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ISOMORPHISMS BETWEEN REPRESENTATIONS OF ALGEBRAS[1]

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A la memoria de Pere Menal, guía, compañero y amigo

Abstract

In this paper we study the precise relation between two representations of a given split basic finite dimensional algebra A as a factor of the free path algebra over its quiver (A). After defining the notion of strongly acyclic quiver, we apply the results obtained to develop a method of calculating the group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ in the case when (A) is strongly acyclic.

The representation of algebras by quivers and relations has a long tradition. It is well-known that if A is a finite dimensional algebra which is split over the field K (i.e., $A = B \oplus J(A)$ as a K-vector space, where B is a subalgebra of A isomorphic to a direct product of full matrix algebras over K and $J(A)$ is the Jacobson radical of A), then A is Morita equivalent to an algebra, usually called its basic algebra, that is isomorphic to a factor of the path algebra $K[\Gamma]$ of its quiver Γ (see, e.g., Section 27 of [**1**] and [**2**]). The study of the connection between any two factors of $K[\Gamma]$ that are both isomorphic to the basic algebra of A is, together with some consequences of it, the aim of this work. This study is divided into two parts. In the present article we study the connection itself (Theorem 3 and Corollary 5) and use what is found to approach the group of automorphisms of A, when the quiver is of a special type (Theorem 7). We shall show further consequences of this connection in a subsequent work, that we hope will be soon in preprint.

In what follows, unless otherwise stated, A will be a split finite dimensional algebra which is basic, i.e., the above mentioned subalgebra B is a direct product of copies of K. Since we want to emphasize a

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certain *uniqueness* of B in that decomposition of A, we will assume a condition on K, like that of being a perfect field, that guarantees that the Wedderburn-Malcev's Theorem (see 11.6 of [**3**]) applies. For all the terminology concerning the relation between a finite dimensional algebra and its associated basic one or, more generally, between an Artinian ring and its associated basic one, we refer the reader to ([**1**, Section 27]). When dealing with maps between algebras, homomorphism and automorphism will always mean K-homomorphism and K-automorphism (i.e., inducing the identity on K), respectively. In reference to the representation of A by its quiver and relations, we follow the terminology of Gabriel ([**2**]), except for the composition of arrows. We will write $\alpha\beta$ for a path $\cdot \overset{\alpha}{\rightarrowtail} \cdot \overset{\beta}{\rightarrowtail}$. We will denote by $\Gamma(A)$, or simply Γ if no confusion appears, the quiver of A and by $V(\Gamma)$ (resp. $A(\Gamma)$) the set of vertices (resp. arrows) of Γ. The symbol $K[\Gamma]$ will stand for the corresponding free path algebra and J for the ideal of $K[\Gamma]$ consisting of all linear combinations of paths of length ≥ 1. It is well-known (see [**2**]) that A is isomorphic to a quotient of $K[\Gamma]$ by an ideal I satisfying $J^m \subseteq I \subseteq J$, for some $m \geq 1$. Every such an ideal, which is usually not unique, is called an *adequate ideal for A in $K[\Gamma]$*. Being $K[\Gamma]/J^m$ a finite dimensional algebra, hence Artinian, every adequate ideal I is finitely generated modulo J^m and from that follows easily that I is itself finitely generated. Every finite set ρ of generators for an adequate ideal is called an *adequate set of relations for A in $K[\Gamma]$*. For our purposes, it will not be restrictive to assume that A is also indecomposable (as a proper direct product of algebras) or, equivalently, that Γ is connected (i.e. there is a not necessarily oriented path between any two vertices). In that case, either A is isomorphic to K or, otherwise, every adequate ideal for A in $K[\Gamma]$ is contained in J^2. We will always assume this situation in the sequel. On the other hand, we shall denote by $\mathcal{P}_n$, for every $n \geq 0$, the set of all paths of length exactly n in Γ. By convention, a path of length 0 is just a vertex.

In the following result and later on in these notes, we will sometimes use the notation x^σ to denote the action of certain *automorphism* on the element x and X^σ to denote the image of the subset X by σ.

Lemma 1. *Let I be an adequate ideal for A in $K[G]$. For every basic set of idempotents $\{e_1, \dots, e_n\}$ of A, there is a surjective homomorphism of algebras $p : K[\Gamma] \rightarrowtail A$ such that* $\ker p = I$ *and, as sets,* $p(V(\Gamma)) = \{e_1, \dots, e_n\}$.

