HAUSDORFF MEASURES AND THE MORSE-SARD
THEOREM

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Abstract
Let $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function and $p < m$ an
integer. If $k \geq 1$ is an integer, $\alpha \in [0, 1]$ and $F \in C^{k+(\alpha)}$, if we set
$C_p(F) = \{x \in U \mid \text{rank}(DF(x)) \leq p\}$ then the Hausdorff measure
of dimension $(p + \frac{n-p}{k+\alpha})$ of $F(C_p(F))$ is zero.

1. Introduction

The Morse-Sard theorem is a fundamental theorem in analysis that
is in the basis of transversality theory and differential topology. The
classical Morse-Sard theorem states that the image of the set of critical
points of a function $F: \mathbb{R}^n \to \mathbb{R}^m$ of class $C^{n-m+1}$ has zero Lebesgue
measure in $\mathbb{R}^m$. It was proved by Morse ([M]) in the case $m = 1$ and by
Sard ([S1]) in the general case.

Due to its theoretical importance, the Morse-Sard theorem was gen-
eralized in many directions. Many of these generalizations are related
with Hausdorff measures and Hausdorff dimensions.

Given a metric space $X$ and a positive real number $\alpha$, we define the
Hausdorff measure of dimension $\alpha$ associated to a covering $\mathcal{U} = (U_\lambda)_{\lambda \in L}$
of $X$ by bounded sets $U_\lambda$ by $m_\alpha(\mathcal{U}) = \sum_{\lambda \in L} (\text{diam } U_\lambda)^\alpha$, where $\text{diam } U_\lambda$
denotes the diameter of $U_\lambda$, and, if we define the norm of a covering $\mathcal{U}$
by $||\mathcal{U}|| = \sup_{U \in \mathcal{U}} (\text{diam } U)$, then the Hausdorff measure of dimension $\alpha$
of $X$ is $m_\alpha(X) = \liminf_{||\mathcal{U}|| \to 0} m_\alpha(\mathcal{U})$.

It is not difficult to see that there is a unique $d \in [0, +\infty]$ such that if
$\alpha > d$ then $m_\alpha(X) = 0$ and if $\alpha < d$ then $m_\alpha(X) = +\infty$. This number $d$
is called the Hausdorff dimension of $X$. It is easy to see that if $X \subset \mathbb{R}^n$
then its Hausdorff dimension $d := HD(X)$ belongs to $[0, n]$. 
Sard himself proved that if \( C_p(F) = \{x \in \mathbb{R}^n \mid \text{rank}(DF(x)) \leq p\} \) then for any \( \varepsilon > 0 \) there is \( k \in \mathbb{N} \) such that if \( F \) is \( C^k \) then \( F(C_p(F)) \) has zero Hausdorff measure of dimension \( p + \varepsilon \) ([S2]). This result was made more precise by Federer ([F]), who proved that if \( k \in \mathbb{N} \) then the Hausdorff measure of dimension \( p + \frac{n-p}{k+\alpha} \) of \( F(C_p(F)) \) is zero. We should also mention the works of Church ([Ch1], [Ch2]), which gave more results about the structure of the set of critical values of differentiable maps. Later, Yomdin ([Y]) proved that the Hausdorff dimension of \( F(C_p(F)) \) is at most \( p + \frac{n-p}{k+\alpha} \left(\frac{n}{2} - \alpha\right) \), provided that \( F \in C_{k+\alpha}^{n-m+1} \), where \( k \in \mathbb{N} \) and \( 0 \leq \alpha < 1 \). More recently, Bates ([B2]) proved that if \( F \in C_{k+\alpha}^{n-m} \) with \( k \in \mathbb{N} \), \( 0 < \alpha \leq 1 \) and \( p + \frac{n-p}{k+\alpha} = m \) then \( F(C_p(F)) \) has zero Lebesgue measure in \( \mathbb{R}^m \) (this in particular improves the hypothesis of the classical Morse-Sard theorem from \( F \in C^{n-m+1} \) to \( F \in C^{n-m+Lips.} \), i.e., \( F \in C^{n-m} \) and \( D^{n-m}F \) Lipschitz).

