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LINEAR TRANSFORMATIONS OF TWO INDEPENDENT BROWNIAN MOTIONS AND ORTHOGONAL DECOMPOSITIONS OF BROWNIAN FILTRATIONS

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Abstract

Brownian motions defined as linear transformations of two independent Brownian motions are studied, together with certain orthogonal decompositions of Brownian filtrations.

1. Introduction

Let $W = (W_t)_{t\geq 0}$ and $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ be two independent Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider semimartingales of the form

$$dX_t = dW_t + Y_t\,dt,$$

where Y_t is linear in W and $\tilde{W}$, and we investigate when X is again a Brownian motion (relative to its own filtration). Our purpose is to extend in this setup involving W and $\tilde{W}$, some results about non-canonical representations of Brownian motion, in terms of only *one* Brownian motion (W_t). In Section 2, we shall consider the special case

$$Y_t = f(t)\tilde{W}_t + g(t)W_t, \tag{1}$$

where f and g are two continuous functions satisfying some integral conditions. We characterize those cases where X is a Brownian motion. This extends a result of Deheuvels [**3**] for $f \equiv 0$. In addition, we extend results of Jeulin-Yor [**6**] concerning the distribution of the process

$$W_t - \nu \int_0^t \frac{W_s}{s}\,ds. \tag{2}$$

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For $\nu \notin [0,1]$, we show that the law of (2) coincides with that of

$$W_t \pm \sqrt{\nu^2 - \nu} \int_0^t \frac{\tilde{W}_s}{s}\, ds. \tag{3}$$

In the case $\nu = 1$, it is well-known that the process in (2) is a Brownian motion, and then the representation (2) is a so-called non-canonical representation of Brownian motion. Many studies about such representations have been made, starting with Lévy [**8**], [**9**]; see, e.g. [**5**] for many references; also [**2**], [**7**] . . . . In a somewhat different vein, de Chávez [**12**] considers some processes obtained from Brownian motion by "formal" Girsanov transformation, i.e.,

$$W_t - \int_0^t \frac{d\langle W, D\rangle_s}{D_s},$$

where (D_t) is a martingale, but is not necessarily integrable, nor positive. In the last part of Section 2, we examine such examples, only to conclude that the situation is quite different from that of the preceding non-canonical representations of Brownian motions, which nonetheless preserve the Gaussian structure.

In Section 3, we characterize Brownian motions X of the form

$$X_t = W_t + \int_0^t (f(u)\tilde{W}_u + g(u)X_u)\, du, \tag{4}$$

with $X_0 = 0$, where f and g satisfy some integrability conditions. Following [**13**] we describe in Section 3.1 a basic orthogonal decomposition of the Brownian filtration. In Section 3.2 we will construct a Brownian motion X of the form (4), and then another Brownian motion Y which is represented in terms of W and $\tilde{W}$ and is independent of X. Using iteration, we get two sequences of Brownian motions $X^{(n)}$ and $Y^{(n)}$, which are independent of each other. This leads to the construction of new orthogonal decompositions of Brownian filtrations. In Section 3.3 a similar decomposition of a Brownian motion related to X will be investigated.

2. A class of linear transformations of two Brownian motions

Let (W_t) and $(\tilde{W}_t)$ be two independent Brownian motions and let $T \in \mathbb{R}_+ \cup \{+\infty\}$. Denote $\mathcal{A}(0,T)$ the set of all measurable functions $\varphi\colon (0,T) \to \mathbb{R}$ satisfying

$$\int_0^t \sqrt{u}|\varphi(u)|\, du < \infty,$$

for all $t < T$. Deheuvels [**3**] has shown that if $(W_t)_{t\geq 0}$ is a Brownian motion and if g belongs to $\mathcal{A}(0,\infty)$, then the process $(X_t)_{t\geq 0}$ defined by

$$X_t = W_t + \int_0^t g(u) W_u \, du,$$

is again a Brownian motion if and only if $g(t) \equiv 0$ or $g(t) = -1/t$. We would like to generalize this result with two functions f and g in $C(0,\infty) \cap \mathcal{A}(0,\infty)$, by considering the process X given by

$$X_t = W_t + \int_0^t (f(u)\tilde{W}_u + g(u) W_u) \, du, \tag{5}$$

and asking for which functions f and g X is again a Brownian motion.

Theorem 2.1. *Denote*

$$U_t = \int_0^t \frac{W_s}{s} \, ds \quad \textit{and} \quad \tilde{U}_t = \int_0^t \frac{\tilde{W}_s}{s} \, ds,$$

then

(a) *for any $\nu \in \mathbb{R}$,*

$$\{W_t - \nu U_t;\, t \geq 0\} \overset{(law)}{=} \{W_t - (1-\nu) U_t;\, t \geq 0\},$$

(b) *for $\nu \notin [0,1]$,*

$$\{W_t - \nu U_t;\, t \geq 0\} \overset{(law)}{=} \{W_t \pm \sqrt{\nu^2 - \nu}\,\tilde{U}_t;\, t \geq 0\},$$

(c) *if the functions $f, g \in C(0,\infty) \cap \mathcal{A}(0,\infty)$, then the process $(X_t)_{t\geq 0}$ given by (5) is a Brownian motion if and only if $f(t) = \pm\sqrt{\nu - \nu^2}/t$ and $g(t) = -\nu/t$, for some $\nu \in [0,1]$; in particular, both processes*

$$X_t^{\pm} := W_t - \int_0^t \left(\frac{\nu}{s} W_s \pm \frac{\sqrt{\nu - \nu^2}}{s} \tilde{W}_s \right) ds, \tag{6}$$

are Brownian motions.

Proof: Concerning (a) and (b), we only need to check that the covariances on both sides are equal.

(i) The proof of (a) can be found in [**6**].