Proof: Since I is an adequate ideal for A, we have an isomorphism of algebras $h : K[\Gamma]/I \rightarrowtail A$ and $h(\overline{V(\Gamma)})$ is a basic set of idempotents for A, where $\overline{V(\Gamma)}$ denotes the set of classes of the vertices modulo I. If we denote by B the subalgebra of A generated by $h(\overline{V(\Gamma)})$ and by B^* that generated by $\{e_1, \ldots, e_n\}$, then we know that $A = B \oplus J(A) = B^* \oplus J(A)$. By the Wedderburn-Malcev Theorem (see, e.g., [**3**, Theor. 11.6]), there is a inner automorphism σ of A such that $B^\sigma = B^*$. But then $h(\overline{V(\Gamma)})^\sigma$ is a basic set of idempotents of B^* that must necessarily coincide with $\{e_1, \ldots, e_n\}$. Therefore the composition $K[\Gamma] \overset{\pi}{\rightarrowtail} K[\Gamma]/I \overset{h}{\rightarrowtail} A \overset{\sigma}{\rightarrowtail} A$, where π is the canonical projection, is the desired surjective homomorphism of algebras. ∎

Remark 2. According to the above lemma, we do not need to worry about the basic set of idempotents of A since, up to a permutation in the set, all the representations of A in the form $K[\Gamma]/I$ are obtained by sending the vertices of Γ to that set.

Definition. *An automorphism of the quiver* Γ is a pair of bijective maps, denoted by the same letter σ, $V(\Gamma) \overset{\sigma}{\rightarrowtail} V(\Gamma)$ and $A(\Gamma) \overset{\sigma}{\rightarrowtail} A(\Gamma)$, so that if α is an arrow from ν to $\omega(\nu, \omega \in V(\Gamma))$ then α^σ is an arrow from ν^σ to ω^σ.

It is obvious that an automorphism of the quiver induces an automorphism of the path algebra $K[\Gamma]$. To avoid too much notation, we denote it also by σ. If now I is an adequate ideal for A in $K[\Gamma]$, then I^σ is another adequate ideal and σ induces an isomorphism of adequate representations $K[\Gamma]/I \cong K[\Gamma]/I^\sigma$. Any isomorphism of adequate representations obtained in this manner will be referred as *induced from the quiver*.

In order to describe completely the isomorphisms between adequate representations of A, we need the following definition, in which $A(\nu, \omega)$ denotes the set of arrows in Γ starting at the vertex ν and ending at ω.

Definition. *A change of variables in* $K[\Gamma]$ is an algebra homomorphism $f : K[\Gamma] \rightarrowtail K[\Gamma]$ such that:

i) f induces the identity in $V(\Gamma)$.
ii) If $A(\nu, \omega) = \{\alpha_1, \ldots, \alpha_{m_{\nu\omega}}\}$ (an ordering of the α_k's being fixed), then $f(\alpha_l) = \sum_{k=1}^{m_{\nu\omega}} \lambda_{kl}\alpha_k$ (modulo J^2) for each l, where, for each pair of vertices ν, ω, the $m_{\nu\omega} \times m_{\nu\omega}$ matric (with coefficients in K) $\Lambda = (\lambda_{kl})$ is invertible.

Note that, if condition i) above holds, then condition ii) is equivalent to the following:

ii*) For each pair of vertices ν, ω, the K-linear endomorphism of $\nu J\omega/\nu J^2\omega$ induced by f is an automorphism.

Now comes the main result of this first part, where, for every ideal I of $K[\Gamma]$, $V(\Gamma) + I/I$ stands for the projection of $V(\Gamma)$ via the canonical projection $p : K[\Gamma] \rightarrowtail K[\Gamma]/I$.

Theorem 3. *Let A be of Loewy length m, I an adequate ideal for A in $K[\Gamma]$ and L an arbitrary ideal of $K[\Gamma]$. The following assertions are equivalent for a homomorphism of algebras $\phi : K[\Gamma]/I \rightarrowtail K[\Gamma]/L$:*

a) *ϕ is an isomorphism of K-algebras such that $\phi(V(\Gamma) + I/I) = V(\Gamma) + L/L$.*

b) *$J^k \subseteq L$, for some integer $k \geq m$, and there are an automorphism σ of $k[\Gamma]$ induced from the quiver and a change of variables f in $K[\Gamma]$ such that $I = \sigma^{-1}(f^{-1}(L))$ and ϕ is induced by $f \circ \sigma$.*

b') *As in b), but with $I = f^{-1}(\sigma^{-1}(L))$ and $\sigma \circ f$ instead of $I = \sigma^{-1}(f^{-1}(L))$ and $f \circ \sigma$, respectively.*

c) *There are σ and f as in b) such that $L = f(I^\sigma) + J^k$, for some $k \geq m$, and ϕ is induced by $f \circ \sigma$.*

c') *As in c), but with $L = f(I)^\sigma + J^k$ and $\sigma \circ f$ instead of $L = f(I^\sigma) + J^k$ and $f \circ \sigma$, respectively.*