The aim of this work is to generalize the mentioned results by proving a general version of the Morse-Sard Theorem involving Hausdorff measures. Let \( k \geq 1 \) be an integer and \( \alpha \in [0,1] \). We say that a function \( F: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) is of class \( C_k^{k+\alpha} \) at a subset \( A \) of \( U \) if \( F \) is \( C^k \) in \( U \) and for each \( x \in A \) there are \( \varepsilon_x > 0 \), \( K_x > 0 \) such that \( |y-x| < \varepsilon_x \Rightarrow |D^kF(y) - D^kF(x)| \leq K_x|y-x|^{\alpha} \) (this is less restrictive than supposing \( F \in C_{k+\alpha}^{n-m+1} \)). Our main result is the following

**Theorem.** Let \( F: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) and let \( p < m \) be an integer. If \( C_p(F) := \{x \in U \mid \text{rank}(DF(x)) \leq p\} \) and if \( F \) is of class \( C_k^{k+\alpha} \) at \( C_p(F) \) then the Hausdorff \((p + \frac{n-p}{k+\alpha})\)-measure of \( F(C_p(F)) \) is zero.

In particular, if \( k + \alpha = \frac{n-p}{m-p} \), we recover the result of [B2], with a weaker hypothesis. We remark that if \( p + \frac{n-p}{k+\alpha} < m \), the Hausdorff \((p + \frac{n-p}{k+\alpha})\)-measure is not the Lebesgue measure or a product measure in \( \mathbb{R}^m \), and so we can not use Fubini’s Theorem. This difficulty is solved in the present paper by replacing the use of Fubini’s theorem by a careful decomposition of the critical set, combined with a parametrized strong version of the main lemma of Morse’s paper ([M, Theorem 2.1]).

We shall also give examples that show that our result is quite sharp, by giving counterexamples to slight changes of the hypothesis or of the conclusion.

### 2. Functions whose zeros include a given set

We shall prove here a version of Theorem 3.6 of [M] and Lemma 3.4.2 of [F], which will be fundamental for the later results.
Theorem 2.1. Let \( k \geq 1, \alpha \in [0,1], n > p \) and \( A \subset U \subset \mathbb{R}^n \), where \( U \) is an open set. Then there are sets \( A_1, A_2, \ldots \subset A \) such that \( A = \bigcup_{i=1}^{\infty} A_i \), where for each \( i = 1, 2, \ldots \) there is a function \( \psi_i : B_i \times V_i \to U \) where \( B_i \) is a ball in some \( \mathbb{R}^r \), \( r_i > 0 \) and \( V_i \) is a ball in \( \mathbb{R}^p \) such that \( \psi_i(x,y) = (\psi_i(x,y), y) \), and \( |\psi_i(x,y_1) - \psi_i(x,y_2)| \geq |(x_1, y_1) - (x_2, y_2)| \), \( \forall (x_1, y_1), (x_2, y_2) \in B_i \times V_i \) and \( A_i \subset \psi_i(B_i \times V_i) \), with the following property: We can write \( A_i = A_i' \cap A_i'' \) so that \( \psi_i^{-1}(A_i') \) has measure zero in \( B_i \times V_i \), and if \( f : U \to \mathbb{R} \) is \( C^{k+\alpha} \) at \( A \) we have:

- \( \limsup_{(x,y) \to (x_0,y_0)} f(\psi_i(x,y)) = +\infty, \forall (x_0, y_0) \in B_i \times V_i \) such that \( \psi_i(x_0,y_0) \in A_i \),
- \( \limsup_{(x,y) \to (x_0,y_0)} f(\psi_i(x,y)) = 0, \forall (x_0, y_0) \in B_i \times V_i \) such that \( \psi_i(x_0,y_0) \in A_i' \).