(ii) The covariance of (U_t) or $(\tilde{U}_t)$ is given by

$$\begin{aligned}
E[U_sU_t] &= E[\tilde{U}_s\tilde{U}_t] \\
&= E\left[\left(\int_0^s \frac{W_u}{u}\,du\right)^2\right] + E\left[\left(\int_0^s \frac{W_u}{u}\,du\right)\left(\int_s^t \frac{W_v}{v}\,dv\right)\right] \\
&= 2s + s\log(t/s),
\end{aligned}$$

for $s < t$. The covariance of Γ, which denotes here either the right-hand side, or left-hand side in (b), is equal to

$$E(\Gamma_s\Gamma_t) = s + (\nu^2 - \nu)\varphi(s,t),$$

where $\varphi(s,t) = 2s + s\log(t/s)$ is indeed the covariance of U_t or $\tilde{U}_t$.

(iii) Denote $Z_t = W_t + i\sqrt{\nu - \nu^2}\tilde{U}_t$, and $\Gamma_t = W_t - \nu U_t$. Therefore, essentially from the previous computations, we find

$$E(\Gamma_s\Gamma_t) = E(Z_sZ_t) = E(W_sW_t) - (\nu - \nu^2)E(\tilde{U}_s\tilde{U}_t).$$

Hence, the covariance of the process $(W_t - \nu U_t \pm \sqrt{\nu - \nu^2}\tilde{U}_t)_{t\ge 0}$ is

$$E(\Gamma_s\Gamma_t) + (\nu - \nu^2)E(\tilde{U}_s\tilde{U}_t) = s + (\nu^2 - \nu)(\varphi(s,t) - E[\tilde{U}_s\tilde{U}_t]) = s,$$

which implies that the processes $X^{\pm}$ are Brownian motions.

(iv) Conversely, since in [**3**] the case $f \equiv 0$ has been proved, here we may assume $f \not\equiv 0$. Suppose $(X_t)_{t\ge 0}$ is a Brownian motion, then from Lemma 2.3 in [**4**], we know that, for $s \le t$,

$$f(t)E(X_s\tilde{W}_t) + g(t)E(X_sW_t) = 0.$$

Due to (5) we can compute $E(X_s\tilde{W}_t)$ and $E(X_sW_t)$, which yields:

$$f(t)\int_0^s uf(u)\,du + g(t)\left(s + \int_0^s ug(u)\,du\right) = 0.$$

Taking derivatives with respect to s, we get

$$sf(s)f(t) + (1 + sg(s))g(t) = 0. \tag{7}$$

Since f is continuous, there exists a countable collection of disjoint component intervals $\{(a_i, b_i) : i \in \mathbb{N}\}$ in $(0,\infty)$, such that

$$f(t)\begin{cases} \neq 0, & \forall\, t \in \bigcup_{i=1}^{\infty}(a_i,b_i), \\ = 0, & \forall\, t \in (0,\infty)\setminus\bigcup_{i=1}^{\infty}(a_i,b_i). \end{cases}$$

Without loss of generality, we only need to look at the case: $f \in C(0,\infty) \cap \mathcal{A}(0,\infty)$, $f(t) \neq 0$ for all $t \in (a,b)$, and $f(t) \equiv 0$ on the set $(0,\infty) \setminus (a,b)$. Then for all $s,t \in (a,b)$, $s < t$, we can rewrite (7) as

$$sf(s) + \frac{g(t)}{f(t)}(1 + sg(s)) = 0,$$

which implies $g(t) = cf(t)$ for some nonzero constant c, for all $t \in (a,b)$. Taking this result in (7) it follows that

$$f(s) = -\frac{c}{(1+c^2)s},$$

which is nonzero on $\mathbb{R}^+$. Since f is continuous, we get $(a,b) = (0,\infty)$, which gives the results. ☐

Now, we look for some extension of the third assertion in Theorem 2.1 to the n-dimensional case:

Theorem 2.2. *Let $(W_t)_{t\geq 0}$ be an n-dimensional Brownian motion and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^{n-1}$ $(\subset \mathbb{R}^n)$, i.e., $\sum_{j=1}^n \alpha_j^2 = 1$. Then the ($n$-dimensional) process (W_t^α), defined as*

$$W_t^\alpha := W_t - \left(\int_0^t l_\alpha(W_s)\frac{ds}{s}\right)\alpha, \tag{8}$$

is an n-dimensional Brownian motion, where $l_\alpha(x) = \sum_{j=1}^n \alpha_j x_j$ with $x = (x_1, x_2, \ldots, x_n)$. Moreover, for each $t > 0$, the process $(W_s^\alpha;\, s \leq t)$ is independent of $l_\alpha(W_t)$.

Proof: We only need to complete α into an orthonormal basis $\{\alpha, \beta_1, \ldots, \beta_{n-1}\}$ of $\mathbb{R}^n$. Then $(l_\alpha(W_t), l_{\beta_1}(W_t), \ldots, l_{\beta_{n-1}}(W_t))_{t\geq 0}$ is an n-dimensional Brownian motion, and

$$l_\alpha(W_t^\alpha) := l_\alpha(W_t) - \int_0^t l_\alpha(W_s)\frac{ds}{s},$$

whereas: for all $\beta \perp \alpha$,

$$l_\beta(W_t^\alpha) := l_\beta(W_t),$$

which proves the first assertion. The second assertion follows directly from $E[W_s^\alpha \cdot l_\alpha(W_t)] = 0$ for all $t > 0$ and for all $s \leq t$. ☐

Remark 2.1. The second assertion of this theorem can be seen as an extension of Theorem 1.1 in [**13**].