Before tackling the proof of this theorem, we give a result upon which we shall lean.

Lemma 4. *If f is a change of variables in $K[\Gamma]$, then the induced homomorphism of algebras $\hat{f} : K[\Gamma]/J^r \rightarrowtail K[\Gamma]/J^r$ is an automorphism, for every $r \geq 1$.*

Proof: It will be enough to prove that, modulo J^r, every path is in the image of f. The result is obvious, by the definition of a change of variables, for $r = 1, 2$. If $r \geq 3$ we proceed by decreasing induction on the length s of the path $p = \alpha_1 \ldots \alpha_s$ in question. If $s \geq r$ it is evident, since $p \equiv 0 \pmod{J^r}$. Suppose now that $s \leq r - 1$ and that every path of length $\geq s + 1$ is in the image of f modulo J^r. By the definition of a change of variables again, every arrow α_i appearing in p is in the image of f modulo J^2. From that follows that p is in the image of f modulo J^{s+1}, and the induction hypothesis applies to show that p is in the image of f modulo J^r, as desired. ■

Proof of Theorem 3: a) $\Rightarrow$ b). Since $\phi(V(\Gamma) + I/I) = V(\Gamma) + L/L$ we have a permutation σ of $V(\Gamma)$ such that $\phi(\nu + I) = \nu^\sigma + L$, for each vertex ν. It is clear that the number of arrows from ν to ω coincides with that of arrows from ν^σ to ω^σ, so that σ can be extended to a not necessarily unique automorphism of the quiver Γ. We fix one among the possible automorphisms of Γ that extend σ and, as usual, we also denote by σ the corresponding automorphism induced in $K[\Gamma]$. Now we consider the decomposition $\phi = \phi \circ \bar{\sigma}^{-1} \circ \bar{\sigma}$, where $\bar{\sigma} : K[\Gamma]/I \rightarrowtail K[\Gamma]/I^\sigma$ is the isomorphism induced by σ. Since clearly $\phi \circ \bar{\sigma}^{-1}(\nu + I^\sigma) = \nu + L$, by replacing ϕ by $\phi \circ \bar{\sigma}^{-1}$ and I by I^σ if necessary, it is not restrictive to assume that $\phi(\nu + I) = \nu + L$, for every vertex ν. All we have to prove now is that $J^k \subseteq L$, for some $k \geq m$, and ϕ is induced by a change of variables f such that $I = f^{-1}(L)$. To prove the inclusion it is enough to check that $J^m \subseteq L$, but this follows in a straightforward way from the fact that $K[\Gamma]/L$ is isomorphic to A, by assumption, and hence the dimensions and Loewy lengths of both algebras must coincide. On the other hand, if $\alpha \in A(\nu, \omega)$ then $\alpha + I \in (\nu + I)(J/I)(\omega + I)$, so that $\phi(\alpha + I) \in (\nu + L)(J/L)(\omega + L) = \nu J\omega + L/L$. Let us fix an element $\eta(\alpha)$ in $\nu J \omega$, i.e. a linear combination of paths of length ≥ 1 starting at ν and ending at ω, such that $\phi(\alpha + I) = \eta(\alpha) + L$ and do the same for all vertices ν, ω and all arrows in $A(\nu, \omega)$. The freeness of the path algebra allows us to extend the assignments $\alpha \rightarrowtail \eta(\alpha)$ to a homomorphism of algebras $f : K[\Gamma] \rightarrowtail K[\Gamma]$ that induces $\phi : K[\Gamma]/I \rightarrowtail K[\Gamma]/L$. It only remains to prove that f is actually a change of variables. But that is easy, because the inverse ϕ^{-1} is also induced by a homomorphism of algebras $g : K[\Gamma] \rightarrowtail K[\Gamma]$ that is defined in the same way that f was defined from Φ. It is a mere routine to check that the K-linear endomorphisms of $\nu J\omega/\nu J^2\omega$ induced by f and g are inverse from each other, thus proving that f (and also g) is a change of variables.

