Publicacions Matemàtiques, Vol 36 (1992), 827–836.

CHAOS EXPANSIONS AND LOCAL TIMES

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Abstract

In this note we prove that the Local Time at zero for a multiparametric Wiener process belongs to the Sobolev space $\mathbb{D}^{k-\frac{1}{2}-\epsilon,2}$ for any $\epsilon > 0$. We do this computing its Wiener chaos expansion. We see also that this expansion converges almost surely. Finally, using the same technique we prove similar results for a renormalized Local Time for the autointersections of a planar Brownian motion.

0. Introduction and notations

In this note we first obtain the Wiener chaos decomposition of the local time at zero for a multiparameter Wiener process. We also show that the Wiener chaos series converges almost surely, and the local time belongs to the Sobolev space $\mathbb{D}^{k-1/2-\epsilon,2}$, for any $\epsilon > 0$, where k is the number of parameters of the Wiener process. The last part of the paper is devoted to show the existence of a renormalized local time for the autointersections of a planar Brownian motion (Varadhan renormalization), by means of the Wiener chaos expansion.

Let $(T, \mathcal{B}, \mu)$ be a σ-finite atomless measure space. We will denote by H the Hilbert space $L^2(T, \mathcal{B}, \mu)$ which is assumed to the separable. Let $W = \{W(h), h \in H\}$ be a zero-mean Gaussian process with covariance function $E[W(f)\,W(g)] = \langle f, g\rangle_H$ defined on some probability space $(\Omega, \mathcal{F}, P)$. We will suppose that $\mathcal{F}$ is the σ-field generated by $\{W(h), h \in H\}$. It is well-known that any square-integrable functional on Ω has an orthogonal decomposition of the form

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{1}$$

where $f_n \in L^2_s(T^n)$ (symmetric square integrable kernel), and I_n denotes the multiple Wiener-Itô stochastic integral.

In this framework we can consider the derivative operator D which acts on multiple stochastic integrals in the following form,

$$D_t I_n(f_n(t_1,\dots,t_n)) = n\, I_{n-1}(f_n(t_1,\dots,t_{n-1},t))$$

for $n \geq 1$, $t \in T$. We can introduce the Sobolev spaces $\mathbb{D}^{\alpha,2}$ for $\alpha \in \mathbb{R}$, as it is done in [11]. A functional $F \in L^2(\Omega)$ with the development (1) belongs to $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geq 1} n!(1+n)^\alpha \, \| f_n \|_2^2 < \infty.$$

Set $\mathbb{D}^{\infty,2} = \cap_{\alpha\in\mathbb{R}} \mathbb{D}^{\alpha,2}$ and $\mathbb{D}^{\alpha-,2} = \cap_{\gamma<\alpha} \mathbb{D}^{\gamma,2}$ for all $\alpha \in \mathbb{R}$.

1. Preliminaries

Let us first recall the Stroock formula (cf. [8]) that gives the Wiener chaos decomposition of a functional F belonging to $\mathbb{D}^{\infty,2}$:

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n\left(E[D^n F]\right). \tag{2}$$

We will also make use of the Hermite polynomials. For each $n \geq 0$, we will denote by $H_n(x)$, the nth Hermite polynomial defined by

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2}\right), \quad n \geq 0. \tag{3}$$

Let $p_\varepsilon(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. The following equality, which follows immediately from (3), relates the derivatives $p_\varepsilon^{(n)}(x)$ with the Hermite polynomials:

$$p_\varepsilon^{(n)}(x) = (-1)^n \sqrt{n!}\, \varepsilon^{-n/2} p_\varepsilon(x)\, H_n\left(\frac{x}{\sqrt{\varepsilon}}\right), \quad n \geq 1. \tag{4}$$

Lemma 1.1.
Let Y be a random variable with distribution $N(0,\sigma^2)$. Then

$$E[H_{2m}(Y)] = \frac{\sqrt{2m!}\,(\sigma^2-1)^m}{2^m\, m!},$$

and $E[H_n(Y)] = 0$ if n is odd.

Proof:
It follows easily from the explicit formula for Hermite polynomials:

$$H_n(x) = \sqrt{n!} \sum_{k=0}^{[n/2]} \frac{(-1)^k\, x^{n-2k}}{k!\,(n-2k)!\,2^k}, \tag{5}$$

and the moments of a Gaussian random variable, $E[Y^{2m}] = \frac{(2m)!}{m!2^m}$. ∎

Lemma 1.2.

Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of square integrable random variables with the expansions

$$F_\varepsilon = \sum_{n=0}^{\infty} I_n(f_n^\varepsilon) , \quad f_n^\varepsilon \in L_s^2(T^n) .$$

Assume that

i) *f_n^ϵ converges in $L^2(T^n)$, when $\varepsilon \downarrow 0$, to some function $f_n \in L_s^2(T^n)$.*

ii) $$\sum_{n=0}^{\infty} \sup_{\varepsilon} \left\{ n! \, \| f_n^\varepsilon \|_2^2 \right\} < \infty.$$

Then the family F_ε converges in $L^2(\Omega)$ to $F = \sum_{n=0}^{\infty} I_n(f_n)$.

Proof:

It is an immediate consequence of the Lebesgue dominated convergence theorem. ■

2. Chaos expansion of $\delta_0(W(h))$

Let δ_0 be the Dirac delta function at zero. We can consider $\delta_0(W(h))$ as a distribution on the Wiener space in the sense of Watanabe (cf. [11]). Using the integration by parts formula on the Wiener space one can show that $p_\varepsilon(W(h))$ converges in $\mathbb{D}^{-1,2}$ to $\delta_0(W(h))$ (see [5]). We will first compute the Wiener chaos expansion of $p_\varepsilon(W(h))$, and from it we will deduce the expansion of $\delta_0(W(h))$. By formulas (2) and (3) we have

$$\begin{aligned} p_\varepsilon(W(h)) &= \sum_{n=0}^{\infty} \frac{1}{n!} E\left[p_\varepsilon^{(n)}(W(h))\right] I_n(h^{\otimes n}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}\,\varepsilon^{n/2}} E\left[p_\varepsilon(W(h))\, H_n\left(\frac{W(h)}{\sqrt{\varepsilon}}\right)\right] I_n(h^{\otimes n}) . \end{aligned} \tag{6}$$

The expectation appearing in the above formula vanishes if n is odd because p_ε and H_n are even functions. On the other hand, using Lemma 1.1 for $n = 2m$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}} H_{2m}\left(\frac{x}{\sqrt{\varepsilon}}\right) p_\varepsilon(x) p_{\|h\|^2}(x) dx \\ &= \left(2\pi(\|h\|^2+\varepsilon)\right)^{1/2} \int_{\mathbb{R}} H_{2m}\left(\frac{x}{\sqrt{\varepsilon}}\right) p_{\varepsilon\|h\|^2/(\varepsilon+\|h\|^2)}(x) dx \\ &= \left(2\pi(\|h\|^2+\varepsilon)\right)^{-1/2} \frac{\sqrt{2m!}}{2^m \; m!} \left(\frac{-\varepsilon}{\|h\|^2+\varepsilon}\right)^m . \end{aligned} \tag{7}$$

Finally, from (6) and (7), we get the following expansion

$$p_\varepsilon(W(h)) = \sum_{m=0}^{\infty} \frac{(-1)^m \ I_{2m}(h^{\otimes 2m})}{\sqrt{2\pi}\ 2^m\ m!\, \left(\| h \|^2 + \varepsilon\right)^{m+1/2}} . \tag{8}$$

Letting ε tend to zero we deduce the Wiener chaos expansion of $\delta_0(W(h))$:

$$\delta_0(W(h)) \ = \ \sum_{m=0}^{\infty} \frac{(-1)^m \ I_{2m}\left(h^{\otimes 2m}\right)}{\sqrt{2\pi}\ 2^m\, m!\, \| h \|^{2m+1}} . \tag{9}$$

This series does not converge in $L^2(\Omega)$, because

$$\| \delta_0(W(h)) \|_2^2 = \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}\ (m!)^2\ 2\pi \| h\|^2} = \infty, \tag{10}$$

by the Striling formula. Observe that from (9) and (10) we obtain

i) $\delta_0(W(h)) \in \mathbb{D}^{-1/2-,2}$

ii) $\delta_0(W(h)) \notin \mathbb{D}^{-1/2,2}$,

and the series (9) converges in the norm of the space $\mathbb{D}^{-1/2-\epsilon,2}$, for any $\epsilon > 0$.

Remark. More generally we can obtain the chaos expansion of $\delta_x(W(h))$ when $x \neq 0$:

$$\delta_x(W(h)) \ = \ \sum_{n=0}^{\infty} p_{\| h \|^2}(x)\ H_n\left(\frac{x}{\| h \|}\right)\ \frac{I_n(h^{\otimes n})}{\| h \|^n \sqrt{n!}} .$$

3. Wiener chaos expansion for the local time of a multiparametric Wiener process

In this section we will assume that T is $[0,1]^k$, with $k \geq 1$. Then $W = \{W(\underline{t}),\ \underline{t} \in T\}$ will be the standard Wiener process on T. We will denote by $[0,\underline{t}]$ the rectangle $[0,t_1] \times \cdots \times [0,t_k]$, where $\underline{t} = (t_1,\ldots,t_k)$. We will also set $|\underline{t}| = t_1 \cdot \ldots \cdot t_k$.