Remark 2.2. We look at the special case: $n = 2$. Using the third assertion of Theorem 2.1 we already knew that each component of W^α is a Brownian motion, since

$$l_\alpha(W_s) = \sqrt{\nu} W_s^{(1)} + \sqrt{1-\nu} W_s^{(2)},$$

for some $\nu \in [0,1]$. It is also not difficult to check the independence of each component of W_s^α.

Proposition 2.1. *The processes $Y^\pm$ defined by*

$$Y_t^\pm := \tilde{W}_t - \int_0^t \left(\frac{1-\nu}{s} \tilde{W}_s \pm \frac{\sqrt{\nu - \nu^2}}{s} W_s \right) ds,$$

are Brownian motions; each of them is independent of the corresponding process $(X_t^\pm)$ as defined in (6).

Proof: If we exchange the roles of W and $\tilde{W}$ in the processes $X^\pm$, we obtain that the resulting processes are still Brownian motions. Thus, we see that the processes $Y^\pm$ are Brownian motions. It remains to check $E[X_t^+ Y_s^+] = E[X_s^+ Y_t^+] = 0$ for all $s \le t$, and the same for X^- and Y^-. This follows from direct computation. □

Now, we discuss questions concerning random predictable integrands $\eta(s, W) \in \{-1, +1\}$, where W is a one-dimensional Brownian motion. We consider the process

$$W_t^\eta := W_t - \int_0^t \eta(s, W) \left(\int_0^s \eta(u, W)\, dW_u \right) \frac{ds}{s}.$$

This process $(W_t^\eta)_{t \ge 0}$ may be thought of as the Girsanov transform of the original Brownian motion W when one changes the original probability with the "density"

$$D_t \equiv h\left(t, \int_0^t \eta(u, W)\, dW_u\right),$$

where

$$h(t, x) = \frac{1}{\sqrt{t}} \exp\left(\frac{x^2}{2t}\right), \quad (t > 0).$$

Note that $h(t, x)$ is space-time harmonic, hence $(D_t)_{t>0}$ is a positive local martingale. However, $E[D_t] = \infty$, for every $t > 0$, so that our "application" of Girsanov theorem can only be formal (for another interesting formal Girsanov transformation, see the appendix in [**1**], and also [**14**]). Nonetheless, in the case $\eta(u, W) \equiv 1$, this formal transform indeed yields the Brownian motion

$$W_t^\star := W_t - \int_0^t \frac{W_u}{u}\, du.$$

More generally, for any deterministic Borel function $\eta(s)$ taking values only in $\{-1, +1\}$, the process $(W_t^\eta)_{t\geq 0}$ is a Brownian motion. Indeed, with our previous notation, we find

$$\int_0^t \eta(s)\, dW_s^\eta = \left(\int_0^\cdot \eta(s)\, dW_s\right)_t^\star.$$

Hence, the left-hand side is a Brownian motion, and so is $(W_t^\eta)_{t\geq 0}$. Now, we ask whether, when $\eta(s, W)$ is a random predictable integrand taking values in $\{-1, +1\}$, W^η is still a Brownian motion. The previous arguments, in case η is deterministic, do not apply, since we should expect a priori that the filtration of W^η is strictly smaller than that of W. Indeed, we also ask whether, for a fixed t, $\mathcal{F}_t^\eta = \sigma\{W_s^\eta,\, s \leq t\}$ is independent of the variable $\int_0^t \eta(s, W)\, dW_s$. The answers to both questions are "no" in general.

Concerning the first question, remark that, if W^η were a Brownian motion, then we would obtain

$$E\left[\eta(s, W) \int_0^s \eta(u, W)\, dW_u\right] = 0, \quad ds\text{-a.s.} \tag{9}$$

However, this condition cannot be satisfied for

$$\eta(s, W) := I_{(s\leq t_0)}(s) + \operatorname{sign}(W_{t_0}) I_{(t_0,\infty]}(s),$$

for $t_0 \in (0, \infty)$, since, for this process η, the left-hand side of (9) converges, as $s \downarrow t_0$, towards:

$$E[\operatorname{sign}(W_{t_0}) W_{t_0}] = E[|W_{t_0}|] > 0.$$

We have also considered the case where $\eta(s, W) = \operatorname{sign}(W_s) := \sigma_s$. In this case, we write $(\bar{W}_t)$ for this W^η, and we have

$$\bar{W}_t = W_t - \int_0^t \frac{C_s}{s}\, ds,$$

where $C_s = W_s - \sigma_s L_s$ with L_s, the local time at 0 of W_s. And we have remarked that assuming: $E[(\bar{W}_t)^2] = t$ is equivalent to:

$$E\left[|W_1| L_1 \int_0^1 \Phi\left(\sqrt{\frac{v}{1-v}}|W_1|\right)\, dv\right] = E\left[L_1^2 \int_0^1 \Phi\left(\sqrt{\frac{v}{1-v}}|W_1|\right)\, dv\right], \tag{10}$$

with Φ, the distribution function of $|N|$, where $N \sim \mathcal{N}(0,1)$. In fact, with the help of the auxiliary variable N^2, now assumed to be independent of the pair (W_1, L_1), (10) is equivalent to

$$(10') \qquad E\left[\frac{|W_1|^3 L_1}{W_1^2 + N^2}\right] = E\left[\frac{W_1^2 L_1^2}{W_1^2 + N^2}\right].$$

Since the density function of $(|W_1|, L_1)$ is $\sqrt{2/\pi}(x+y)\exp(-(x+y)^2/2)$, we need only to check whether

$$E\int_0^\infty \int_0^\infty \frac{x^2 y(x^2 - y^2)}{N^2 + x^2} \exp\left(-\frac{(x+y)^2}{2}\right) dx\, dy \stackrel{?}{=} 0$$

holds, but F. Petit obtained that the left-hand side is equal to $\sqrt{\frac{\pi}{2}}(2 - 3\log 2)$, hence not equal to 0. This implies that the process $(\bar{W}_t)_{t\geq 0}$ is not a Brownian motion.

