Publicacions Matemàtiques, Vol 36 (1992), 1001–1010.

BILINEAR FORMS FOR $SL(2,q)$, $\tilde{A}_n$ AND SIMILAR GROUPS

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Article dedicat a la memòria del bon amic Pere Menal

Abstract

The set of invariant symmetric bilinear forms on irreducible modules over fields of characteristic zero for certain groups is studied. Results are obtained under the presence in a finite group of elements of order four whose square is central. In particular, we find that the relevant modules for the groups mentioned in the title always accept an invariant symmetric bilinear form under which the module admits an orthonormal basis.

Introduction

Let G be a finite group and χ some complex irreducible character of G with real values. If F is any real number field containing $\mathbb{Q}(\chi)$, (here $\mathbb{Q}(\chi)$ is $\mathbb{Q}$ extended by all the values of χ), then there is a unique (up to isomorphism) FG-module M which affords the character $m_F(\chi)\chi$, where $m_F(\chi)$ is the Schur index of χ with respect to F. A basic problem in representation theory of finite groups is to describe these modules. Since F is a real field, the standard averaging argument shows that M will afford some (positive definite) symmetric G-invariant bilinear form f. What can be said about f?

Symmetric bilinear forms f of M are classified up to isomorphism in $GL(M)$ by the signature of f under each embedding of F into $\mathbb{R}$, the determinant of f (defined up to squares in F^*) and the Hasse invariant of f, see for example Corollary 3.3 in p. 168 of [1]. For every $\lambda \in F^*$, λf will also be a non-degenerate G-invariant symmetric bilinear form on M and its determinant will be $\det(\lambda f) = \lambda^{\dim_F(M)} \det(f)$ up to squares in F^*. It follows that the problem will be more tractable if $\dim_F(M)$ is

*Partially supported by a grant from the NSA.

even, for then, at least f and λf will then have the same determinant. There are various conditions that force $\dim_F(M)$ to be even. If the Schur index of χ with respect to F is not one, for example, then $\dim_F(M)$ is even. This case is analyzed in **[3]**. Another example is that if M is faithful and $G' \cap Z(G)$ is of even order then $\chi(1)$ must be even. The present paper analyses situations that occur frequently in this case, and in particular our results yield the answer for $G = SL(2,q)$, $\tilde{A}_n$ or $\tilde{S}_n$ and χ a faithful character of G.

Before stating our results, we describe our notation. We denote by $Br(F)$ the Brauer group of F. If $a, b \in F^*$ we denote by (a,b) the element of $Br(F)$ which has as a representative the quaternion algebra of dimension 4 over F generated by $\vec{i}$ and $\vec{j}$ satisfying $\vec{i}^{\,2} = a$, $\vec{j}^{\,2} = b$ and $\vec{i}\,\vec{j} = -\vec{j}\,\vec{i}$. If f is a symmetric non-singular form on M we denote by Hasse(f) its Hasse invariant. Hasse(f) is an element of $Br(F)$ and is calculated as follows. Let $e_1, \ldots, e_n$ be an orthogonal basis for M and set $a_i = f(e_i, e_i)$. Then

$$\text{Hasse}(f) = \prod_{i<j} (a_i, a_j).$$

Here the product is in the Brauer group of F and Hasse(f) does not depend on the orthogonal basis chosen.

Theorem A. *Let G be a finite group and $x \in G$ be an element of order 4 such that $x^2 \in Z(G)$. Let F be a real field and χ an irreducible character of G with values in F. Let M be an FG-module affording $m_F(\chi)\chi$ and assume that x^2 acts non-trivially on M. Let f be a G-invariant non-zero symmetric bilinear form on M. Then the following hold.*

1) $\det(f) = 1$ *up to squares in F^*.*
2) Hasse(f) $= (\lambda_o, -1)$ *for some $\lambda_o \in F^*$, with $\lambda_o > 0$ if f is positive definite.*

Furthermore, if $\dim_F(M) \equiv 2$ (mod 4), then for every $\lambda \in F^*$ there is some $\mu \in F^*$ with Hasse(μf) $= (\lambda, -1)$.

