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DISTRIBUTION-VALUED ITERATED GRADIENT AND CHAOTIC DECOMPOSITIONS OF POISSON JUMP TIMES FUNCTIONALS

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Abstract

We define a class of distributions on Poisson space which allows to iterate a modification of the gradient of [**1**]. As an application we obtain, with relatively short calculations, a formula for the chaos expansion of functionals of jump times of the Poisson process.

1. Introduction

Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with jump times $(T_k)_{k\geq 1}$, and $T_0=0$. The underlying probability space is denoted by $(\Omega,\mathcal{F},P)$, so that $L^2(\Omega,\mathcal{F},P)$ is the space of square-integrable functionals of $(N_t)_{t\in\mathbb{R}_+}$. Any $F\in L^2(\Omega,\mathcal{F},P)$ can be expanded into the series

$$F = E[F] + \sum_{n=1}^{\infty}\frac{1}{n!}I_n(f_n) \tag{1}$$

where $I_n(f_n)$ is the iterated stochastic integral

$$I_n(f_n) = n!\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\dots,t_n)\,d(N_{t_1}-t_1)\cdots d(N_{t_n}-t_n)$$

of the symmetric function $f_n\in L^2(\mathbb{R}_+^{\circ n})$ (stochastic integrals are taken in the Itô sense, thus diagonal terms have no influence in the above expression), with the isometry

$$\langle I_n(f_n), I_m(g_m)\rangle_{L^2(\Omega)} = n!1_{\{n=m\}}\langle f_n,g_m\rangle_{L^2(\mathbb{R}_+,dt)^{\circ n}},$$
$$f_n\in L^2(\mathbb{R}_+,dt)^{\circ n},\quad g_m\in L^2(\mathbb{R}_+,dt)^{\circ m}.$$

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If f_n is not symmetric we let $I_n(f_n) = I_n(\tilde{f}_n)$, where $\tilde{f}_n$ denotes the symmetrization of f_n in n variables, hence (1) can be written as

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n 1_{\Delta_n})$$

where

$$\Delta_n = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : 0 \le t_1 < \dots < t_n\}.$$

Let $D\colon L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$ denote the linear unbounded operator defined on multiple stochastic integrals as

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)), \quad \text{a.e. } t \in \mathbb{R}_+.$$

The formula of Y. Ito [**3**, Relations (7.4) and (7.5), pp. 26–27], allows in principle to compute f_n as

$$f_n(t_1, \dots, t_n) = E[D_{t_1} \cdots D_{t_n} F], \quad \text{a.e. } t_1, \dots, t_n \in \mathbb{R}_+.$$

Given the probabilistic interpretation of D as a finite difference operator (cf. [**3**] and [**6**]), we have for $F = f(T_1, \dots, T_d)$:

$$D_t F = \sum_{k=1}^{k=d} 1_{]T_{k-1}, T_k]}(t)(f(T_1, \dots, T_{k-1}, t, T_k, \dots, T_{d-1}) - f(T_1, \dots, T_d)), \quad t \in \mathbb{R}_+,$$

thus $D_{t_1} \cdots D_{t_n} F$ is well defined and explicit computations can be carried out but may be complicated due to the recursive application of a finite difference operator, cf. [**4**]. See [**8**] for an elementary approach using only orthogonal expansions in Charlier polynomials.

On the other hand, the gradient $\tilde{D}$ of [**1**] (see also [**2**]), defined as

$$\tilde{D}_t = -\sum_{k=1}^{k=d} 1_{[0, T_k]}(t) \partial_k f(T_1, \dots, T_d),$$

has some properties in common with D, namely its adapted projection coincides with that of D, and in particular we have

$$E[D_t F] = E[\tilde{D}_t F], \quad t \in \mathbb{R}_+.$$

Since the operator $\tilde{D}$ has the derivation property it is easier to manipulate than the finite difference operator D in recursive computations. Its disadvantage is that it can not be iterated in L^2 due to the non-differentiability of $1_{[0,T_k]}(t)$ in T_k, thus an expression such as $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ makes a priori no sense, moreover $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F]$ may differ from $E[D_{t_1} \cdots D_{t_n} F]$ for $n \ge 2$ (see Relations (13) and (14) below).

In [**7**] the combined use of D^n and $\tilde{D}$ in L^2 sense has led to the computation of the expansion of the jump time T_d, $d \geq 1$. A direct calculation using only the operator D can be found in [**5**], concerning a Poisson process on a bounded interval.

In this paper we show that the gradient $\tilde{D}$ can be iterated in a precise sense of distributions on Poisson space, to be introduced in Section 3. For example we have for $(t_1, \dots, t_n) \in \Delta_n$:

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) = (-1)^n f^{(n)}(T_d) 1_{[0,T_d]}(t_1 \vee \cdots \vee t_n) + (-1)^n 1_{\{0 \leq t_1, \dots, t_{n-1} < t_n\}} \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \delta_{t_n}^{(j-1)}(T_d),$$

where $\delta_{t_n}(T_d)$ is a generalized functional, i.e. the composition of the Dirac distribution δ_{t_n} at t_n with the jump time T_d, cf. Proposition 3, $f^{(n)}$ denotes the n-th derivative of the function $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$, and $t_1 \vee \cdots \vee t_n = \max(t_1, \dots, t_n)$. Moreover we obtain the equality

$$E[D_{t_1} \cdots D_{t_n} F \mid \mathcal{F}_a] = E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a], \qquad 0 \leq a < t_1 < \cdots < t_n, \quad n \geq 2,$$

where we make sense of the conditional expectation $E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \mid \mathcal{F}_a]$ using the pairing $\langle \cdot, \cdot \rangle$ between distributions and test functions. This implies

$$f_n(t_1, \dots, t_n) = E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F], \quad 0 < t_1 < \cdots < t_n.$$

This gives an expression for the decomposition of $f(T_1, \dots, T_n)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, with relatively short computations, cf. Proposition 6, for example

$$f(T_d) = \sum_{n=0}^{\infty} I_n(h_n 1_{\Delta_n}),$$

with

$$h_n(t_1, \dots, t_n) = E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)]$$
$$= (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{d-1}(t)\, dt + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n),$$

$0 < t_1 < \cdots < t_n$, where $p_{d-1}(t) = \frac{t^{d-1}}{(d-1)!} e^{-t}$, $t \in \mathbb{R}_+$, $d \geq 1$.

2. Integration by parts

In this section we review the definition of the three main gradient operators on Poisson space, and present an elementary derivation of integration by parts formulas. All $\mathcal{C}^\infty$ functions on Δ_d are extended by continuity to the closure of Δ_d.