The local time of W can be formally defined as

$$L(\underline{t}\,, x) = \int_{[0,\underline{t}]} \delta_x(W_{\underline{s}})\, d\underline{s}\,, \quad \underline{t} \in T, \quad x \in \mathbb{R}. \tag{11}$$

Although for any fixed $\underline{s}$, $\delta_x(W_{\underline{s}})$ is not an ordinary random variable but a distribution on the Wiener space, it turns out that the integral in (11) has a smoothing effect, and $L(\underline{t}, x)$ is a well-defined random variable for any fixed point $\underline{t}$, not on the axes. We will restrict our analysis to the case $x = 0$, and we will set $L(\underline{t}) = L(\underline{t}, 0)$. We know that $L(\underline{t}) = \int_{[0,\underline{t}]} \delta_0(W_{\underline{s}})\, d\underline{s}$ can be obtained as the L^2-limit of

$$L_\varepsilon(\underline{t}) = \int_{[0,\underline{t}]} p_\varepsilon(W_{\underline{s}})\, d\underline{s} \tag{12}$$

when ε tends to 0 (see, for instance, [**2**]). In the next theorem we will compute the Wiener chaos expansion of $L(\underline{t})$.

Theorem 3.1.

We have that $L(\underline{t})$ belongs to the space $\mathbb{D}^{k-\frac{1}{2}-,2}$, for any point t not on the axes, and it holds that

$$L(\underline{t}) = \sum_{m=0}^{\infty} \frac{(-1)^m 2^k}{\sqrt{2\pi}\; 2^m m!(1-m)^k} I_{2m} \left[\prod_{i=1}^{k} \left((t_i)^{(1-m)/2} - (t_{1,i} \vee \cdots \vee t_{2m,i})^{(1-m)/2}\right)\right].$$

Moreover, $L(\underline{t})$ does not belong to $\mathbb{D}^{k-\frac{1}{2},2}$.

Proof:

We will first compute the Wiener chaos expansion of $L_\varepsilon(\underline{t})$ applying the results of the previous section. From (8) and (11) we obtain

$$L_\varepsilon(\underline{t}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{2\pi}\; 2^m\; m!} \int_{[0,\underline{t}]} \frac{I_{2m}\left(\mathbf{1}_{[0,\underline{s}]}^{\otimes 2m}\right)}{\left(|\underline{s}| + \varepsilon\right)^{m+1/2}}\, d\underline{s}\,. \tag{13}$$

Then the series $\sum_{n=1}^{\infty} X_n$ *converges a.s.*

As a consequence of this theorem, if F is a square integrable random variable with the development (1), and

$$\sum_{n=0}^{\infty} n!(\log n)^2 \|f_n\|_2^2 < \infty, \tag{14}$$

then the Wiener chaos expansion (1) converges a.s. In particular the condition (14) is satisfied if F belongs to the Sobolev space $\mathbb{D}^{\epsilon,2}$ for any $\epsilon > 0$. Consequently, applying Theorem 3.1, and the above criterion (14), we deduce the almost sure convergence of the Wiener chaos expansion of the local time of the multiparameter Wiener process.

4. Renormalized local time for the autointersections of a planar Brownian motion

Consider now $W = \{ (W_t^1, W_t^2),\ t \in [0,1] \}$ a standard planar Brownian motion. Let us write $[X] = X - E(X)$ for any integrable random variable X. It is known from **[6]** that

$$L_\varepsilon = \int_{0<s<t<1} \left[p_\varepsilon(W_t^1 - W_s^1)\, p_\varepsilon\left(W_t^2 - W_s^2\right)\right] ds\, dt \tag{15}$$

converges in $L^2(\Omega)$, as ε tends to zero. The purpose of this section is to give a new proof of this fact by means of the results obtained on Section 2.

Theorem 4.1.

The family of random variables L_ϵ *converges as* ε *tends to zero, in* $\mathbb{D}^{1/2-\delta,2}$, *for any* $\delta > 0$. *In particular, this implies the convergence in* $L^2(\Omega)$.