Now, concerning the second question, let us consider the particular case: $\eta(s, W) = z_{g_s}$, where $g_s := \sup\{u \leq s : W_u = 0\}$ and $(z_v)_{v\geq 0}$ is predictable, valued in $\{-1, +1\}$. Then the balayage formula (see, for example, [**11**]) yields

$$\eta(s, W)W_s = \int_0^s \eta(u, W)\, dW_u,$$

so that, in this case:

$$W_t^\eta = W_t - \int_0^t W_s \frac{ds}{s} \equiv W_t^\star.$$

If the independence were true, we could get

$$E[W_t^\eta(\eta(t, W)W_t)] = 0,$$

from which we deduce, by differentiating

$$(11) \qquad E[\eta(t, W)] = \frac{1}{t} E[\eta(t, W)W_t^2].$$

Now, using $\eta(s, W) = z_{g_s}$ and conditioning on $\mathcal{F}_{g_t}$, we obtain, from (11)

$$E[z_{g_t}] = \frac{2}{t} E[z_{g_t}(t - g_t)].$$

But, this certainly cannot hold in this generality (i.e. for all z predictable, $z \in \{-1, +1\}$), since it would imply the absurd result that $2(1-(g_t/t)) = 1$.

3. An orthogonal decomposition of Brownian filtrations

3.1. The basic example of an orthogonal decomposition.

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. In [**6**] and Chapter 1 of [**13**] it has been shown that the natural filtration generated by $(B_t)_{t\geq 0}$ can be decomposed into the direct sum of two independent σ-algebras

$$\mathcal{F}_t^B = \mathcal{G}_t \oplus \sigma(B_t) \tag{12}$$

for all $t \geq 0$, where the σ-algebra $\mathcal{G}_t$ is given by

$$\mathcal{G}_t := \sigma\left(B_u - \int_0^u \frac{B_v}{v}\,dv;\, u \leq t\right).$$

Define an operator T (acting on Brownian trajectories) by

$$T(B)_t := B_t - \int_0^t \frac{B_u}{u}\,du. \tag{13}$$

It has been established that the process $(T(B)_t)_{t\geq 0}$ is a Brownian motion; see [**3**] and Chapter 1 in [**13**]. For the sake of convenience, we write $T^0(B) = B$ and $T^n(B) := T(T^{n-1}(B))$ for $n \geq 1$. Consequently, for any non-negative integer n, the process $T^n(B)$ is a Brownian motion relative to its natural filtration. Using this notation we can rewrite the decomposition (12) as

$$\mathcal{F}_t^B = \sigma(B_t) \oplus \sigma(T(B)_u;\, u \leq t) = \sigma(B_t) \oplus \mathcal{F}_t^{T(B)}.$$

Using the same argument as above iteratively, we can get an orthogonal decomposition of the σ-algebra $\mathcal{F}_t^B$ in the following form:

$$\mathcal{F}_t^B = \sigma(B_t) \oplus \sigma(T(B)_t) \oplus \cdots \oplus \sigma(T^n(B)_t) \oplus \mathcal{F}_t^{T^{n+1}(B)}. \tag{14}$$

Given the independence of $B_t, T(B)_t, \ldots, T^n(B)_t$ and of the σ-field $\mathcal{F}_t^{T^{n+1}(B)}$, and also that $\mathcal{F}_t^{T^{n+1}(B)}$ decreases, as $n \to \infty$, to the trivial σ-field, (see [**6**]), one can conclude from (14) that

$$\mathcal{F}_t^B = \bigoplus_{n=0}^{\infty} \sigma\left(T^n(B)_t\right). \tag{15}$$

These remarks have already been made in [**6**].

Remark 3.1. Since $(t^{-\frac{1}{2}} T^n(B)_t)_{n\in\mathbb{N}}$ is an orthonormal system in $L^2(\mathbb{P})$, we conclude that for fixed $t \geq 0$, the sequence $(T^n(B)_t)_{n\geq 0}$ is not strongly L^2-convergent, but converges weakly to 0 in L^2.

Remark 3.2. In the following, we shall use again the sequence $\{T^n\}$ of the iterates of T, but this time with respect to two Brownian motions.

3.2. Construction of orthogonal decompositions of Brownian filtrations.

Let W, $\tilde{W}$ be two independent Wiener processes. Consider the process X satisfying the stochastic differential equation

$$dX_t = dW_t + (f(t)\tilde{W}_t + g(t)X_t)\,dt, \tag{16}$$

where f and g are continuously differentiable functions on $(0,1)$. In Corollary 5.2 in [**4**] we saw that the solution of (16) cannot be a Brownian motion, provided that f and g satisfy

$$\int_0^t uf^2(u)\,du < \infty \quad \text{and} \quad \int_0^t ug^2(u)\,du < \infty \tag{17}$$

for all $t < 1$. However, if we release condition (17) to $f, g \in \mathcal{A}(0,1)$, we get the following result.

Proposition 3.1. *For any constant c such that* $0 \le |c| < 1$ *the process* X *satisfying the stochastic differential equation*

$$dX_t = dW_t + \frac{c\tilde{W}_t - c^2 X_t}{(1-c^2)t}\,dt, \tag{18}$$

with $X_0 = 0$*, is a Brownian motion with respect to its own filtration* $(\mathcal{F}_t^X)$*.*

Proof: For the case $c = 0$, it is clear, since $X = W$ is a Brownian motion. If $c \neq 0$, the solution to (18) is given by

$$X_t = \int_0^t \left(\frac{u}{t}\right)^a dW_u + \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) d\tilde{W}_u, \tag{19}$$

with the constant a defined by

$$a := \frac{c^2}{1-c^2}. \tag{20}$$

Applying (19) we get $E[X_s X_t] = s$ for $s \le t$. This ensures that the Gaussian process X is a standard Brownian motion with respect to its own filtration. □

In the following discussion of (18), we will always exclude the trivial case $c = 0$.