Definition 1. Let $a \geq 0$. Let $\mathcal{S}_d(\Omega \times [a,\infty[^l)$ denote the test function space

$$\mathcal{S}_d(\Omega \times [a,\infty[^l) = \{h_1 \otimes \cdots \otimes h_l \otimes f(T_1,\ldots,T_d) : \\ f \in \mathcal{C}_b^\infty(\Delta_d),\ h_1,\ldots,h_l \in \mathcal{C}_b([a,\infty[)\},$$

with $\mathcal{S}_d(\Omega) = \mathcal{S}_d(\Omega \times \mathbb{R}_+^0)$ for $l = 0$.

We recall that if $f \in L^2(\Delta_d, e^{-t_d}\, dt_1 \cdots dt_d)$ then

$$E[f(T_1,\ldots,T_d)] = \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} f(t_1,\ldots,t_d)\, dt_1 \cdots dt_d,$$

which follows e.g. from the fact that $(\tau_n)_{n\geq 1} = (T_n - T_{n-1})_{n\geq 1}$ is a family of independent exponential random variables.

2.1. Intrinsic gradient.

The intrinsic gradient $\hat{D}$ on Poisson space is defined on $\mathcal{S}_d(\Omega)$ as

$$\hat{D}_t F = \sum_{k=1}^{k=d} 1_{\{T_k\}}(t)\partial_k f(T_1,\ldots,T_d), \quad dN_t\text{-a.e.},$$

with $F = f(T_1,\ldots,T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, where $\partial_k f$ represents the partial derivative of f with respect to its k-th variable, $1 \leq k \leq d$.

Lemma 1. *Let $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0) = 0$. We have the integration by parts formula*

$$(2) \quad E[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+,dN_t)}] = E\big[FU_{h'}^d\big] = -E\left[F\int_0^\infty h'(t)\, d(N_t - t)\right],$$

where $U_{h'}^d = -\big(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t)\, dt\big) \in \mathcal{S}_d(\Omega)$.

Proof: We have by integration by parts on Δ_d:

$$
\begin{aligned}
E[\langle \hat{D}F, h\rangle_{L^2(\mathbb{R}_+, dN_t)}] &= \sum_{k=1}^{k=d} \int_0^\infty \int_0^{t_d} \cdots \int_0^{t_2} e^{-t_d} h(t_k) \partial_k f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&= \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1) \partial_1 f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad + \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h(t_k) \frac{\partial}{\partial t_k} \int_0^{t_k} \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h(t_k) \int_0^{t_k} \int_0^{t_{k-2}} \cdots \int_0^{t_2} f(t_1, \ldots, t_{k-2}, t_k, t_k, \ldots, t_d) \\
&\qquad\qquad dt_1 \cdots d\hat{t}_{k-1} \cdots dt_d \\
&= - \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_1) f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad + \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_3} h(t_2) f(t_2, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_k) f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad + \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad + \sum_{k=2}^{k=d-1} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdots \int_0^{t_2} h(t_{k+1}) f(t_1, \ldots, t_{k-1}, t_{k+1}, t_{k+1}, \ldots, t_d) \\
&\qquad\qquad dt_1 \cdots d\hat{t}_k \cdots dt_d \\
&\quad - \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_k} \int_0^{t_{k-2}} \cdots \int_0^{t_2} h(t_k) f(t_1, \ldots, t_{k-2}, t_k, t_k, \ldots, t_d)\, dt_1 \cdots dt_d \\
&= - \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h'(t_k) f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&\quad + \int_0^\infty e^{-t_d} h(t_d) \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_d \\
&= -E\left[F \left(\sum_{k=1}^{k=d} h'(T_k) - \int_0^{T_d} h'(t)\, dt \right) \right],
\end{aligned}
$$

where $d\hat{t}_k$ denotes the absence of dt_k. Concerning the second part of the equality it suffices to notice that if $k > d$,

$$\begin{aligned}
&E[Fh'(T_k)] \\
&\quad = \int_0^\infty e^{-t_k} h'(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_k \\
&\quad = \int_0^\infty e^{-t_k} h(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_k \\
&\qquad - \int_0^\infty e^{-t_{k-1}} h(t_{k-1}) \int_0^{t_{k-1}} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)\, dt_1 \cdots dt_{k-1} \\
&\quad = E\big[F(h(T_k) - h(T_{k-1}))\big] = E\left[F \int_{T_{k-1}}^{T_k} h'(t)\, dt\right]. \qquad \square
\end{aligned}$$

Relation (2) implies immediately for $F, G \in \mathcal{S}_d(\Omega)$:

$$\begin{aligned}
&E[\langle \hat{D}F, hG\rangle_{L^2(\mathbb{R}_+, dN_t)}] \\
&\qquad = E\big[\langle \hat{D}(FG), h\rangle_{L^2(\mathbb{R}_+, dN_t)} - F\langle \hat{D}G, h\rangle_{L^2(\mathbb{R}_+, dN_t)}\big] \\
&\qquad = E\big[F(GU^d_{h'} - \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+, dN_t)})\big] \\
&\qquad = -E\left[F\left(G \int_0^\infty h'(t)\, d(N_t - t) + \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+, dN_t)}\right)\right].
\end{aligned}$$

2.2. Damped gradient.

Let $r(s,t) = -s \vee t$ denote the Green function associated to the Laplacian $\mathcal{L}$ on $\mathbb{R}_+$:

$$\mathcal{L}f = -f'', \quad f \in \mathcal{C}_c^\infty(]0, \infty[),$$

i.e. we have, with $g = -f''$:

$$\int_0^\infty r(s,t) g(t)\, dt = -\int_0^s \int_0^t g(u)\, du\, dt, \quad s \in \mathbb{R}_+.$$

Definition 2. Given $F \in \mathcal{S}_d(\Omega)$, $F = f(T_1, \ldots, T_d)$, we let

$$r^{(1)}(s,t) = \frac{\partial}{\partial s} r(s,t) = -1_{]-\infty, s]}(t), \quad s, t \in \mathbb{R}_+,$$

and

$$\tilde{D}_t F = \int_0^\infty r^{(1)}(s,t)\hat{D}_s F \, dN_s.$$

We have

$$\tilde{D}_t F = \sum_{k=1}^{k=d} r^{(1)}(T_k,t)\partial_k f(T_1,\ldots,T_d) = -\sum_{k=1}^{k=d} 1_{[0,T_k]}(t)\partial_k f(T_1,\ldots,T_d).$$

In fact $\tilde{D}$ is (up to a minor modification) the gradient introduced in [**1**]. This presentation of $\tilde{D}$ using the Green function $r(s,t)$ is motivated by [**9**].

Proposition 1. *We have for $F \in \mathcal{S}_d(\Omega)$ and $h \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$E[\langle \tilde{D}F, h\rangle_{L^2(\mathbb{R}_+,dt)}] = E\big[FU_h^d\big] = E\left[F\int_0^\infty h(t)\,d(N_t - t)\right], \tag{3}$$

where $U_h^d = \sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)\,dt$.