Our theorem has a consequence about local Schur indices which we now proceed to describe. Recall that, if χ is an irreducible character, the local Schur indices of χ are the positive integers $m_\infty(\chi)$ and $m_p(\chi)$ (for p a rational prime), where $m_\infty(\chi) = m_{\mathbb{R}}(\chi)$ and $m_p(\chi) = m_{\mathbb{Q}_p}(\chi)$ ($\mathbb{Q}_p$ being the field of p-adic numbers). The Frobenius-Schur indicator gives a straightforward (and well known) formula for $m_\infty(\chi)$. Namely, if χ has real values, $m_\infty(\chi) = 1$ if $\dfrac{1}{|G|} \sum_{g \in G} \chi(g^2) = 1$, and $m_\infty(\chi) = 2$

and $\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = -1$, otherwise. There is no known similar formula for $m_p(\chi)$. Furthermore, knowing $m_\infty(\chi)$ provides in general very little information about $m_p(\chi)$. For example, for every finite subset S of $\{\infty, 2, 3, 5, 7, 11, \ldots\}$ of even cardinality there is a rational valued irreducible character χ of a double cover of some alternating group such that $m_p(\chi) = 2$ for $p \in S$ and $m_p(\chi) = 1$ for $p \notin S$, see **[2]**. However, some further conditions on χ and G do imply some further relationship between $m_\infty(\chi)$ and the $m_p(\chi)$ for p a rational prime. If F is a field containing the values of χ we denote by $[\chi]$ the element of $Br(F)$ represented by $End_{FG}(M)$ where M is an irreducible FG-module affording the character $m_F(\chi)\chi$.

Corollary 1. *Let G be a finite group and $x \in G$ be an element of order 4 such that $x^2 \in Z(G)$. Let χ be an irreducible character of G which does not contain x^2 in its kernel and such that $\chi(1) \equiv 2$ (mod 4) and $m_\infty(\chi) = 2$. Let F be a real field that contains $\mathbb{Q}(\chi)$. Then*

$$[\chi] = (\lambda_o, -1) \quad \textit{in} \quad Br(F) \quad \textit{for some negative} \quad \lambda_o \in F^*.$$

In particular, if χ is rational valued then

$$m_p(\chi) = 1 \quad \textit{for every prime} \quad p \equiv 1 (mod\ 4).$$

Although Theorem A fully describes Hasse(f) when $\dim_F(M) \equiv 2$ (mod 4), if $\dim_F(M) \equiv 0$ (mod 4) then, as we shall see, Hasse$(\mu f) =$ Hasse(f) for all $\mu \in F^*$. Hence, Hasse(f) will be determined by G and χ at least if $\chi(1) \equiv 0$ (mod 4) and M is absolutely irreducible. However, the conclusion of Theorem A can not be made more precise even in this case, as our next result shows.

Theorem B. *Given $\lambda_o \in \mathbb{Q}^*$, $\lambda_o > 0$, then there exist $G, x, F = \mathbb{Q}, \chi, M$ and f satisfying the hypotheses of Theorem A with the following properties:*

a) *$m_{\mathbb{Q}}(\chi) = 1$ and $\chi(1) \equiv 0$ (mod 4).*
b) Hasse$(f) = (\lambda_o, -1)$ *in $Br(\mathbb{Q})$.*
c) *Every G-invariant symmetric non-singular bilinear form g on M satisfies* Hasse$(g) = (\lambda_o, -1)$ *in $Br(\mathbb{Q})$.*

If in addition to assuming that G has a certain element of order 4 we assume further that G contains a certain copy of the quaternion group of order 8, then all Hasse invariants are trivial.

Theorem C. *Let G be a finite group and let Q be a subgroup of G isomorphic to the quaternion group of order 8 and such that $Z(Q) \subseteq Z(G)$. Let F be a real field and χ be an irreducible character of G with values in F. Let M be an FG-module affording $m_F(\chi)\chi$ and assume that $Z(Q)$ acts non-trivially on M. Let f be a G-invariant non-zero symmetric bilinear form on M. Then the following hold.*

1) $\det(f) = 1$ *up to squares in* F^*.
2) $\mathrm{Hasse}(f) = 1$ *in* $Br(F)$.

This theorem has a consequence that is analogous to Corollary 1, namely that with the hypothesis of Theorem C if $m_\infty(\chi) = 2$ and $\chi(1) \equiv 2 \pmod 4$ then $m_p(\chi) = 1$ for all odd primes p. However, this can also be proved easily by noting that $m_p(\chi)|2$ and the irreducible faithful character of Q has odd multiplicity in $\chi_{|Q}$.