Now, we want to construct a new Brownian motion from W and $\tilde{W}$ which is independent of X. Our first attempt is a Brownian motion, say $\tilde{Y}$, of the form

$$d\tilde{Y}_t = d\tilde{W}_t + \frac{-cW_t - c^2\tilde{Y}_t}{(1-c^2)t}\,dt.$$

We can easily check that for any given t the variables X_t and $\tilde{Y}_t$ are independent. But the processes X and $\tilde{Y}$ are not independent. Hence we have to look for other Brownian motions which might be independent of X. The following proposition gives us one example.

Proposition 3.2. *The process Y satisfying the stochastic differential equation*

$$dY_t = d\tilde{W}_t + \frac{cW_t - (1-c^2)\tilde{W}_t - c^2 Y_t}{(1-c^2)t}\, dt, \tag{21}$$

is a standard Brownian motion independent of X.

Proof: The solution to (21) is given by

$$Y_t = \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) dW_u - \int_0^t \left(\frac{1}{a} - \frac{a+1}{a}\left(\frac{u}{t}\right)^a\right) d\tilde{W}_u. \tag{22}$$

For $s \le t$, it can be shown that $E[Y_s Y_t] = s$. It means that Y is a Brownian motion. Furthermore, we have $E[X_s Y_t] = E[X_t Y_s] = 0$, for all $s \le t$. This implies that X and Y are independent. □

Remark 3.3. At the beginning of Section 3.1 we saw that if $\tilde{W}$ is a Brownian motion, the process $(T(\tilde{W})_t)_{t\ge 0}$ is a Brownian motion and its natural filtration $(\mathcal{F}_t^{T(\tilde{W})})$ is strictly smaller than $(\mathcal{F}_t^{\tilde{W}})$. Using this notation, (21) can be written in the form

$$dY_t = dT(\tilde{W})_t + \frac{cW_t - c^2 Y_t}{(1-c^2)t}\, dt.$$

Since W and $\tilde{W}$ are independent, the processes $T(\tilde{W})$ and W are therefore also independent. In the same way, we know that the process $(T(W)_t)_{t\ge 0}$ is a Brownian motion independent of $\tilde{W}$ as well as $T(\tilde{W})$, and that $\mathcal{F}_t^{T(W)} \subsetneqq \mathcal{F}_t^W$. Using again the same argument as in Proposition 3.2, we know that the process $\tilde{X}$ satisfying

$$d\tilde{X}_t = dT(W)_t + \frac{cT(\tilde{W})_t - c^2\tilde{X}_t}{(1-c^2)t}\, dt, \tag{23}$$

is a Brownian motion independent of Y. Looking at the processes X and $\tilde{X}$, we see that the equations (23) and (18) have the same form. Only the σ-algebras generated by the driving Brownian motions $T(W)$ and $T(\tilde{W})$ are strictly smaller than those generated by W and $\tilde{W}$, respectively. Hence, we get also $\mathcal{F}_t^{\tilde{X}} \subseteq \mathcal{F}_t^X$. Furthermore, we deduce $E[X_t\tilde{X}_s] = 0$, for all $s \le t$. This implies that the variable X_t is independent from $\mathcal{F}_t^{\tilde{X}}$, hence that $\mathcal{F}_t^{\tilde{X}} \subsetneqq \mathcal{F}_t^X$.

Iterating this procedure, we define

$$X_t^{(n)} := \int_0^t \left(\frac{u}{t}\right)^a dT^n(W)_u + \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) dT^n(\tilde{W})_u, \tag{24}$$

and

$$Y_t^{(n)} := \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) dT^n(W)_u + \int_0^t \left(\frac{u}{t}\right)^a dT^{n+1}(\tilde{W})_u, \tag{25}$$

for $n \geq 0$ and $0 < |c| < 1$. In other words, the processes $X^{(n)}$ and $Y^{(n)}$ satisfy the stochastic differential equations

$$dX_t^{(n)} = dT^n(W)_t + \frac{cT^n(\tilde{W})_t - c^2 X_t^{(n)}}{(1-c^2)t}\, dt, \tag{26}$$

and

$$dY_t^{(n)} = dT^{n+1}(\tilde{W})_t + \frac{cT^n(W)_t - c^2 Y_t^{(n)}}{(1-c^2)t}\, dt. \tag{27}$$

The next proposition gives us more information about these two sequences of stochastic processes.

Proposition 3.3. *For $n \geq 0$ and $t \geq 0$, we have*

$$X_t^{(n+1)} = T(X^{(n)})_t \quad \textit{and} \quad Y_t^{(n+1)} = T(Y^{(n)})_t.$$

Proof: From (24), (25), the definition of $T(X^{(n)})$, $T(Y^{(n)})$ and the stochastic Fubini Theorem, we get the desired results. □

Due to this proposition we can rewrite (26) and (27) as

$$T^n(X)_t = T^n(W)_t + \int_0^t \frac{cT^n(\tilde{W})_u - c^2 T^n(X)_u}{(1-c^2)u}\, du,$$

and

$$T^n(Y)_t = T^{n+1}(\tilde{W})_t + \int_0^t \frac{cT^n(W)_u - c^2 T^n(Y)_u}{(1-c^2)u}\, du.$$

Corollary 3.1. *For $f, g \in C(0,1) \cap \mathcal{A}(0,1)$, the expectations*

$$E\left[X_t^{(n+1)}\left(\int_0^1 f(u)\, dX_u^{(n)} + \int_0^1 g(u)\, dY_u^{(n)}\right)\right]$$

and

$$E\left[Y_t^{(n+1)}\left(\int_0^1 f(u)\, dX_u^{(n)} + \int_0^1 g(u)\, dY_u^{(n)}\right)\right]$$

are equal to 0 for all $t \leq 1$ if and only if f and g are constant.