Proof: We have

$$\begin{aligned} E\big[\langle \tilde{D}F, h\rangle_{L^2(\mathbb{R}_+,dt)}\big] &= E\left[\int_0^\infty \int_0^\infty r^{(1)}(s,t)\hat{D}_s F h(t)\,dN_s\,dt\right] \\ &= -E\left[\Big\langle \hat{D}F, \int_0^{\cdot} h(t)\,dt\Big\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] \\ &= E\left[F\int_0^\infty h(t)\,d(N_t - t)\right]. \end{aligned}$$

□

Relation (3) also implies that for $F, G \in \mathcal{S}_d(\Omega)$,

$$\begin{aligned} &E[\langle \tilde{D}F, hG\rangle_{L^2(\mathbb{R}_+,dt)}] \\ &\quad = E\big[\langle \tilde{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dt)} - F\langle \tilde{D}G, h\rangle_{L^2(\mathbb{R}_+,dt)}\big] \\ &\quad = E\big[F(GU_h^d - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)})\big] \\ &\quad = E\left[F\left(G\int_0^\infty h(t)\,d(N_t - t) - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)}\right)\right]. \end{aligned} \tag{4}$$

2.3. Finite difference gradient.

For completeness we mention the gradient D which is associated to the Fock space structure, and whose properties have been discussed in the introduction.

3. Distribution-valued gradient

For $n \geq 2$ and $F \in \mathcal{S}_d(\Omega)$ we let $dN_{t_1} \otimes \cdots \otimes dN_{t_n}$-a.e.:

$$\hat{D}^n_{t_1,\ldots,t_n} F = \sum_{1 \leq j_1,\ldots,j_n \leq d} 1_{\{T_{j_1}\}}(t_1) \cdots 1_{\{T_{j_n}\}}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d).$$

This is not the n-th iteration of $\hat{D}$, in fact we have

$$\|\hat{D}^n F\|^2_{L^2(\mathbb{R}_+, dN_t)^{\otimes n}} = \sum_{1 \leq j_1,\ldots,j_n \leq d} (\partial_{j_1} \cdots \partial_{j_n} f(T_1, \ldots, T_d))^2.$$

Definition 3. Let $a \in \mathbb{R}_+$ and $l \in \mathbb{N}$.

i) We denote by $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ the space of continuous linear forms (distributions) on $\mathcal{S}_d(\Omega \times [a, \infty[^l)$, i.e. $F \in \mathcal{S}'_d(\Omega \times [a, \infty[^l)$ if there exists $k \geq 0$ and $C > 0$ such that

$$|\langle F, h_1 \otimes \cdots \otimes h_l \otimes G \rangle| \leq C \sum_{i=0}^{i=k} \|h_1 \otimes \cdots \otimes h_l\|_\infty \|\hat{D}^i G\|_{L^\infty(\Omega, L^2(\mathbb{R}_+, dN_t)^{\otimes i})},$$

$G \in \mathcal{S}_d(\Omega \times \mathbb{R}^l_+)$, $h_1, \ldots, h_l \in \mathcal{C}_c([a, \infty[)$.

ii) A sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}'_d(\Omega \times [a, \infty[^l)$ is said to converge in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ if the sequence $(\langle F_n, G \rangle)_{n \in \mathbb{N}}$ converges to $\langle F, G \rangle$ for all $G \in \mathcal{S}_d(\Omega \times [a, \infty[^l)$.

The notation $\langle \cdot, \cdot \rangle$ will be used to denote the pairing between $\mathcal{S}_d(\Omega \times [a, \infty[^l)$ and $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$, for all values of $l \in \mathbb{N}$. Every $F \in \mathcal{S}_d(\Omega \times [a, \infty[^l)$ is identified to an element of $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$ by letting

$$\langle F, G \rangle = \int_\Omega \langle F(\omega, \cdot), G(\omega, \cdot) \rangle_{L^2([a,\infty[, dt)^{\otimes l}} P(d\omega), \quad G \in \mathcal{S}_d(\Omega \times [a, \infty[^l).$$

The closability property in L^2 of the operator $\tilde{D}$ is a well-known statement which extends to distributions in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$.

Proposition 2. *Let $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}_d(\Omega \times [a, \infty[^l)$ such that*

i) *$(F_n)_{n \in \mathbb{N}}$ converges to 0 in $\mathcal{S}'_d(\Omega \times [a, \infty[^l)$,*

ii) *$(\tilde{D} F_n)_{n \in \mathbb{N}}$ converges in $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$.*

Then $(\tilde{D} F_n)_{n \in \mathbb{N}}$ converges to 0 in $\mathcal{S}'_d(\Omega \times [a, \infty[^{l+1})$.

Proof: For $l = 0$ this is a direct consequence of the integration by parts formula (4), which shows that

$$\langle \tilde{D} F_n, hG \rangle = E[F_n(G U^d_h - \langle \tilde{D} G, h \rangle_{L^2(\mathbb{R}_+, dt)})], \; h \in \mathcal{C}_c([a, \infty[), \; G \in \mathcal{S}_d(\Omega),$$

with $G U^d_h - \langle \tilde{D} G, h \rangle_{L^2(\mathbb{R}_+, dt)} \in \mathcal{S}_d(\Omega)$. The generalization to $l \geq 1$ is straightforward. □

This proposition justifies the following extension of $\tilde{D}$ to generalized functionals.

Definition 4. Let $a \in \mathbb{R}_+$. We let $\mathbb{D}_l([a,\infty[)$ denote the subspace of $F \in \mathcal{S}'_d(\Omega \times [a,\infty[^l)$ such that

i) there exists $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_d(\Omega \times [a,\infty[^l)$ that converges to F in $\mathcal{S}'_d(\Omega \times [a,\infty[^l)$,

ii) $(\tilde{D}F_n)_{n\in\mathbb{N}}$ converges in $\mathcal{S}'_d(\Omega \times [a,\infty[^{l+1})$.