Proof:

Set $\Delta = (s,\, t]$. Applying the results of Section 2, we have

(16)

$$L_\varepsilon = \sum_{n=1}^{\infty} \frac{(-1)^n}{2\pi\, 2^n} \sum_{\ell+p=n} \frac{1}{\ell!\, p!} \int_{0<s<t<1} \frac{I_{2\ell}^1\left(1_\Delta^{\otimes 2\ell}\right) I_{2p}^2\left(1_\Delta^{\otimes 2p}\right)}{(|\Delta| + \varepsilon)^{n+1}}\, ds\, dt,$$

where $I_{2\ell}^1$ and I_{2p}^2 denote, respectively, the multiple stochastic integrals with respect to the Brownian motions W^1 and W^2. When n varies the terms appearing in the above sum are orthogonal. The square of the L^2-norm of the nth term is given by

$$\frac{1}{(2\pi)^2\, 2^{2n}} \sum_{\ell+p=n} \frac{1}{(\ell\,!)^2\,(p\,!)^2}$$
$$\int_{\substack{s<t\\u<v}} \frac{E\left[I^1_{2\ell}(1_\Delta)\, I^1_{2\ell}(1_{\Delta^*})\right]\, E\left[I^2_{2p}(1_\Delta)\, I^2_{2p}(1_{\Delta^*})\right]}{\left[(|\Delta|+\epsilon)\,(|\Delta^*|+\epsilon)\right]^{n+1}}\, ds\, dt\, du\, dv,$$

where $\Delta^* = (u, v]$. We can estimate this term by

$$\frac{(2n)\,!}{(2\pi)^2 2^{2n}(n\,!)^2} \sum_{\ell+p=n} \left(\frac{n\,!}{\ell\,!p\,!}\right)^2 \int_{\substack{s<t\\u<v}} \frac{(2\ell)\,!(2p)\,!\langle 1_\Delta, 1_{\Delta^*}\rangle^{2n}}{(2n)\,!\left(|\Delta|\,|\Delta^*|\right)^{n+1}}\, ds\, dt\, du\, dv$$
$$= \frac{(2n)\,!}{(2\pi)^2 2^{2n}\,(n\,!)^2} \left[\sum_{\ell+p=n} \frac{\binom{n}{\ell}\binom{n}{p}}{\binom{2n}{2\ell}}\right] \int_{\substack{s<t\\u<v}} \frac{|\Delta\cap\Delta^*|^{2n}}{\left(|\Delta|\,|\Delta^*|\right)^{n+1}}\, ds\, dt\, du\, dv\,.$$

Observe that

$$\sum_{\ell+p=n} \frac{\binom{n}{\ell}^2}{\binom{2n}{2\ell}} \le (n+1) \max_{0\le\ell\le n} \frac{\binom{n}{\ell}^2}{\binom{2n}{2\ell}} \le n+1\,. \tag{17}$$

On the other hand we claim that

$$\int_{\substack{s<t\\u<v}} \frac{|(s,t]\cap(u,v]|^{2n}}{(t-s)^{n+1}\,(v-u)^{n+1}}\, ds\, dt\, du\, dv \le \frac{3}{n^2}. \tag{18}$$

In order to show (18) we will decompose the integral by considering the different positions of s, t, u and v. We have that the left hand side of (18) is equal to

$$2\int_{u<s<v<t} \frac{(v-s)^{2n}}{(t-s)^{n+1}\,(v-u)^{n+1}}\, ds\, dt\, du\, dv + 2\int_{u<s<t<v} \frac{(t-s)^{n-1}}{(v-u)^{n+1}}\, ds\, dt\, du\, dv. \tag{19}$$

The second summand in (19) can be estimated as follows

$$\frac{2}{n} \int_{u<s<v} \frac{(v-s)^n}{(v-u)^{n+1}}\, du\, ds\, dv = \frac{1}{n\,(n+1)} \le \frac{1}{n^2}.$$

For the first term, we have

$$\begin{aligned}
&\frac{2}{n}\int_{s<v<t}\frac{(v-s)^n}{(t-s)^{n+1}}\,ds\,dt\,dv-\frac{2}{n}\int_{s<v<t}\frac{(v-s)^{2n}}{(t-s)^{n+1}\,v^n}\,ds\,dv\,dt\\
&=\frac{1}{n\,(n+1)}-\frac{2}{n^2}\int_{s<v}\frac{(v-s)^{2n}}{(1-s)^n\,v^n}\,ds\,dv+\frac{2}{n^2}\int_{s<v}\frac{(v-s)^n}{v^n}\,ds\,dv\\
&=\frac{1}{n\,(n+1)}+\frac{2}{n^2}\int_{s<v}\frac{(v-s)^n}{v^n}\left[1-\frac{(v-s)^n}{(1-s)^n}\right]ds\,dv\\
&\le\frac{1}{n\,(n+1)}+\frac{1}{n^2}\le\frac{2}{n^2},
\end{aligned}$$

which completes the proof of (18). Therefore the square of the L^2 norm of each term of (16) can be estimated by

$$\frac{(2n)!}{(2\pi)^2 2^{2n}\,(n!)^2}\,\frac{3(n+1)}{n^2},$$

which is equivalent to a constant times $n^{-3/2}$. Then Lemma 1.2 allows to complete the proof of the theorem. ■

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Rebut el 2 de Març de 1992