More importantly, it should be noted that Theorem C applies for faithful χ, whenever $G \simeq SL(2,q)$, the special linear group of dimension 2 over the field of q elements, $G \simeq \tilde{A}_n$ $(n \geq 4)$, the double cover of an alternating group, or $G \simeq \tilde{S}_n$ $(n \geq 4)$ some double cover of a symmetric group. In particular, we have the following.

Corollary 2. *Let G be isomorphic to $SL(2,q)$ for q odd, or some double cover of S_n or A_n for $n \geq 4$. Let M be an FG-irreducible faithful G-module, where F is a real field and the character of M is a multiple of some irreducible character of G.*

Then there is a symmetric G-invariant bilinear form on M under which M has an orthonormal basis.

1. Preliminary Lemmas

In this section we review some results that we need for our proofs. Unless otherwise stated our vector spaces and forms are over a fixed but arbitrary real number field F. Recall that a hyperbolic plane is a two dimensional vector space with a non-singular symmetric bilinear form that has a non-zero vector whose product with itself is zero.

Lemma 1.1. *Let V and W be vector spaces and let f and g be non-singular symmetric bilinear forms on V and W respectively. Then the following hold.*

a) $\mathrm{Hasse}(f \perp g) = \mathrm{Hasse}(f)\,\mathrm{Hasse}(g)(\det(f), \det(g))$, *where $f \perp g$ is the orthogonal sum of f and g.*
b) *For every $\lambda \in F$,* $\mathrm{Hasse}(\lambda f) = \mathrm{Hasse}(f)(\lambda, (-1)^{\frac{n(n-1)}{2}} d^{n-1})$ *where $n = \dim_F(M)$ and $d = \det(f)$.*

c) *If* W *is a hyperbolic plane then* $\det(g) = -1$ *and* $\mathrm{Hasse}(g) = 1$.

Proof: These facts are easily verified. They can also be found in [**1**], for example b) appears as Exercise 8 on page 140. ∎

Lemma 1.2. *Let* C *be a cyclic group of order 4 and* M *be an irreducible* FC*-module faithful for* C*. Let* f *be a non-singular* C*-invariant symmetric bilinear form on* M*. Then* $\det(f) = 1$ *and* $\mathrm{Hasse}(f) = (\lambda, -1)$ *for some* $\lambda \in F^*$.

Proof: Since F is real, the character afforded by M will be the sum of the two faithful irreducible characters of C. Any C-invariant bilinear form g on $M \otimes \mathbb{C}$ satisfies

$$g(e_1, e_1) = g(ie_1, ie_1) = -g(e_1, e_1) = 0 = g(e_2, e_2),$$

where $e_1, e_2 \in M \otimes \mathbb{C}$ are eigenvectors for a generator of C corresponding to i and $-i$ respectively. It follows that the $\mathbb{C}$-space of C-invariant symmetric bilinear forms on $M \otimes \mathbb{C}$ is one dimensional. This, in turn, implies that the F-space of C-invariant symmetric bilinear forms on M is one dimensional. Hence, by Lemma 1.1, b), and the well known facts that $(\alpha, 1) = 1$ and $(\alpha, \beta)(\alpha', \beta) = (\alpha\alpha', \beta)$ in $Br(F)$ for $\alpha, \alpha', \beta \in F^*$, it is enough to show that Lemma 1.2 holds for some f.

Let W be the one dimensional faithful module for the cyclic subgroup of order 2 of C over F. Then W affords a symmetric invariant bilinear form b and a basis vector e such that $b(e, e) = 1$. M is isomorphic to the C-module induced from W, and it follows that M affords a C-invariant symmetric bilinear form f and a basis e_1, e_2 such that $f(e_1, e_1) = f(e_2, e_2) = 1$ and $f(e_1, e_2) = 0$. Obviously, for this f, $\det(f) = 1$ and $\mathrm{Hasse}(f) = 1$. Hence, the Lemma holds. ∎

Theorem 1.3. *Let* G *be a finite group,* F *some real field and* χ *some irreducible character of* G *with values in* F *such that* $m_\infty(\chi) \neq 1$*. Let* M *be an* FG*-module affording the character* $m_F(\chi)\chi$ *and* f *be some* G*-invariant non-singular symmetric bilinear form on* M*. Then the following hold.*

1) $\det(f) = 1$ *(up to squares in* F^**).*
2) $\mathrm{Hasse}(f) = (-1, -1)[\chi]$ *if* $4 \nmid \chi(1)$*, and* $\mathrm{Hasse}(f) = 1$ *if* $4 | \chi(1)$.