Given F as above we define $\tilde{D}F$ as the limit in $\mathcal{S}'_d(\Omega \times [a,\infty[^{l+1})$

$$\tilde{D}F = \lim_{n\to\infty} \tilde{D}F_n, \quad F \in \mathbb{D}_l([a,\infty[) \subset \mathcal{S}'_d(\Omega \times [a,\infty[^l).$$

4. Iterated gradient in distribution sense

We let for $n \geq 2$:

$$r^{(n)}(T_k,t) = \partial_1^n r(T_k,t) = -\delta_t^{(n-2)}(T_k),$$

in distribution sense, i.e. $r^{(n)}(T_k,t)$ belongs to $\mathcal{S}'_d(\Omega)$ with for $k = 1,\ldots,d$, and $f \in \mathcal{C}_b^\infty(\Delta_d)$:

$$\begin{aligned}
&\langle r^{(n)}(T_k,t), f(T_1,\ldots,T_d)\rangle(-1)^{n+1}1_{\{k<d\}} \\
&\times\int_0^\infty e^{-s_d}\int_0^{s_d}\cdots\int_0^{s_{k+2}}\left(\frac{\partial^{n-2}}{\partial s_k^{n-2}}\int_0^{s_k}\cdots\int_0^{s_2} f(s_1,\ldots,s_d)\,ds_1\cdots ds_{k-1}\right)_{|s_k=t} ds_{k+1}\cdots ds_d \\
&+(-1)^{n+1}1_{\{k=d\}}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t\int_0^{s_{d-1}}\cdots\int_0^{s_2} e^{-t} f(s_1,\ldots,s_{d-1},t)\,ds_1\cdots ds_{d-1},
\end{aligned}$$

and for $n = 1$:

$$\begin{aligned}
&\langle r^{(1)}(T_k,t), f(T_1,\ldots,T_d)\rangle \\
&\qquad = -\int_0^\infty\int_0^{s_d}\cdots\int_0^{s_2} e^{-s_d}1_{[t,\infty[}(s_k) f(s_1,\ldots,s_d)\,ds_1\cdots ds_d.
\end{aligned}$$

Let $\phi \in \mathcal{C}_c^\infty([-1,1])$, $\phi \geq 0$, such that $\int_{-1}^1 \phi(t)\,dt = 1$, and let

$$\phi_\varepsilon(t) = \varepsilon^{-1}\phi(\varepsilon^{-1}t), \quad t \in \mathbb{R}, \quad \varepsilon > 0.$$

Let $\phi_\varepsilon * r^{(n)}(T_k,t)$, $n \geq 1$, denote the convolution of ϕ_ε with $r^{(n)}(T_k,t)$ in the first variable, i.e. for $n=1$:

$$\begin{aligned}\phi_\varepsilon * r^{(1)}(T_k,t) &= -\int_{-\infty}^{\infty} \phi_\varepsilon(u) 1_{]-\infty,T_k-u]}(t)\,du \\ &= -\int_{-\infty}^{\infty} \phi_\varepsilon(T_k+u) 1_{]-\infty,u]}(t)\,du, \quad s,t \in \mathbb{R}_+,\end{aligned}$$

and for $n \geq 2$, $l \in \mathbb{N}$:

$$\phi_\varepsilon^{(l)} * r^{(n)}(T_k,t) = \phi_\varepsilon * r^{(n+l)}(T_k,t) = -\phi_\varepsilon^{(n+l-2)}(T_k - t),$$

which converges in $\mathcal{S}'_d(\Omega)$ to $r^{(n+l)}(T_k,t)$ if $n+l \geq 1$ (i.e. to $-\delta_t^{(n+l-2)}(T_k)$ if $n+l \geq 2$), $k=1,\dots,d$. Let $t_1 \vee \cdots \vee t_n$, $t_1,\dots,t_n \in \mathbb{R}_+$.

Proposition 3. *Let $k \geq 1$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$. Then for all $n \geq 1$, $\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for a.a. $(t_1,\dots,t_n) \in \mathbb{R}_+^n$, and*

$$\begin{aligned}(5)\quad &\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n f^{(n)}(T_k) 1_{[0,T_k]}(t_1 \vee \cdots \vee t_n) \\ &+(-1)^n \sum_{j=1}^{n-1} 1_{\{t_1 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_n\}} f^{(n-j)}(t_{j+1} \vee \cdots \vee t_n) \delta^{(j-1)}_{t_{j+1} \vee \cdots \vee t_n}(T_k).\end{aligned}$$

Proof: For $n=1$ this is the definition of $\tilde{D}$. We proceed by induction, assuming that $\tilde{D}_{t_3} \cdots \tilde{D}_{t_{n+1}} f(T_k) \in \mathbb{D}_0(\mathbb{R}_+)$ for some $n \geq 1$, and

$$\begin{aligned}&\tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} f(T_k) = (-1)^n f^{(n)}(T_k) 1_{[0,T_k]}(t_2 \vee \cdots \vee t_{n+1}) \\ &+(-1)^n \sum_{j=1}^{n-1} 1_{\{t_2 \vee \cdots \vee t_{j+1} < t_{j+2} \vee \cdots \vee t_{n+1}\}} f^{(n-j)}(t_{j+2} \vee \cdots \vee t_{n+1}) \delta^{(j-1)}_{t_{j+2} \vee \cdots \vee t_{n+1}}(T_k).\end{aligned}$$

Let for $\varepsilon > 0$. We define a smooth approximation of $\tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} f(T_k)$ by letting

$$\begin{aligned}F_\varepsilon(t_2,\dots,t_{n+1}) &= (-1)^n f^{(n)}(T_k) \phi_\varepsilon * r^{(1)}(T_k, t_2 \vee \cdots \vee t_{n+1}) \\ &+ (-1)^n \sum_{j=1}^{n-1} 1_{\{t_2 \vee \cdots \vee t_{j+1} < t_{j+2} \vee \cdots \vee t_{n+1}\}} f^{(n-j)}(t_{j+2} \vee \cdots \vee t_{n+1}) \\ &\qquad \times \phi_\varepsilon * r^{(j+1)}(T_k, t_{j+2} \vee \cdots \vee t_{n+1}).\end{aligned}$$

Then $F_\varepsilon(t_2,\dots,t_{n+1}) \in \mathcal{S}_d(\Omega)$ and

$$\begin{aligned}
&\tilde{D}_{t_1} F_\varepsilon(t_2,\dots,t_{n+1})\\
&=(-1)^{n+1}1_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1}) f^{(n+1)}(T_k)\\
&\quad +(-1)^{n+1}1_{[0,T_k]}(t_1) f^{(n)}(T_k)\phi'_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1})\\
&\quad +(-1)^{n+1}1_{[0,T_k]}(t_1)\sum_{j=1}^{n-1} 1_{\{t_2\vee\cdots\vee t_{j+1}<t_{j+2}\vee\cdots\vee t_{n+1}\}} f^{(n-j)}(t_{j+2}\vee\cdots\vee t_{n+1})\\
&\quad \times\phi'_\varepsilon * r^{(j+1)}(T_k, t_{j+2}\vee\cdots\vee t_{n+1})\\
&=(-1)^{n+1}1_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1}) f^{(n+1)}(T_k)\\
&\quad +(-1)^{n+1}1_{[0,T_k]}(t_1) f^{(n)}(T_k)\phi_\varepsilon * r^{(2)}(T_k, t_2\vee\cdots\vee t_{n+1})\\
&\quad +(-1)^{n+1}1_{[0,T_k]}(t_1)\sum_{j=2}^{n} 1_{\{t_2\vee\cdots\vee t_{j}<t_{j+1}\vee\cdots\vee t_{n+1}\}} f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})\\
&\quad \times\phi_\varepsilon * r^{(j+1)}(T_k, t_{j+1}\vee\cdots\vee t_{n+1})\\
&=(-1)^{n+1}1_{[0,T_k]}(t_1)\phi_\varepsilon * r^{(1)}(T_k, t_2\vee\cdots\vee t_{n+1}) f^{(n+1)}(T_k)\\
&\quad +(-1)^{n+1}1_{[0,T_k]}(t_1)\sum_{j=1}^{n} 1_{\{t_2\vee\cdots\vee t_{j}<t_{j+1}\vee\cdots\vee t_{n+1}\}} f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})\\
&\quad \times\phi_\varepsilon * r^{(j+1)}(T_k, t_{j+1}\vee\cdots\vee t_{n+1}),
\end{aligned}$$