a) $\Rightarrow$ b') As the above implication, but considering the decomposition $\phi = \bar{\sigma} \circ \bar{\sigma}^{-1} \circ \phi$ instead of the above taken.

b) $\Rightarrow$ c) From b) follows at once that $f(I^\sigma) + J^k \subseteq L$ and ϕ is induced by $f \circ \sigma$. On the other hand, by considering the isomorphism of algebras $\hat{f} : K[\Gamma]/J^k \rightarrowtail K[\Gamma]/J^k$ induced by f, it follows from Lemma 4 that $\hat{f}(\hat{f}^{-1}(L/J^k) = L/J^k$ and hence $f(f^{-1}(L)) + J^k = L$. But $f^{-1}(L)$ is I^σ and thus we are done.

b') $\Rightarrow$ c') Completely analogous to the above one, by making the suitable changes.

c) $\Rightarrow$ a) From c) we derive a homomorphism of algebras $\bar{f} : K[\Gamma]/I^\sigma \rightarrowtail K[\Gamma]/L$. If now p_1 and p_2 are the canonical projections of $K[\Gamma]/J^k$, where k is the integer given by c), onto $K[\Gamma]/I^\sigma$ and $K[\Gamma]/L$, respectively, we have that $p_2 \circ \hat{f} = \bar{f} \circ p_1$, from which we deduce that $\bar{f}$ is surjective. On the other hand, its kernel is $f^{-1}(L)/I^\sigma$ and from the fact that $\hat{f}$ is monic follows that $I^\sigma/J^k = f^{-1}[f(I^\sigma) + J^k]/J^k = = f^{-1}(L)/J^k$. Therefore $\bar{f}$ is also injective and, since $\phi = \bar{f} \circ \bar{\sigma}$, σ is an isomorphism.

c') $\Rightarrow$ a) As c) $\Rightarrow$ a) with the appropriate changes. ■

As a consequence of the above theorem, it will be easy to express any adequate ideal for A in terms of a given adequate set of relations. Let us recall first that the quiver of A is said to be *acyclic* in case it has neither loops nor oriented cycles.

Corollary 5. *Let A have Loewy length m and ρ be an adequate set of relations for A in $K[\Gamma]$. For an ideal L of $K[\Gamma]$ the following assertions are equivalent:*

a) *L is adequate for A.*
b) *There exist an integer $k \geq m$ and f, σ as in the theorem such that L is generated (as an ideal of $K[\Gamma]$!) by $f(\rho^\sigma) \cup \mathcal{P}_k$ (or by $f(\rho)^\sigma \cup \mathcal{P}_k$).*
c) *There exist f and σ as in the theorem such that, for every $k \geq m$, L is generated by $f(\rho^\sigma) \cup \mathcal{P}_k$ (or by $f(\rho)^\sigma \cup \mathcal{P}_k$).*
d) *There exist f and σ as in the theorem such that L is generated by $f(\rho^\sigma) \cup \mathcal{P}_m$ (or by $f(\rho)^\sigma \cup \mathcal{P}_m$).*

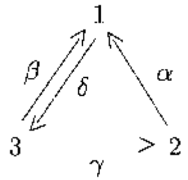
If, moreover, A has an acyclic quiver, then the above conditions are equivalent to the simpler one:

e) *There exist f and σ as in the theorem such that J is generated by $f(\rho^\sigma)$.*

Proof: Let $I = \langle \rho \rangle$ be the ideal of $K[\Gamma]$ generated by ρ. It is clear that, for f and σ as in the theorem, $f(I^\sigma) + J^k$ (resp. $f(I)^\sigma + J^k$) is just the ideal of $K[\Gamma]$ generated by $f(\rho^\sigma) \cup \mathcal{P}_k$ (resp. $f(\rho)^\sigma \cup \mathcal{P}_k$), for every $k \geq m$. With that in mind, the equivalence of a), b), c) and d) follows in a straightforward way from Theorem 3 and Lemma 1, once we notice that, in the proof of a) $\Rightarrow$ b) in that theorem, we actually proved statement b) for every $k \geq m$. On the other hand, A has an acyclic quiver if and only if $J^k = 0$ for some $k \geq 1$. From it follows the equivalence with e) in this particular situation. ■