Proof: See Theorem B of [**3**]. ∎

Lemma 1.4. *Let χ be an irreducible character of some finite group G and assume that the values of χ are in F. Suppose $[\chi] = (\lambda, -1)$ in $Br(F)$ for some $\lambda \in F^*$. Then $m_\infty(\chi) = 2$ if and only if λ is negative. Furthermore, if $F = \mathbb{Q}$, then $m_p(\chi) = 1$ for every rational finite prime p such that $p \equiv 1$ (mod 4).*

Proof: The local Schur indices of χ are 1 or 2 depending on whether or not the tensor of $(\lambda, -1)$ with the completion of F splits. For example, $m_\infty(\chi) = 1$ if and only if $(\lambda, -1) = 1$ in $Br(\mathbb{R})$. However, the latter condition holds if and only if λ is positive, so the first assertion of the lemma holds. If p is any finite prime and $F = \mathbb{Q}$ then $m_p(\chi) = 1$ if and only if $(\lambda, -1) = 1$ in $Br(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers. Now $(\lambda, -1) = 1$ in $Br(\mathbb{Q}_p)$ if and only if $\lambda x^2 - y^2 = z^2$ has a solution with $x, y, z \in \mathbb{Q}_p$ and $z \neq 0$. Suppose p is a finite rational prime and $p \equiv 1$ (mod 4). Then p is the sum of two rational squares. So, in this case we may assume that p is not involved in the prime factorization of λ. But then $(\lambda, -1) = 1$ in $Br(\mathbb{Q}_p)$ by, for example, Exercise 10 in p. 186 of [**1**]. This completes the proof of the lemma. ■

Lemma 1.5. *Let $n > 1$ be an integer, let $G = S_n$ be the symmetric group of degree n. Then there is an absolutely irreducible $\mathbb{Q}G$-module W of dimension $n - 1$ and a G-invariant symmetric bilinear form f on W such that $\det(f) = n$ up to squares in $\mathbb{Q}^*$.*

Proof: Let N be the natural permutation module for S_n over $\mathbb{Q}$. Then $\dim_{\mathbb{Q}}(N) = n$ and the permutation basis $e_1, ..., e_n$ of N can be taken to be an orthonormal basis of an S_n-invariant symmetric bilinear form g on N. Then $\det(g) = 1$. The vector $v = e_1 + ... + e_n$ is S_n-invariant and $g(v, v) = n$. The space $v^\perp$ of vectors in N which are orthogonal to v is an S_n-submodule of N of dimension $n - 1$. We set $W = v^\perp$. Then $N = W \perp < v >$. It follows that if we set f to be the restriction of g to W then $\det(f) = n$, since $\det(f)n = 1$ up to squares in $\mathbb{Q}^*$. Since S_n acts doubly transitively on $e_1, ..., e_n$, W is absolutely irreducible, and the lemma holds. ■

Lemma 1.6. *let G be a finite group and M an FG-module. Suppose M is endowed with a non-singular G-invariant symmetric bilinear form f. Then we can write*

$$M = M_o \oplus M_1 \oplus M_2$$

as FG-modules where the following hold:

1) *M_o is an orthogonal sum of irreducible submodules on which f is non-singular.*

2) *f is totally isotropic on both M_1 and M_2.*
3) *As quadratic spaces $M = M_o \perp (M_1 + M_2)$ and $M_1 + M_2$ is the orthogonal sum of $\dim_F(M_1) = \dim_F(M_2)$ copies of the hyperbolic plane.*

Proof: Suppose the lemma is false, and pick a counterexample with $\dim_F(M)$ as small as possible. Let N_1 be an irreducible submodule of M. Suppose f is non-singular on N_1. Then $M = N_1 \perp N_1^{\perp}$ and $N_1^{\perp}$ is a G-submodule on which f is non-singular. Hence, by the minimality of our counterexample, the lemma holds for $N_1^{\perp}$, and it follows that it also holds for M, a contradiction.