where we used the relation $\phi'_\varepsilon * r^{(j+1)} = \phi_\varepsilon * r^{(j+2)}$. As $\varepsilon \to 0$, $\tilde{D}_{t_1} F_\varepsilon(t_2,\dots,t_{n+1})$ converges in $\mathcal{S}'_d(\Omega)$ to

$$\begin{aligned}
&(-1)^{n+1} f^{(n+1)}(T_k) 1_{[0,T_k]}(t_1\vee\cdots\vee t_{n+1})\\
&\quad + (-1)^{n+1}\sum_{j=1}^{n} 1_{\{t_1\vee\cdots\vee t_j<t_{j+1}\vee\cdots\vee t_{n+1}\}} f^{(n+1-j)}(t_{j+1}\vee\cdots\vee t_{n+1})\\
&\qquad\qquad\qquad \times r^{(j+1)}(T_k, t_{j+1}\vee\cdots\vee t_{n+1}). \quad \square
\end{aligned}$$

In particular, for $n \geq 2$:

$$\begin{aligned}\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} T_k &= (-1)^{n+1} 1_{\{0 \leq t_1, \dots, t_{n-1} < t_n\}} r^{(n)}(T_k, t_n) \\ &= (-1)^n 1_{\{0 \leq t_1, \dots, t_{n-1} < t_n\}} \delta_{t_n}^{(n-2)}(T_k).\end{aligned}$$

We note that since $f \in \mathcal{C}_b^\infty(\mathbb{R}_+)$, there exists $C > 0$ such that

$$\begin{aligned}&|\langle \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k), h_1 \otimes \cdots \otimes h_l \otimes G \rangle| \\ &\qquad \leq C \sum_{i=0}^{i=n} \|h_1 \otimes \cdots \otimes h_l\|_\infty \|\hat{D}^i G\|_{L^\infty(\Omega, L^2(\mathbb{R}_+, dN_t)^{\otimes i})},\end{aligned}$$

$dt_1 \cdots dt_n$-a.e., for all $G \in \mathcal{S}_d(\Omega)$ and $h_1, \dots, h_l \in \mathcal{C}_c(\mathbb{R}_+)$. Hence $\tilde{D}^n f(T_k) \in \mathbb{D}_n(\mathbb{R}_+) \subset \mathcal{S}'_d(\Omega \times \mathbb{R}_+^n)$, and (5) can be written as

$$\begin{aligned}&\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) \\ &= (-1)^n \sum_{j=0}^{j=n} 1_{\{t_1 \vee \cdots \vee t_j < t_{j+1} \vee \cdots \vee t_n\}} f^{(n-j)}(T_{k,j}) (-r^{(j+1)}(T_k, t_{j+1} \vee \cdots \vee t_n))\end{aligned}$$

with $t_0 = 0$ and

$$T_{k,j} = \begin{cases} T_k & \text{if } j = 0, \\ t_{j+1} \vee \cdots \vee t_n & \text{if } j \geq 1, \end{cases}$$

i.e. if $0 \leq t_1 < \cdots < t_n$:

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_k) = (-1)^n \sum_{j=0}^{j=n} f^{(n-j)}(T_{k,j}) (-r^{(j+1)}(T_k, t_n)).$$

If $t_1 > \cdots > t_n$ then

$$\begin{aligned}&\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F \\ &= (-1)^n \sum_{1 \leq j_1, \dots, j_n \leq d} 1_{[0, T_{j_1}]}(t_1) \cdots 1_{[0, T_{j_n}]}(t_n) \partial_{j_1} \cdots \partial_{j_n} f(T_1, \dots, T_d).\end{aligned}$$

Given $j_1, \dots, j_n \in \{1, \dots, d\}$ and $i \in \{1, \dots, d\}$, let

$$\begin{aligned}a_i(j_1, \dots, j_n) &= \text{Card}\{l \in \{1, \dots, n\} : j_l = i\}, \\ c_i(j_1, \dots, j_n) &= \max\{l : j_l = i\}.\end{aligned}$$

With this notation we obtain the following formula, in which the indices $j_1, \dots, j_n$ are omitted in $a_i(j_1, \dots, j_n)$ and $c_i(j_1, \dots, j_n)$.

Theorem 1. *Let* $F = f(T_1, \ldots, T_d)$*, with* $f \in \mathcal{C}_b^\infty(\Delta_d)$*. We have* $\tilde{D}^n F \in \mathbb{D}_n(\mathbb{R}_+)$ *for all* $n \geq 0$*, and:*

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F = (-1)^n \sum_{\substack{1 \le j_1, \ldots, j_n \le d \\ 0 \le i_1 \le (a_1 - 1) \vee 0 \\ 0 \le i_d \le (a_d - 1) \vee 0}} \partial_1^{a_1 - i_1} \cdots \partial_d^{a_d - i_d} f(T_{1,i_1}, \ldots, T_{d,i_d}) \times \prod_{\substack{l=1 \\ a_l \neq 0}}^{l=d} (-r^{(1+i_l)}(T_l, t_{c_l})), \tag{6}$$

$(t_1, \ldots, t_n) \in \Delta_n$*,* $n \geq 2$*, where*

$$T_{l,i_l} = \begin{cases} T_l & \text{if } i_l = 0, \\ t_{c_l} & \text{if } i_l \geq 1. \end{cases}$$

Proof: Let $\tilde{D}_{t,k}$ denote the partial gradient with respect to the k-th variable, i.e.

$$\tilde{D}_{t,k} f(T_1, \ldots, T_d) = -1_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_d), \quad 1 \leq k \leq d.$$

Then $\tilde{D}_{t,k}$ and $\tilde{D}_{s,l}$, are commuting operators, $1 \leq k < l \leq d$, and for $0 < t_1 < \cdots < t_l$,

$$\tilde{D}_{t_1,k} \cdots \tilde{D}_{t_l,k} F = (\tilde{D}_{t_l,k})^l F.$$

Consequently,

$$\begin{aligned} \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F &= \sum_{1 \le j_1, \ldots, j_n \le d} \tilde{D}_{t_1,j_1} \cdots \tilde{D}_{t_n,j_n} F \\ &= \sum_{1 \le j_1, \ldots, j_n \le d} (\tilde{D}_{t_{c_1},1})^{a_1} \cdots (\tilde{D}_{t_{c_d},d})^{a_d} F. \end{aligned}$$