Examples 6. a) When A does not have an acyclic quiver in the above corollary, $\mathcal{P}_m$ is necessary to guarantee that J is an adequate ideal. This can be seen with the simple example $A = K[X]/\langle X^3 \rangle$, which has a quiver consisting of a unique vertex and a loop, which is denoted by X in the sequel. Obviously $\rho = \{X^3\}$ and the change of variables f taking X to $X + X^2$ yields $f(\rho) = \{(X + X^2)^3\}$, that generates an ideal of $K[\Gamma] = K[X]$ containing no power of $J = \langle X \rangle$. However, as a confirmation of condition c) of the corollary, $f(\rho) \cup \{X^k\}$ generates the ideal $\langle X^3 \rangle$, for every $k \geq 3$ (which is the Loewy length of A).

b) Let A have quiver



and relations $\rho = \{\beta\delta, \gamma\alpha\delta, \alpha\delta\beta - \alpha\delta\gamma\alpha\}$. It is no hard to see that its Loewy length is 5. If now we apply the change of variables f that takes $\beta \rightarrowtail \beta + \gamma\alpha$ and fixes all the other arrows, then we see that $f(\rho) = \{\beta\delta + \gamma\alpha\delta, \gamma\alpha\delta, \alpha\delta\beta\}$ so that the ideal L of $K[\Gamma]$ generated by $f(\rho)$ can be generated by $\{\beta\delta, \gamma\alpha\delta, \alpha\delta\beta\}$ as well. With this and the previous corollary we can already assert that A is monomial algebra, a fact not easily deducible from the first representation given for A. In a further step, one can easily check that every path of length 5 in Γ is in L, thus proving that $A \cong K[\Gamma]/L$.

By using Lemma 1 and Theorem 3, one can deduce, in a straightforward way, that every automorphism ϕ of A that leaves invariant (although not necessarily fixing its elements) a given basic set of idempotents (i.e. $\phi(\{e_1, \ldots, e_n\}) = \{e_1, \ldots, e_n\}$) is induced by a composition $f \circ \sigma$ or $\sigma \circ f$, where σ is an automorphism of $K[\Gamma]$ induced from the quiver and f is a change of variables and, for certain adequate ideal I of $K[\Gamma]$, $f(I^\sigma)$ (or $f(I)^\sigma$) in included in I. Since, by the Wedderburn-Malcev Theorem, up to composition by a inner automorphism all the automorphisms of A leave invariant a given basic set of idempotents, it is then reasonable to expect, in particular cases, some information about the group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ coming from our Theorem 3. That is our next goal.

Let us introduce some notation. We will write S_Γ for the group of all permutations σ of the set $V(\Gamma)$ satisfying that (# arrows $\nu \rightarrowtail \omega$) = (# arrows $\nu^\sigma \rightarrowtail \omega^\sigma$), for every pair of vertices ν, ω. Intuitively, S_Γ is the group of all permutations of $V(\Gamma)$ which potentially can be extended to automorphisms of Γ. We will denote by V_Γ the set of all changes of variables in $K[\Gamma]$ and, if ρ is an adequate set of relations for A in $K[\Gamma]$, $V_{(\Gamma,\rho)}$ will stand for the subset $\{f \in V_\Gamma / f(\rho) \subseteq \langle\rho\rangle\}$ (one should notice that $V_{(\Gamma,\rho)}$ actually depends on $\langle\rho\rangle$ rather than ρ itself). V_Γ is a semigroup with the composition of maps as operation and $V_{(\Gamma,\rho)}$ is a subsemigroup. Moreover, we have a canonical semigroup homomorphism $V_{(\Gamma,\rho)} \rightarrowtail \mathrm{Aut}(A) = \mathrm{Aut}(K[\Gamma]/\langle\rho\rangle)$ whose image turns out to be a subgroup of the $\mathrm{Aut}(A)$, by Theorem 3. We will denote by i_ρ to the composition of the latter homomorphism followed by the canonical projection $\pi : \operatorname{Aut}(A) \rightarrowtail \operatorname{Aut}(A)/\operatorname{Inn}(A)$. We shall introduce now a certain type of algebras for which the technics developed here will help determine $\operatorname{Aut}(A)/\operatorname{Inn}(A)$.

Definition. The quiver Γ of A will be said *strongly acyclic* if it satisfies the following property:

(&) For all $\nu, \omega \in V(\Gamma)$, if there is an arrow $\nu \rightarrowtail \omega$, then there is no path of length ≥ 2 from ν to ω.