Proof: Without loss of generality, we may assume $n = 0$. From

$$E\left[T(X)_t \int_0^1 f(u)\, dX_u\right] = \int_0^t f(u)\, du - \int_0^t \frac{1}{u} \int_0^u f(v)\, dv\, du,$$

we see that this expectation is equal to 0 if and only if f is a constant. In the same way, we get

$$E\left[T(Y)_t \int_0^1 g(u)\, dY_u\right] = 0 \quad \text{if and only if} \quad g \text{ is constant},$$

and this completes the proof. □

From Section 3.1 we know that the processes $(T^n(X)_t)_{n\geq 0}$ and $(T^n(Y)_t)_{n\geq 0}$ do not converge strongly in L^2, but converges weakly to 0 in L^2. For each $n \geq 0$ the processes $X^{(n)}$ and $Y^{(n)}$ are Brownian motions and

$$\mathcal{F}_t^X = \mathcal{F}_t^{X^{(0)}} \supsetneq \mathcal{F}_t^{X^{(1)}} \supsetneq \cdots \supsetneq \mathcal{F}_t^{X^{(n)}} \supsetneq \cdots,$$

$$\mathcal{F}_t^Y = \mathcal{F}_t^{Y^{(0)}} \supsetneq \mathcal{F}_t^{Y^{(1)}} \supsetneq \cdots \supsetneq \mathcal{F}_t^{Y^{(n)}} \supsetneq \cdots.$$

Explicitly, the orthogonal decompositions of the filtrations generated by $(X_t)_{t\geq 0}$ and by $(Y_t)_{t\geq 0}$, respectively, are given by:

$$\mathcal{F}_t^X = \mathcal{F}_t^{X^{(0)}} = \bigoplus_{n=0}^{\infty} \sigma(X_t^{(n)}) \quad \text{and} \quad \mathcal{F}_t^Y = \mathcal{F}_t^{Y^{(0)}} = \bigoplus_{n=0}^{\infty} \sigma(Y_t^{(n)}),$$

for all $t > 0$. In addition, the processes $X^{(n)}$ and $Y^{(n)}$, $Y^{(n)}$ and $X^{(n+1)}$ are mutually independent. Now, we look at some more relations between the natural filtrations of $X^{(n)}$, $Y^{(n)}$, $T^n(W)$ and $T^n(\tilde{W})$.

Proposition 3.4. (i) *The filtration generated by $X^{(n)}$ and $Y^{(n)}$ is strictly smaller than that generated by $T^n(W)$ and $T^n(\tilde{W})$, i.e., for all $n \geq 0$ and $t \geq 0$,*

$$\mathcal{F}_t^{X^{(n)}} \oplus \mathcal{F}_t^{Y^{(n)}} \subsetneq \mathcal{F}_t^{T^n(W)} \oplus \mathcal{F}_t^{T^n(\tilde{W})}.$$

Moreover, we have

$$\mathcal{F}_t^{X^{(n+1)}} \oplus \mathcal{F}_t^{Y^{(n)}} \subsetneq \mathcal{F}_t^{T^n(W)} \oplus \mathcal{F}_t^{T^{n+1}(\tilde{W})}.$$

(ii) *For all $0 \leq n < m$, the σ-algebra $\mathcal{F}_t^{X^{(n)}} \oplus \mathcal{F}_t^{Y^{(n)}}$ is not contained in $\mathcal{F}_t^{T^m(W)} \oplus \mathcal{F}_t^{T^m(\tilde{W})}$. The same negative result is also true for the σ-algebras $\mathcal{F}_t^{X^{(n+1)}} \oplus \mathcal{F}_t^{Y^{(n)}}$ and $\mathcal{F}_t^{T^m(W)} \oplus \mathcal{F}_t^{T^{m+1}(\tilde{W})}$.*

(iii) *For $|c| < 1$ and $t \geq 0$,*

$$\mathcal{F}_t^{X+cY} = \mathcal{F}_t^{W+c\tilde{W}}.$$

Proof: Here we show only the case

$$\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}} \subsetneq \mathcal{F}_t^W \oplus \mathcal{F}_t^{T(\tilde{W})}.$$

The general case can be proved by a similar method. We can easily check the inclusion

$$\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}} \subseteq \mathcal{F}_t^W \oplus \mathcal{F}_t^{T(\tilde{W})},$$

due to the definitions of $(X_t^{(1)})$ and $(Y_t^{(0)})$. Suppose the σ-algebras $\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}}$ and $\mathcal{F}_t^W \oplus \mathcal{F}_t^{T(\tilde{W})}$ coincide. Then we know from the above proposition that the random variable $X_t^{(0)}$ is independent of $\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}}$. It is therefore also independent of $\mathcal{F}_t^W \oplus \mathcal{F}_t^{T(\tilde{W})}$. But it is easy to compute

$$E[X_t^{(0)} T(\tilde{W})_t] = -c(1-c^2)t,$$

which obviously contradicts the above assumption. The second assertion follows by the same argument and the property $E[X_t^{(n)} T^{m-1}(W)_t] \neq 0$ for all $m > n$. The last statement follows directly from the relation:

$$X_t + cY_t = W_t + c\tilde{W}_t,$$

for all $t \geq 0$. □

In Chapter 1 in [**13**] it has been shown that

$$T^n(B)_1 = \int_0^1 L_n\left(\log\left(\frac{1}{u}\right)\right) dB_u,$$

for a sequence of orthonormal polynomials $(L_n(u))$ for the measure $e^{-u}\,du$ in $\mathbb{R}^+$. In the following we want to find an analogical argument for our two new sequences of Brownian motions $X^{(n)}$ and $Y^{(n)}$. We will show a representation of them as stochastic integrals with respect to W and $\tilde{W}$ and get that the corresponding integrands are no more orthonormal. Nonetheless, they have some interesting properties.