It remains to apply Proposition 3 under the form

$$\begin{aligned} &(\tilde{D}_{t_{c_l},l})^{a_l} F \\ &= (-1)^{a_l} \sum_{0 \le i_l \le (a_l - 1) \vee 0} \partial_l^{a_l - i_l} f(T_1, \ldots, T_{l-1}, T_{l,i_l}, T_{l+1}, \ldots, T_d)(-r^{(i_l+1)}(T_l, t_{c_l})), \end{aligned}$$

if $a_l \geq 1$, and

$$\begin{aligned} &(\tilde{D}_{t_{c_l},l})^{a_l} F \\ &\quad = (-1)^{a_l} \sum_{0 \le i_l \le (a_l - 1) \vee 0} \partial_l^{a_l - i_l} f(T_1, \ldots, T_{l-1}, T_{l,i_l}, T_{l+1}, \ldots, T_d) \end{aligned}$$

if $a_l = 0$. □

5. Equality of adapted projections in distribution sense

We recall that the adjoint of D extends the compensated Poisson stochastic integral, cf. [**3**, Theorem 6.9, p. 23], i.e. for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have

$$E[\langle DF, u\rangle_{L^2(\mathbb{R}_+,dt)}] = E\left[F\int_0^\infty u(t)\,d(N_t - t)\right]. \tag{7}$$

Since the adapted projections of $\tilde{D}$ and D coincide, cf. [**7**, Proposition 20]:

$$E[\tilde{D}_t F \mid \mathcal{F}_a] = E[D_t F \mid \mathcal{F}_a], \quad 0 < a < t, \tag{8}$$

the same property hold for $\tilde{D}$:

$$E[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+,dt)}] = E\left[F\int_0^\infty u(t)\,d(N_t - t)\right].$$

We now show that Relation (8) can be extended in distribution sense to $\tilde{D}^n$ and D^n, $n \geq 2$. The next proposition will be interpreted in terms of generalized conditional expectations as

$$E[\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F \mid \mathcal{F}_a] = E[D_{t_1}\cdots D_{t_n}F \mid \mathcal{F}_a],\ (t_1,\dots,t_n) \in \Delta_n \cap [a,\infty[^n.$$

Proposition 4. *Let $F \in \mathcal{S}_d(\Omega)$ and $G \in \mathcal{S}_d(\Omega)$ be $\mathcal{F}_a$-measurable. We have*

$$\langle G\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F, 1\rangle = E[GD_{t_1}\cdots D_{t_n}F],\ (t_1,\dots,t_n) \in \Delta_n \cap [a,\infty[^n. \tag{9}$$

Proof: The proposition holds for $n = 1$. We assume that it holds for some $n \geq 1$. Let $F_\varepsilon(t_2,\dots,t_{n+1})$ denote the regularization of $\tilde{D}_{t_2}\cdots\tilde{D}_{t_{n+1}}F$ constructed as in the proof of Proposition 3:

$$F_\varepsilon(t_2,\dots,t_{n+1}) = (-1)^n \sum_{\substack{1\leq j_1,\dots,j_n\leq d\\ 0\leq i_1\leq (a_1-1)\vee 0\\ 0\leq i_d\leq (a_d-1)\vee 0}} \partial_1^{a_1-i_1}\cdots\partial_d^{a_d-i_d} f(T_{1,i_1},\dots,T_{d,i_d}) \times \prod_{\substack{l=1\\ a_l\neq 0}}^{l=d} (-\phi^\varepsilon * r^{(1+i_l)}(T_l, t_{1+c_l})).$$

Let $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1} \cap [a,\infty[^{n+1})$. Since G is $\mathcal{F}_a$ measurable we have $\tilde{D}_{t_1} G = 0$, $t_1 > a$, hence from Proposition 1:

$$
\begin{aligned}
E\left[\int_a^{t_2} \tilde{D}_{t_1} F_\varepsilon(t_2,\ldots,t_{n+1}) f_{n+1}(t_1,\ldots,t_{n+1})\, dt_1 G\right]
&= E\left[\int_a^{t_2} \tilde{D}_{t_1}(G F_\varepsilon(t_2,\ldots,t_{n+1})) f_{n+1}(t_1,\ldots,t_{n+1})\, dt_1\right] \\
&= E\left[G F_\varepsilon(t_2,\ldots,t_{n+1}) \left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) \right.\right. \\
&\qquad \left.\left. - \int_0^{T_d} f_{n+1}(t,t_2,\ldots,t_{n+1})\, dt\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \tilde{D}^{n+1} F, 1_{\Delta_{n+1}} f_{n+1} G\rangle \\
&= \lim_{\varepsilon\to 0} \left\langle \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} \tilde{D}_{t_1} F_\varepsilon(t_2,\ldots,t_{n+1}) f_{n+1}(t_1,\ldots,t_{n+1})\, dt_1 \cdots dt_{n+1}, G\right\rangle \\
&= \lim_{\varepsilon\to 0} E\left[\int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} \tilde{D}_{t_1} F_\varepsilon(t_2,\ldots,t_{n+1}) f_{n+1}(t_1,\ldots,t_{n+1})\, dt_1 \cdots dt_{n+1}, G\right] \\
&= \lim_{\varepsilon\to 0} E\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} F_\varepsilon(t_2,\ldots,t_{n+1}) \left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) \right.\right. \\
&\qquad \left.\left. - \int_0^{T_d} f_{n+1}(t,t_2,\ldots,t_{n+1})\, dt\right) dt_2 \cdots dt_{n+1}\right] \\
&= E\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} \left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) \right.\right. \\
&\qquad \left.\left. - \int_0^{T_d} f_{n+1}(t_1,t_2,\ldots,t_{n+1})\, dt_1\right) \tilde{D}_{t_2} \cdots \tilde{D}_{t_{n+1}} F\, dt_2 \cdots dt_{n+1}\right] \\
&= E\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_3} \left(\sum_{k=1}^{k=d} f_{n+1}(T_k,t_2,\ldots,t_{n+1}) \right.\right. \\
&\qquad \left.\left. - \int_0^{T_d} f_{n+1}(t_1,t_2,\ldots,t_{n+1})\, dt_1\right) D_{t_2} \cdots D_{t_{n+1}} F\, dt_2 \cdots dt_{n+1}\right],
\end{aligned}
$$

where on the last step we used the induction hypothesis with $a = t_2$. This is possible because the functional $\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) - \int_0^{T_d} f_{n+1}(t_1, t_2, \ldots, t_{n+1})\, dt_1$ is $\mathcal{F}_{t_2}$-measurable since it depends on T_k only when $T_k \leq t_2$, $1 \leq k \leq d$, due to the fact that $f_{n+1} \in \mathcal{C}_c^\infty(\Delta_{n+1})$.