Theorem 7. *Let A have a strongly acyclic quiver. The following properties hold:*

a) *V_Γ is a group, which is isomorphic to $\prod_{\nu,\omega \in V(\Gamma)} GL_{m_{\nu\omega}}(K)$, where $m_{\nu\omega}$ is the number of arrows $\nu \rightarrowtail \omega$, for all $\nu, \omega \in V(\Gamma)$.*
b) *$V_{(\Gamma,\rho)}$ is a subgroup of V_Γ.*
c) *i_ρ is an injective map.*
d) *There is an exact sequence of groups and group homomorphisms*

$$1 \rightarrowtail V_{(\Gamma,\rho)} \rightarrowtail \operatorname{Aut}(A)/\operatorname{Inn}(A) \rightarrowtail S_\Gamma$$

Proof: If $\alpha \in A(\nu, \omega)$ and f is a change of variables in $K[\Gamma]$, the fact that Γ is strongly acyclic implies that $f(\alpha)$ must be necessarily a linear combination of arrows $\nu \rightarrowtail \omega$. If we consider the vector subspace $V_{\nu\omega}$ of $K[\Gamma]$ generated by $A(\nu, \omega)$, it is clear that f is completely determined by its *components* $f_{\nu\omega} : V_{\nu\omega} \rightarrowtail V_{\nu\omega}$, i.e., the restrictions of f to these subspaces, so that the assignment $f \rightarrow (f_{\nu\omega}) \in \prod GL(V_{\nu\omega})$ is a bijective map. It is routine to check that it preserves the group operations and thereby a) holds.

To prove b) we should observe that if $f \in V_{(\Gamma,\rho)}$ then f induces an automorphism ϕ in the algebra $K[\Gamma]/\langle\rho\rangle$ whose inverse ϕ^{-1} is also induced by a $g \in V_{(\Gamma,\rho)}$ due to the proof of Theorem 3. The strong acyclic nature of Γ implies that $g = f^{-1}$.

Let us consider now a $f \in V_{(\Gamma,\rho)}$ such that the induced automorphism $\bar{f} : K[\Gamma]/\langle\rho\rangle \rightarrowtail K[\Gamma]/\langle\rho\rangle$ is inner and let u be an element of $A = K[\Gamma]/\langle\rho\rangle$ such that $\bar{f}(a) = uau^{-1}$, for every $a \in A$. If $\alpha \in A(\nu, \omega)$ and $A(\nu, \omega) = \{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_{m_{\nu\omega}}\}$, then $f(\alpha) = \sum_{k=1}^{m_{\nu\omega}} \lambda_k \alpha_k$, with $\lambda_k \in K$ for each k. By writing a bar to denote the class modulo $\langle\rho\rangle$, we have $u\bar{\alpha}u^{-1} = \sum \lambda_k \bar{\alpha}_k$ and so $(0 \neq) u\bar{\alpha} = \sum \lambda_k \bar{\alpha}_k u$. If we view u as the class modulo $\langle\rho\rangle$ of a linear combination of paths in Γ and a coefficient, say, λ_k is nonzero, then a careful look at the latter equality tells us that there is a path of length ≥ 1 in Γ that starts with the arrow α_k and ends with the arrow α. That Γ is acyclic (it does not have to be strongly acyclic here) implies that it can only occur when $\alpha_k = \alpha$ and the path has length exactly 1. From this follows at once that $\lambda_k = 0$ when $k \neq 1$ and $\lambda_1 = 1$, thus implying that f is the identity map and proving c).

