)) = Q_A^r(- \otimes_B P).
\]

Again, we thus obtain a relative right Morita equivalence \(M^r\) given by

\[
(Q_B^r(P), Q_A^r(Q), \mu_r : Q_B^r(P \otimes_A Q) \to Q_B^r(B), \nu_r : Q_A^r(Q \otimes_B P) \to Q_A^r(A))
\]

The previous discussion then implies:

(3.10) Theorem. Any hermitian Morita context

\[
\mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A)
\]

between the \(R\)-algebras with involution \((A, \alpha)\) and \((B, \beta)\) induces a relative hermitian biequivalence, given by

\[
(Q_B^l(P \otimes_A -), Q_A^l(Q \otimes_B -)) : (A, \sigma^l_A)\text{-mod} \cong (B, \sigma^l_B)\text{-mod}
\]

and

\[
(Q_B^r(- \otimes_A Q), Q_A^r(- \otimes_B P)) : \text{mod-}(A, \sigma^*_A) \cong \text{mod-}(B, \sigma^*_B),
\]

and where the isomorphism

\[
\Theta^l : Q_B^l(P \otimes_A -) \cong Q_B^r((-)^{\alpha} \otimes_A Q)^{\beta}
\]

is induced by \(\theta\), as in (2.7).

(3.11) Note. Starting from the hermitian Morita context

\[
\mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A),
\]
we may also construct directly from it a relative hermitian Morita biequivalence (equivalent to the hermitian biequivalence determined by the previous theorem), given by

(3.11.1) the relative left Morita equivalence

\[ M^l = (P, Q, \mu : Q_B^l (P \otimes_A Q) \to Q_B^l B, \nu : Q_A^l (Q_B \otimes_B P) \to Q_A^l A) , \]

where \( P_l = Q_B^l (P) \), \( Q_l = Q_A^l (Q) \) and where \( \mu_l \) is the morphism

\[ Q_B^l (\mu) : Q_B^l (P_l \otimes_A Q_l) \cong Q_B^l (Q_B^l (P_l) \otimes_A Q_A^l (Q)) = Q_B^l (P \otimes_A Q) \to Q_B^l (B) \]

and where, similarly, \( \nu_l \) is the morphism

\[ Q_A^l (\nu) : Q_A^l (Q_l \otimes_B P_l) \cong Q_A^l (Q_A^l (Q_l) \otimes_B Q_B^l (P)) = Q_A^l (Q \otimes_B P) \to Q_A^l (A) \]

and where the isomorphisms (*) hold by the relative flatness of \( P \) and \( Q \) with respect to \( \sigma_A^l \) and \( \sigma_B^l \);

(3.11.2) the relative right Morita equivalence \( M^r \) given by

\[ (P_r, Q_r, \mu_r : Q_B^r (P_r \otimes_A Q_r) \to Q_B^r B, \nu_r : Q_A^r (Q_r \otimes_B P_r) \to Q_A^r A) , \]

defined in a similar way;

(3.11.3) the \( B-A \)-anti-isomorphism \( \theta' : P_l \to Q_r \), obtained by applying the following Lemma to the \( B-A \)-anti-isomorphism \( \theta : P \to Q \):

(3.12) Lemma. Let \( (A, \alpha) \) resp. \( (B, \beta) \) be \( R \)-algebras with involution and be \( \sigma \) be an idempotent kernel functor in \( \text{mod-} A \). Denote by \( \alpha(\sigma) \) the induced idempotent kernel functor in \( \text{mod-} A \). Let \( M \) be an \( A-B \)-bimodule, \( N \) a \( B-A \)-bimodule and let \( \theta : M \to N \) be an \( A-B \)-anti-isomorphism. Then \( \theta \) induces an \( A-B \)-anti-isomorphism \( \theta' : Q_A^\sigma (M) \to Q_A^{\alpha(\sigma)} (N) \).