The proof of Proposition 1 also shows that

$$
\begin{aligned}
E\Bigg[G\Bigg(&\sum_{k=1}^{k=d} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) \\
&\quad - \int_0^{T_d} f_{n+1}(t_1, t_2, \ldots, t_{n+1})\, dt_1\Bigg) D_{t_2} \cdots D_{t_{n+1}} F\Bigg] \\
&= E\Bigg[G\Bigg(\sum_{k=1}^{k=\infty} f_{n+1}(T_k, t_2, \ldots, t_{n+1}) \\
&\quad - \int_0^{\infty} f_{n+1}(t, t_2, \ldots, t_{n+1})\, dt\Bigg) D_{t_2} \cdots D_{t_{n+1}} F\Bigg] \\
&= E\left[G \int_a^{t_2} f_{n+1}(t_1, t_2, \ldots, t_{n+1})\, d(N_{t_1} - t_1) D_{t_2} \cdots D_{t_{n+1}} F\right] \\
&= E\left[G \int_a^{t_2} f_{n+1}(t_1, \ldots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}} F\, dt_1\right],
\end{aligned}
$$

where on the last line we used the duality (7) between D and the Poisson compensated integral on the adapted processes. Hence

$$
\begin{aligned}
&\langle \tilde{D}^{n+1} F, 1_{\Delta_{n+1}} f_{n+1} G\rangle \\
&= E\left[G \int_a^\infty \int_a^{t_{n+1}} \cdots \int_a^{t_2} f_{n+1}(t_1, \ldots, t_{n+1}) D_{t_1} \cdots D_{t_{n+1}} F\, dt_1 \cdots dt_{n+1}\right].
\end{aligned}
$$

This shows the almost-sure equality

$$
\langle G \tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F, 1\rangle = E[G D_{t_1} \cdots D_{t_n} F], \qquad \text{a.e. } (t_1, \ldots, t_n) \in \Delta_n \cap [a, \infty[^n,
$$

which becomes an equality for all $(t_1,\dots,t_n) \in \Delta_n \cap [a,\infty[^n$ since $(t_1,\dots,t_n) \mapsto \langle G\tilde{D}_{t_1}\cdots\tilde{D}_{t_n}F,1\rangle$ and $(t_1,\dots,t_n) \mapsto E[GD_{t_1}\cdots D_{t_n}F]$ are clearly continuous functions on Δ_n when $F,G \in \mathcal{S}_d(\Omega)$. □

Note that Relation (9) does not hold if $(t_1,\dots,t_n) \notin \Delta_n$, see Relation (14) below.

6. Chaos expansions of jump times functionals

Our result is stated for smooth functions $f(T_1,\dots,T_d)$ of a finite number of jump times. We start with the simple case of $f(T_d)$. For $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, let

$$p_n(t) = P(N_t = n) = \frac{t^n}{n!}e^{-t} = e^{-t}\int_0^t\int_0^{s_{n-1}}\cdots\int_0^{s_2} ds_1\cdots ds_{n-1},$$

if $n \geq 0$, and $p_n(t) = P(N_t = n) = 0$ if $n < 0$, i.e. $p_{n-1}\colon \mathbb{R}_+ \to \mathbb{R}_+$, $n \geq 1$, is the density function of T_n, and

$$p_n^{(k)}(t) = \frac{\partial^k}{\partial t^k}p_n(t) = (-\Delta)^k p_n(t) = (-1)^k\frac{p_n(t)}{t^k}C_k^t(n),$$

where Δ is the finite difference operator $\Delta f(n) = f(n) - f(n-1)$ and C_k^t is the Charlier polynomial of order $k \in \mathbb{N}$ and parameter $t \in \mathbb{R}_+$.

Proposition 5. *The decomposition*

$$f(T_d) = \sum_{n=0}^{\infty} I_n(h_n 1_{\Delta_n}),$$

is given for $(t_1,\dots,t_n) \in \Delta_n$ as:

$$\begin{aligned} &h_n(t_1,\dots,t_n)\\ &=(-1)^n\int_{t_n}^{\infty} f^{(n)}(t)p_{d-1}(t)\,dt+(-1)^n\sum_{j=1}^{j=n-1}(-1)^{j-1}f^{(n-j)}(t_n)p_{d-1}^{(j-1)}(t_n). \end{aligned}$$

Proof: From Relation (9) of Proposition 4 and Relation (5) of Proposition 3 we have

$$
\begin{aligned}
& h_n(t_1,\dots,t_n) = E[D_{t_1}\cdots D_{t_n} f(T_d)] = \langle \tilde{D}_{t_1}\cdots \tilde{D}_{t_n} f(T_d), 1\rangle \\
& = (-1)^n \left\langle f^{(n)}(T_d) 1_{[0,T_d]}(t_n) + \sum_{j=1}^{n-1} f^{(n-j)}(t_n)\delta_{t_n}^{(j-1)}(T_d), 1 \right\rangle \\
& = (-1)^n \int_0^\infty 1_{[t_n,\infty[}(s_d) e^{-s_d} f^{(n)}(s_d) \int_0^{s_d}\cdots\int_0^{s_2} ds_1\cdots ds_d \\
& \quad + (-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \int_0^\infty e^{-s_d} \int_0^{s_d}\cdots\int_0^{s_2} ds_1\cdots ds_{d-1}\delta_{t_n}^{(j-1)}(ds_d) \\
& = (-1)^n \int_{t_n}^\infty e^{-s_d} f^{(n)}(s_d) \frac{{s_d}^{d-1}}{(d-1)!}\, ds_d \\
& \quad + (-1)^n \sum_{j=1}^{n-1} f^{(n-j)}(t_n) \int_0^\infty e^{-s_d} \frac{{s_d}^{d-1}}{(d-1)!} \delta_{t_n}^{(j-1)}(ds_d) \\
& = (-1)^n \int_{t_n}^\infty f^{(n)}(t) p_{d-1}(t)\, dt \\
& \quad + (-1)^n \sum_{j=1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n). \qquad \square
\end{aligned}
$$