Finally, for d), we should recall that every element of $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ is represented by a $\phi \in \mathrm{Aut}(A)$ that leaves invariant $\{e_\nu = \nu + \langle\rho\rangle | \nu \in V(\Gamma)\}$. If we assign to that ϕ the permutation σ of $V(\Gamma)$ defined by $\phi(e_\nu) = e_{\sigma(\nu)}$, then (# arrows $\nu \rightarrowtail \omega$) = (# arrows $\nu^\sigma \rightarrowtail \omega^\sigma$), for each pair for vertices ν, ω. We claim that this assignment defines a group homomorphism $\mathrm{Aut}(A)/\mathrm{Inn}(A) \rightarrowtail S_\Gamma$. It is obvious that the above σ is in S_Γ and, in case of being well-defined, the map preserves multiplication. In order to prove that the assignment is a well-defined map, and hence establish our claim, we shall check that an element of $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ is represented by a *unique* $\phi \in \mathrm{Aut}(A)$ that leaves invariant $\{e_\nu | \nu \in V(\Gamma)\}$. Indeed, if $\phi, \psi \in \mathrm{Aut}(A)$ leave invariant that basic set and $\phi \circ \psi^{-1}$ is inner, then an argument similar to the one followed in c) for f shows that $(\phi \circ \psi^{-1})(e_\nu) = e_\nu$, for every $\nu \in V(\Gamma)$. But then, by the proof of Theorem 3, $\phi \circ \psi^{-1}$ is induced by a change of variables f in $K[\Gamma]$ so that $\phi \circ \psi^{-1} = i_\rho(f)$. By part c), $f \equiv \mathrm{id}_{K[\Gamma]}$ and, consequently, $\phi \equiv \psi$ as desired. ■

Example 8. Assume that the quiver of A is

$$\begin{array}{ccccc} & & 2 & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ 1 & & & & 4 \\ & \underset{\gamma}{\searrow} & & \underset{\delta}{\nearrow} & \\ & & 3 & & \end{array}$$

By the foregoing theorem, $V_\Gamma \cong K^* \times K^* \times K^* \times K^*$ and it is obvious that $S_\Gamma = \{1, \tau\} \cong \mathbf{C}_2$ (the multiplicative cyclic group with two elements), where τ is the transposition of vertices 2 and 3. We shall determine $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ for some algebras having the above quiver.

a) $A = K[\Gamma]$. This case can be dealt with greater generality since, whatever the quiver, $V_\Gamma = V_{(\Gamma,\rho)}$ always and, when Γ is also strongly acyclic, the group homomorphism $p : \mathrm{Aut}(A)/\mathrm{Inn}(A) \rightarrowtail S_\Gamma$ is surjective. In our particular situation it is easy to see that p is also split, so that $\mathrm{Aut}(A)/\mathrm{Inn}(A) \cong (K^* \times K^* \times K^* \times K^*) \propto \mathbf{C}_2$, $\propto$ denoting a semidirect product. An element $[(c_{12}, c_{24}, c_{13}, c_{34}), \tau]$ of the second group represents the class modulo $\mathrm{Inn}(A)$ of the composition $(c_{12}, c_{24}, c_{13}, c_{34}) \circ \tau$, where

τ is the above mentioned transposition on the vertices and $\binom{\alpha\beta\gamma\delta}{\gamma\delta\alpha\beta}$ on the arrows, while the quadruple $(c_{\nu\omega})$ takes every arrow $\varepsilon : \nu \rightarrowtail \omega$ to $c_{\nu\omega}\varepsilon$.

b) $A = K[\Gamma]/\langle\alpha\beta\rangle$. Here $V_\Gamma = V_{(\Gamma,\rho)}$ again and it is not hard to see that any $\phi \in \mathrm{Aut}(A)$ that leaves invariant the basic set of idempotents necessarily induces the identity on that set, thus proving that the image of $p : \mathrm{Aut}(A)/\mathrm{Inn}(A) \rightarrowtail S_\Gamma$ is the trivial subgroup. Therefore $\mathrm{Aut}(A)/\mathrm{Inn}(A) \cong K^* \times K^* \times K^* \times K^*$.

c) $A = K[\Gamma]/\langle\alpha\beta, \gamma\delta\rangle$. $V_\Gamma = V_{(\Gamma,\rho)}$ and p is also split, so that the situation is exactly the same as in a).

d) $A = K[\Gamma]/\langle\alpha\beta - \gamma\delta\rangle$. In this case every change of variables (identified with the quadruple $(c_{12}, c_{24}, c_{13}, c_{34})$) that leaves unaltered the adequate ideal $\langle\alpha\beta - \gamma\delta\rangle$ must necessarily satisfy $c_{12}c_{24} = c_{13}c_{34}$. Hence $V_{(\Gamma,\rho)} \cong \{(c_{12}, c_{24}, c_{13}, c_{34}) \in K^* \times K^* \times K^* \times K^*/c_{12}c_{24} = c_{13}c_{34}\}$. On the other hand, p is again a split epimorphism. Therefore $\mathrm{Aut}/(A)\mathrm{Inn}(A) \cong V_{(\Gamma,\rho)} \propto \mathbf{C}_2$.

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