Proof. The morphism \( \theta \) induces an \( A-B \)-linear morphism \( \hat{\theta} : M \to \alpha N^\beta \), which localizes to an \( A-B \)-linear morphism \( Q_A^\sigma (\hat{\theta}) : Q_A^\sigma (M) \to Q_A^{\alpha(\sigma)} (N) \). From [(2.6.), 12], it follows that

\[ Q_A^\sigma (\alpha N^\beta) \cong \alpha Q_A^{\alpha(\sigma)} (N)^\beta \]

in \( \text{B-mod-} A \), so \( Q_A^\sigma (\hat{\theta}) \) may be viewed as a morphism \( \theta' : Q_A^\sigma (M) \to Q_A^{\alpha(\sigma)} (N) \), satisfying the above requirements.

(3.13) Consider a hermitian Morita context

\[ \mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to A, \nu : Q \otimes_B P \to A) . \]
between the algebras with involution \((A, \alpha)\) and \((B, \beta)\). We say that \(\mathcal{H}\) is right normalized if the underlying Morita context

\[
(P, Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A)
\]

is right normalized, in the sense of [15], i.e., that the natural maps \(A \to [Q, B]_B, Q \to [P, A]_A, A \to [Q, Q]_B\) and \(B \to [P, P]_A\) are isomorphisms. (Recall that any Morita context canonically induces a right normalized context, cf. [15, (2.4.)].) From [15, Theorem 7], it then follows that the natural maps \(A \to [J_A, A]_A\) and \(P \to [J_P, P]_A\) are bijective, i.e., that \(A\) and \(P\) are \(\sigma_A\)-closed. So, \(A = Q^r_A(A)\) and \(P = P_P\).

On the other hand, since the equivalence \(\text{mod-}(A, \sigma_A^+) \cong \text{mod-}(B, \sigma_B^+)\) is given by \(F^r = [P, -]_A\) and its adjoint \(G^r = [Q, -]_B\), it follows that \(B \cong [P, P]_A = F^r(P)\) is \(\sigma_A^B\)-closed. From the remarks made in (1.4), it thus follows that \(A\) resp. \(B\) is \(\sigma_A^B\) resp. \(\sigma_B^B\)-closed, as well.

Next, it follows from the existence of the \(B\)-\(A\)-anti-isomorphism \(\theta : P \to Q\) and the induced \(\theta^\sigma : P, Q\to Q, P\), that \(Q \cong Q_1\) as \(A\)-\(B\)-bimodules. But then, up to isomorphism of \(A\)-\(B\)-bimodules, we also obtain that

\[
Q_r = Q^*_B(Q) = Q^*_B(\alpha' P^\beta) = \alpha Q^i_B(P)^\beta = \alpha P^\beta = Q,
\]

and similarly that \(P_l = P\).

We thus have proved that the hermitian Morita biequivalence induced by \(\mathcal{H}\) is involutive in the sense of (2.9), hence:

(3.14) Proposition. Every right normalized hermitian Morita context

\[
\mathcal{H} = (P, Q, \theta : P \to Q, \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A)
\]

between the algebras with involution \((A, \alpha)\) and \((B, \beta)\) induces a relative left hermitian Morita equivalence

\[
\mathcal{H}_l^i = (P, Q, \theta : P \to Q, \mu_l : Q^i_B(P \otimes_A Q) \to B, \nu_r : Q^i_A(Q \otimes_B P) \to A)
\]

between the left torsion triples \((A, \alpha, \sigma)\) and \((B, \beta, \tau)\) and a relative right hermitian Morita equivalence

\[
\mathcal{H}^r = (P, Q, \theta : P \to Q, \mu_r : Q^*_B(P \otimes_A Q) \to B, \nu_r : Q^*_A(Q \otimes_B P) \to A)
\]

between the right torsion triples \((A, \alpha, \alpha(\sigma))\) and \((B, \beta, \beta(\tau))\).
References


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