By induction this gives for $l = 0,\dots,n-1$:

$$
\begin{aligned}
h_n(t_1,\dots,t_n) = (-1)^{n+l} & \int_{t_n}^\infty f^{(n-l)}(t) p_{d-1}^{(l)}(t)\, dt \\
& + (-1)^n \sum_{j=l+1}^{j=n-1} (-1)^{j-1} f^{(n-j)}(t_n) p_{d-1}^{(j-1)}(t_n),
\end{aligned}
$$

and in particular for $l = n-1$:

$$
\begin{aligned}
h_n(t_1,\dots,t_n) &= -\int_{t_n}^\infty f'(s) p_{d-1}^{(n-1)}(s)\, ds \\
&= f(t_n) p_{d-1}^{(n-1)}(t_n) + \int_{t_n}^\infty f(s) p_{d-1}^{(n)}(s)\, ds,
\end{aligned}
$$

hence

$$(10)\quad f(T_d) = \sum_{n\geq 0} \frac{1}{n!} I_n \left(f(t_1 \vee \cdots \vee t_n) p_{d-1}^{(n-1)}(t_1 \vee \cdots \vee t_n) + \int_{t_1\vee\cdots\vee t_n}^{\infty} f(s) p_{d-1}^{(n)}(s)\, ds \right),$$

with the convention $t_1 \vee t_0 = 0$. In order to treat the case of d variables we recall the notation

$$a_i(j_1, \ldots, j_n) = \mathrm{Card}\{l : j_l = i\}, \text{ and } c_i(j_1, \ldots, j_n) = \max\{l : j_l = i\}.$$

Proposition 6. *Let $f \in \mathcal{C}_b^\infty(\Delta_d)$. We have*

$$f(T_1, \ldots, T_d) = \sum_{n=0}^{\infty} I_n(h_n 1_{\Delta_n}),$$

with

$$(11)\quad h(t_1, \ldots, t_n) = (-1)^n \sum_{\substack{1\leq j_1,\ldots,j_n\leq d \\ 0\leq i_1\leq (a_1-1)\vee 0 \\ 0\leq i_d\leq (a_d-1)\vee 0}} \int_0^\infty \int_0^{s_d} \cdots \int_0^{s_2} e^{-s_d} \partial_1^{a_1-i_1} \cdots \partial_d^{a_d-i_d} f(s_{1,i_1}, \ldots, s_{d,i_d}) \times \prod_{\substack{l=1 \\ a_l\neq 0}}^{l=d} (-r^{(1+i_l)}(ds_l, t_{c_l})),$$

where $r^{(1)}(ds, t) = 1_{[t,\infty[}(s)\, ds$, and

$$s_{l,i_l} = \begin{cases} s_l & \text{if } i_l = 0, \\ t_{c_l} & \text{if } i_l \geq 1,\ l = 1, \ldots, d. \end{cases}$$

Proof: We apply Theorem 1 and Relation (9) of Proposition 4. □

This expression can be made more explicit by evaluation of the action of $r^{(1+i_l)}(ds_{j_l}, t_{c_l})$, either as an indicator function or as a derivative in t_{c_l}. However this will not be done here in order to keep formula (11) to a reasonable size.

In [**8**], another expression (different from (11)) has been obtained using elementary orthogonal decompositions in Charlier polynomials. Let $n_1, \ldots, n_l \in \mathbb{N}$ with $1 \leq n_1 < \cdots < n_l$, and let $f \in \mathcal{C}_b^d(\Delta_l)$. As a convention, if $k_1 \geq 0, \dots, k_d \geq 0$ satisfy $k_1 + \cdots + k_d = n$, we let for $(t_1, \dots, t_n) \in \Delta_n$:

$$(t_1^1, \dots, t_{k_1}^1, t_1^2, \dots, t_{k_2}^2, \dots, t_1^d, \dots, t_{k_d}^d) = (t_1, \dots, t_n),$$

with $t_{k_i}^i = 0$ if $k_i = 0$, and $(t_1^i, t_0^i) = ()$. We have

$$f(T_{n_1}, \dots, T_{n_l}) = \sum_{n=0}^{\infty} I_n(1_{\Delta_n} h_n),$$

where

$$\text{(12)}\quad h_n(t_1, \dots, t_n) = (-1)^l \sum_{k_1+\cdots+k_l=n} \int_{t_{k_l}^l}^{\infty} \cdots \int_{t_{k_i}^i}^{t_1^{i+1}} \cdots \int_{t_{k_1}^1}^{t_1^2} \partial_1 \cdots \partial_l f(s_1, \dots, s_l) K_{s_1,\dots,s_l}^{k_1,\dots,k_l} \, ds_1 \cdots ds_l,$$

with, for $0 \leq s_1 \leq \cdots \leq s_l$ and $k_1 \geq 0, \dots, k_l \geq 0$:

$$K_{s_1,\dots,s_l}^{k_1,\dots,k_l} = \sum_{\substack{m_1 \geq n_1, \dots, m_l \geq n_l \\ m_1 \leq \cdots \leq m_l}} p_{m_1-m_0}^{(k_1)}(s_1 - s_0) \cdots p_{m_l-m_{l-1}}^{(k_l)}(s_l - s_{l-1}), \qquad m_0 = 0, \quad s_0 = 0,$$

cf. Theorem 1 of [**8**]. For $l = 1$, i.e. for $f(T_d)$, $n_1 = d$, we have

$$\begin{aligned} K_s^k &= \sum_{m=d}^{\infty} p_m^{(k)}(s) = \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_m^{(1)}(s) \\ &= \frac{\partial^{k-1}}{\partial s^{k-1}} \sum_{m=d}^{\infty} p_{m-1}(s) - p_m(s) = p_{d-1}^{(k-1)}(s) \end{aligned}$$

hence

$$h_n(t_1, \dots, t_n) = -\int_{t_n}^{\infty} f'(s) p_{d-1}^{(n-1)}(s) \, ds,$$

which coincides with (10).

Remarks. i) All expressions obtained above for $f(T_1, \dots, T_d)$, $f \in \mathcal{C}_b^{\infty}(\Delta_d)$, extend to $f \in L^2(\Delta_d, e^{-s_d} ds_1 \cdots ds_d)$, i.e. to square-integrable $f(T_1, \dots, T_d)$, by repeated integrations by parts.

ii) Chaotic decompositions on the Poisson space on the compact interval $[0, 1]$ as in [**4**] or [**5**] can be obtained by considering the functional $f(1 \wedge T_1, \dots, 1 \wedge T_d)$ instead of $f(T_1, \dots, T_d)$.

iii) If $t_1 > \cdots > t_n$ then Relation (9) does not hold, for example we have

$$\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d) = (-1)^n 1_{[0,T_d]}(t_1) f^{(n)}(T_d),$$

and

$$\begin{aligned} E[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} f(T_d)] &= (-1)^n E[1_{[0,T_d]}(t_1) f^{(n)}(T_d)] \\ &= (-1)^n \int_{t_1}^{\infty} f^{(n)}(s) p_{d-1}(s)\, ds, \end{aligned} \tag{13}$$

which differs (if $n \geq 2$) from

$$E[D_{t_1} \cdots D_{t_n} f(T_d)] = -\int_{t_1}^{\infty} f'(s) p_{d-1}^{(n-1)}(s)\, ds. \tag{14}$$

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