Therefore, we must assume that f is singular on N_1 and on every other irreducible G-submodule of M. Since f is G-invariant, it follows that f is totally isotropic on N_1 and on every other irreducible G-submodule of M. In this case, $N_1 \subsetneq N_1^{\perp}$. By Maschke's Theorem, there is an irreducible submodule N_2 of M such that $N_2 \cap N_1^{\perp} = 0$. Now f is totally isotropic on both N_1 and N_2, and

$$(N_1 + N_2) \cap (N_1 + N_2)^{\perp} = (N_1 + N_2) \cap N_1^{\perp} \cap N_2^{\perp} = N_1 \cap N_2 = 0,$$

so f is non-singular on $N_1 + N_2$. The bilinear form provides an isomorphism between the dual of N_1 and $N_2 \simeq N_1 + N_2/N_1^{\perp}$. Hence, if we choose $e_1, ..., e_n$ to be a basis of N_1, we can then choose for N_2 the corresponding "dual" basis $e_1^*, ..., e_n^*$, i.e. $\dim_F(N_1) = \dim_F(N_2)$ and $f(e_i, e_j^*) = \delta_{ij}$ where δ_{ij} is Kronecker's delta. Hence, as quadratic spaces the $< e_i, e_i^* >$ are isomorphic to the hyperbolic plane and $N_1 + N_2 =< e_1, e_1^* >\perp< e_2, e_2^* >\perp \dots \perp< e_n, e_n^* >$. Hence, induction applied to $< N_1 + N_2 >^{\perp}$ completes the proof of the lemma. ∎

2. Proofs of the main results

Proof of Theorem A: Let $C =< x >$ be the subgroup of G generated by x. Then C is a cyclic group of order 4 and x^2 fixes no non-zero vector in M. It follows that, as an FC-module, M is the direct sum of faithful irreducible FC-modules. Apply Lemma 1.6 to M under the action of C. Then, as a quadratic space, M is the orthogonal sum of non-singular quadratic spaces on irreducible C-submodules of M and $\dim_F(M_1)$ copies of the hyperbolic plane, where M_1 is some FC-submodule of M. Since F is real, $\dim_F(M_1)$ is even and it follows from Lemma 1.1 that the sum of all these hyperbolic planes forms a subspace with determinant 1 and Hasse invariant either 1 or $(-1, -1)$ in $Br(F)$. The other orthogonal summands of M all have determinant 1 and Hasse invariant $(\lambda, -1)$ for various $\lambda \in F^*$ by Lemma 1.2. Hence, by Lemma

1.1, $\det(f) = 1$ up to squares in F^* and $\mathrm{Hasse}(f)$ is a product of elements of the form $(\lambda, -1)$ for various $\lambda \in F^*$. Since $(\lambda, -1)(\mu, -1) = (\lambda\mu, -1)$ in $Br(F)$, it follows that $\mathrm{Hasse}(f) = (\lambda_o, -1)$ for some $\lambda_o \in F^*$. If f is positive definite, then $\mathrm{Hasse}(f)$ is trivial over $\mathbb{R}$, so $\lambda_o > 0$ in this case. Hence 1) and 2) of the theorem hold.

Suppose now that $\dim_F(M) \equiv 2 (\mathrm{mod}\ 4)$. Then by Lemma 1.1, b), $\mathrm{Hasse}(\mu f) = (\lambda_o, -1)(\mu, -1) = (\lambda_o\mu, -1)$. It follows that $\mathrm{Hasse}(\mu f) = (\lambda, -1)$ for every $\lambda \in F^*$ if we set $\mu = \lambda_o\lambda$. ■

Proof of Corollary 1: Since $F \supseteq \mathbb{Q}(\chi)$, we can take M to be an irreducible FG-module affording the character $m_F(\chi)\chi$. Since F is real, there is a positive definite G-invariant symmetric bilinear form f on M. The Hasse invariant of f can be calculated in two ways. On the one hand, Theorem A tells us that

$$\mathrm{Hasse}(f) = (\lambda_o, -1) \text{ in } Br(F)$$

for some $\lambda_o \in F^*$. On the other hand, Theorem 1.3 tells us that

$$\mathrm{Hasse}(f) = (-1, -1)[\chi].$$

Solving for $[\chi]$ we obtain $[\chi] = (-\lambda_o, -1)$. By Lemma 1.4, $-\lambda_o$ is negative, since $m_\infty(\chi) = 2$. Furthermore, if $F = \mathbb{Q}$, then $m_p(\chi) = 1$ for every rational finite prime p such that $p \equiv 1$ (mod 4), by Lemma 1.4. Hence Corollary 1 holds. ■