Proposition 3.5. *The processes $X^{(n)}$ and $Y^{(n)}$ can be represented as*

$$X_t^{(n)} = \int_0^t p^{(n)}\left(\log\frac{t}{u}\right) dW_u + \frac{1}{c}\int_0^t q^{(n)}\left(\log\frac{t}{u}\right) d\tilde{W}_u, \tag{28}$$

$$Y_t^{(n)} = \frac{1}{c}\int_0^t q^{(n)}\left(\log\frac{t}{u}\right) dW_u + \int_0^t p^{(n+1)}\left(\log\frac{t}{u}\right) d\tilde{W}_u, \tag{29}$$

where the functions $p^{(n)}(u)$ and $q^{(n)}(u)$ satisfy the recurrence relation

$$\gamma^{(n+1)}(u) = \gamma^{(n)}(u) - \int_0^u \gamma^{(n)}(v)\,dv,$$

with initial conditions $p^{(0)}(u) = e^{-au}$ *and* $q^{(0)}(u) = 1 - e^{-au}$. *More explicitly,* $p^{(n)}(u)$ *and* $q^{(n)}(u)$ *can be represented in the following form:*

$$p^{(n)}(u) = -\frac{1}{a}\sum_{k=0}^{n-1} r_k^n \frac{(-u)^k}{k!} + \left(\frac{a+1}{a}\right)^n e^{-au}, \tag{30}$$

and

$$q^{(n)}(u) = L_n(u) - p^{(n)}(u), \tag{31}$$

where the sequence (r_k^n) *satisfies the recurrence relation:*

$$\begin{cases} r_0^{n+1} = r_0^n + \left(\dfrac{a+1}{a}\right)^n, & \forall\, n \geq 0, \\ r_n^{n+1} = 1, & \forall\, n \geq 1, \\ r_k^{n+1} = r_k^n + r_{k-1}^n, & \forall\, 0 < k < n, \\ r_q^p \equiv 0, & \text{for } p \leq q, \end{cases} \tag{32}$$

and $(L_n(u))_{n\geq 0}$ *is the sequence of Laguerre polynomials given by*

$$L_n(u) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-u)^k}{k!}, \tag{33}$$

which is an orthonormal basis of polynomials for the measure $e^{-u}\,du$ *in* $\mathbb{R}^+$.

Proof: Let $p^{(0)}(u) = e^{-au}$ and $q^{(0)}(u) = 1 - e^{-au}$. From (24) and the stochastic Fubini Theorem (see, e.g., [**10**]) we have

$$\begin{aligned} X_t^{(n)} &= \int_0^t p^{(0)}\left(\log\frac{t}{u}\right) dT^n(W)_u + \frac{1}{c}\int_0^t q^{(0)}\left(\log\frac{t}{u}\right) dT^n(\tilde{W})_u \\ &= \int_0^t p^{(1)}\left(\log\frac{t}{u}\right) dT^{n-1}(W)_u + \frac{1}{c}\int_0^t q^{(1)}\left(\log\frac{t}{u}\right) dT^{n-1}(\tilde{W})_u \\ &= \cdots = \int_0^t p^{(n)}\left(\log\frac{t}{u}\right) dW_u + \frac{1}{c}\int_0^t q^{(n)}\left(\log\frac{t}{u}\right) d\tilde{W}_u, \end{aligned} \tag{34}$$

where

$$\gamma^{(k+1)}\left(\log\frac{t}{u}\right) = \gamma^{(k)}\left(\log\frac{t}{u}\right) - \int_u^t \frac{1}{v}\gamma^{(k)}\left(\log\frac{t}{v}\right) dv,$$

for $\gamma^{(k)} = p^{(k)}$ or $q^{(k)}$, and for all $k \geq 0$. Applying a change of variable, we obtain

$$\gamma^{(n+1)}(u) = \gamma^{(n)}(u) - \int_0^u \gamma^{(n)}(v)\,dv.$$

The relations (30) and (31) follow directly by induction. □

Remark 3.4. We can write the recurrence relation (32) as

$$r_m^n = \sum_{i_1=m}^{n-1} \sum_{i_2=m-1}^{i_1-1} \cdots \sum_{i_m=1}^{i_{m-1}-1} \sum_{i_{m+1}=0}^{i_m-1} \left(\frac{a+1}{a}\right)^{i_{m+1}},$$

for $n > m$, with initial conditions $r_{n-1}^n = 1$ and $r_0^n = a(\frac{a+1}{a})^n - a$.