Proof of Theorem B: Since for every $a \in \mathbb{Q}^*$, $(\lambda_o, -1) = (a^2\lambda_o, -1)$, we assume without loss that λ_o is a positive integer divisible by 9. Furthermore, $(2, -1) = 1$ in $Br(\mathbb{Q})$, so we further assume without loss that λ_o is odd. Now $\lambda_o > 1$ and we set $n = \lambda_o$ and $G = D_8 \times S_n$ to be the direct product of the dihedral group of order 8 and the symmetric group of degree n. We take x to be any element of order 4 in D_8. Then $1 \neq x^2 \in Z(G)$. Let V be a quadratic F-space of dimension 2 with orthonormal basis e_1, e_2. We endow V with the structure of a D_8-module by letting two generators of order 2 of D_8 act on V as linear transformations which have the following matrices,

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the basis e_1, e_2. It is clear that D_8 stabilizes the quadratic form on V. Let W be the S_n module given by Lemma 1.5. Set

$$M = V \otimes W.$$

Then M is an absolutely irreducible G-module and x^2 does not act trivially on M. Let χ be the character afforded by M and f be the symmetric bilinear form obtained by tensoring those of V and W. Clearly $m_{\mathbb{Q}}(\chi) = 1$ and $\chi(1) = 2(\lambda_o - 1) \equiv 0 \pmod 4$. As a quadratic space

$$M \simeq W \perp W,$$

so that $\mathrm{Hasse}(f) = (\det(W), \det(W)) = (\lambda_o, \lambda_o)$ by Lemma 1.1 and Lemma 1.5. Since $(\lambda_o, \lambda_o) = (\lambda_o, -1)$ in $Br(\mathbb{Q})$, this shows a) and b) of Theorem B. Since M is absolutely irreducible all G-invariant bilinear forms on M are multiples of f. Since $\det(f) = 1$, c) follows from Lemma 1.1, b). ■

Proof of Theorem C: Notice that as F is real, Q has only one isomorphism class of faithful irreducible FQ-modules and they have dimension 4. Since $Z(Q)$ fixes no non-zero vector of M, M is a sum of irreducible faithful FQ-modules. Applying Lemma 1.6, it follows that as quadratic Q-modules $M = M_o \perp (M_1 + M_2)$ where M_o is the orthogonal sum of non-singular irreducible FQ-modules and $M_1 + M_2$ is an FQ-module and as orthogonal space it is the orthogonal sum of a multiple of 4 copies of a hyperbolic plane. It follows that $\det(M_1 + M_2) = 1$ and $\mathrm{Hasse}(M_1 + M_2) = 1$, by Lemma 1.1.

Let ψ be the faithful irreducible complex character of Q. It is well known that $[\psi] = (-1, -1)$ in $Br(F)$. It then follows from Theorem 1.3 that if N is any of the orthogonal summands of M_o, then $\det(N) = 1$ and $\mathrm{Hasse}(N) = (-1, -1)(-1, -1) = 1$. Hence, by Lemma 1.1, $\det(f) = 1$ and $\mathrm{Hasse}(f) = 1$, as desired. ■

Proof of Corollary 2: Let f_o be a symmetric bilinear form on M under which M admits an orthonormal basis. Since F is a real field f_o is positive definite, and in fact, it is positive definite under each imbedding of F into $\mathbb{R}$. Define

$$f : M \times M \to F$$
$$f(v, w) = \frac{1}{|G|} \sum_{g \in G} f_o(gv, gw)$$

for $v, w \in M$. Then f is a G-invariant symmetric bilinear form. Furthermore, f is positive definite under every imbedding of F into $\mathbb{R}$. Furthermore, by Theorem C, $\det(f) = 1$ and $\mathrm{Hasse}(f) = 1$. It follows that M admits an orthonormal basis under f. This completes the proof of the corollary. ■

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Rebut el 12 de Febrer de 1992