Proposition 3.6. *Let m, n be nonnegative integers, then the sequences of functions $(p^{(n)}(u))_{n\geq 0}$ and $(q^{(n)}(u))_{n\geq 0}$ possess the following properties:*

(a) $$\int_0^\infty p^{(n)}(u)e^{-u}\,du = \begin{cases} 1-c^2, & n=0, \\ 0, & n\geq 1. \end{cases}$$

(b) $$\int_0^\infty q^{(n)}(u)e^{-u}\,du = \begin{cases} c^2, & n=0, \\ 0, & n\geq 1. \end{cases}$$

(c) $$\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}\,du = \frac{c^{2m}(1-c^2)}{1+c^2}, \text{ for all } m\geq 0.$$

(d) $$\int_0^\infty q^{(n)}(u)q^{(n+m)}(u)e^{-u}\,du = \begin{cases} \dfrac{2c^4}{1+c^2}, & m=0, \\ -\dfrac{c^{2(m+1)}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(e) $$\int_0^\infty p^{(n+m)}(u)q^{(n)}(u)e^{-u}\,du = \begin{cases} \dfrac{c^2(1-c^2)}{1+c^2}, & m=0, \\ -\dfrac{c^{2m}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(f) $$\int_0^\infty p^{(n)}(u)q^{(n+m)}(u)e^{-u}\,du = \begin{cases} \dfrac{c^2(1-c^2)}{1+c^2}, & m=0, \\ -\dfrac{c^{2(m+1)}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(g) $$\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}\,du + \frac{1}{c^2}\int_0^\infty q^{(n)}(u)q^{(n+m)}(u)e^{-u}\,du = \begin{cases} 1, & m=0, \\ 0, & m\neq 0. \end{cases}$$

Proof: Due to (34) and Lemma 3.5 we get the desired results. □

3.3. Some related decompositions.

Let us look at some further properties of the process X. Consider the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t := t\int_t^\infty \frac{1}{u}\, dX_u.$$

From Chapter 1 in [**13**] we see that this process Z is a Brownian motion. Furthermore, it is easy to check that X_t and Z_t are independent for any t, but that the processes X and Z are not. In this section we want to give a representation of Z in terms of W and $\tilde{W}$, and to compare it with the representation of X.

Lemma 3.1. *The process X defined via* (18) *satisfies*

$$Z_t = t\int_t^\infty \frac{dX_u}{u} = V_t^1(W) + V_t^2(\tilde{W}), \tag{35}$$

where

$$V_t^1(W) := (1-c^2)t\int_t^\infty \frac{dW_u}{u} - c^2 t^{-a}\int_0^t u^a\, dW_u,$$

$$V_t^2(\tilde{W}) := \frac{ct}{1-c^2}\int_t^\infty u^{-a-2}\int_0^u v^a\, d\tilde{W}_v\, du.$$

Proof: It follows from Itô's formula and (19) that

$$\begin{aligned}
t^{-a-1}\int_0^t u^a\, dW_u - s^{-a-1}\int_0^s u^a\, dW_u &= \int_s^t d\left(u^{-a-1}\int_0^u v^a\, dW_v\right)\\
&= \int_s^t \left(-(a+1)u^{-a-2}\int_0^u v^a\, dW_v\, du + u^{-1}\, dW_u\right) \qquad (36)\\
&= -\frac{1}{a}\int_s^t \frac{dW_u}{u} + \frac{a+1}{a}\int_s^t \frac{dX_u}{u} - \frac{a+1}{c}\int_s^t u^{-a-2}\int_0^u v^a\, d\tilde{W}_v\, du.
\end{aligned}$$

There exists a standard Brownian motion Γ such that

$$t^{-a-1}\int_0^t u^a\, dW_u = t^{-(a+1)}\Gamma_{\frac{1}{2a+1}t^{2a+1}}.$$

Applying the law of large numbers we get

$$\lim_{t\to\infty} t^{-(\frac{1}{2}r+\epsilon)}\Gamma_{t^r} = 0,$$

for any $\epsilon > 0$, hence

$$\lim_{t\to\infty} t^{-a-1} \int_0^t u^a \, dW_u = 0.$$

Letting t go to ∞, it follows that (36) can be written in the form (35). □

From the decomposition in the previous lemma, we can derive another representation for (35) and we will see that Z_t has the same form as X_t.

Proposition 3.7. *If the process X satisfies* (18)*, then*

$$Z_t = t\int_t^\infty \frac{dX_u}{u} = \int_0^t \left(\frac{u}{t}\right)^a dB_u + \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) d\tilde{B}_u, \tag{37}$$

where B and $\tilde{B}$ are two independent Brownian motions given by

$$B_t = -W_t + \int_0^t \int_u^\infty \frac{dW_v}{v}\, du \quad \textit{and} \quad \tilde{B}_t = -\tilde{W}_t + \int_0^t \int_u^\infty \frac{d\tilde{W}_v}{v}\, du.$$

Proof: The covariance function of $(V_t^1(W))_{t\geq 0}$ is given by

$$E[V_s^1(W)V_t^1(W)] = \left(\frac{1-c^2}{1+c^2}\right) s^{a+1}t^{-a}.$$

This is exactly the covariance function of the process $(\int_0^t (\frac{u}{t})^a \, dB_u)_{t\geq 0}$ for some standard Brownian motion B. Similarly, we have

$$E[V_s^2(\tilde{W})V_t^2(\tilde{W})] = s - \left(\frac{1-c^2}{1+c^2}\right) s^{a+1}t^{-a},$$

which coincides with the covariance function of $(\frac{1}{c}\int_0^t (1-(\frac{u}{t})^a)\, d\tilde{B}_u)_{t\geq 0}$ for some standard Brownian motion $\tilde{B}$. Since the processes $(V_t^1(W))$ and $(V_t^2(\tilde{W}))$ are independent and generate the same filtration respectively as B and $\tilde{B}$, B and $\tilde{B}$ are therefore independent. Hence, we get the representation (37). Furthermore, from

$$\int_0^t \left(\frac{u}{t}\right)^a dB_u = V_t^1(W) = (1-c^2)t\int_t^\infty \frac{dW_u}{u} - c^2 t^{-a}\int_0^t u^a \, dW_u,$$

and

$$\frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) d\tilde{B}_u = V_t^2(\tilde{W}) = \frac{ct}{1-c^2}\int_t^\infty u^{-a-2}\int_0^u v^a \, d\tilde{W}_v \, du,$$

we get the representations of B and $\tilde{B}$ in terms of W and $\tilde{W}$, respectively. □

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