Proof: Let us consider first the case \( k = 1 \) and \( df(x) \cdot v = 0 \) \( \forall x \in A, v \in \mathbb{R}^{n-p} \times \{0\} \). In this case we take \( A = (A' \cap A) \cup A'' \) where \( A' \) is the set of density points of \( A \) in the direction of \( \mathbb{R}^{n-p} \times \{0\} \) \( (x,y) \in A' \Rightarrow \lim_{\varepsilon \to 0} \frac{m((B(x) \times \{y\}) \cap A)}{m(B(x))} = 1 \), where \( m \) is the \( (n-p) \)-dimensional measure. The measure of \( A'' = A - A' \) is zero, since it is zero in each plane \( \mathbb{R}^{n-p} \times \{y\} \).

For \( (x_0, y_0) \in A \) take \( B((x_0, y_0), \varepsilon(x_0, y_0)) \) a ball contained in \( U \) and \( \psi = \text{Id}_{|B((x_0, y_0), \varepsilon(x_0, y_0))|} \). We have \( \limsup_{(x,y) \to (x_0,y_0)} \frac{f(x,y)}{|x-x_0|^{1+\alpha}} < +\infty \), since \( f(x,y) = f(x,y) - f(x_0,y_0) + f(x_0,y_0) = df(tx_0 + (1-t)x)(x-x_0), t \in (0,1) \Rightarrow |f(x,y)| \leq K|x-x_0|^{1+\alpha} \). For \( (x_0, y_0) \in A' \),

\[
\lim_{\delta \to 0} \frac{1}{\text{vol}(S^{n-p-1})} \int_{S^{n-p-1}} \left( \frac{1}{\gamma} \int_0^\delta \chi_A(x_0 + tv, y_0) dt \right) dv = 1,
\]

so \( \forall \varepsilon > 0 \exists \delta_0 > 0 \) s.t. \( |x-x_0| < \delta_0 \Rightarrow \exists v \in S^{n-p-1} \) with

\[
|v - \frac{x-x_0}{|x-x_0|}| < \varepsilon
\]

and

\[
\left| \frac{1}{|x-x_0|} \int_0^{|x-x_0|} \chi_A(x_0 + tv, y_0) dt - 1 \right| < \varepsilon,
\]
so, if \( \tilde{x} = x_0 + |x - x_0|v \),
\[
|f(x, y_0) - f(x_0, y_0)| \leq |f(x, y_0) - f(\tilde{x}, y_0)| + |f(\tilde{x}, y_0) - f(x_0, y_0)|,
\]
but
\[
f(x, y_0) - f(\tilde{x}, y_0) = df(\theta t + (1 - \theta)\tilde{x}, y_0) \cdot (x - \tilde{x}), \quad \theta \in (0, 1)
\]
\[

\Rightarrow |f(x, y_0) - f(\tilde{x}, y_0)| \leq K_{x_0}|x - x_0|^\alpha \cdot \varepsilon |x - x_0| = \varepsilon K_{x_0}|x - x_0|^{1 + \alpha}
\]
and
\[
f(\tilde{x}, y_0) - f(x_0, y_0)
\]
\[
= \int_0^{||\tilde{x} - x_0||} df(x_0 + tv, y_0) \cdot v \, dt
\]
\[
\leq K_{x_0}|x - x_0|^\alpha \cdot m \left\{ t \in [0, |\tilde{x} - x_0|] \mid \frac{\partial f}{\partial x}(x_0 + tv, y_0) \neq 0 \right\}
\]
\[
\leq K_{x_0}|x - x_0|^\alpha \cdot \varepsilon |x - x_0| = \varepsilon K_{x_0}|x - x_0|^{1 + \alpha}.
\]
So
\[
|f(x, y_0)| = |f(x, y_0) - f(x_0, y_0)| \leq 2\varepsilon K_{x_0}|x - x_0|^{1 + \alpha}
\]
\[
\Rightarrow \lim_{(x, y_0) \to (x_0, y_0)} \frac{f(x, y_0)}{|x - x_0|^{1 + \alpha}} = 0.
\]

We can take a countable subcovering of \( A \) by the \( B((x_0, y_0), \varepsilon(x_0, y_0)) \) to finish the proof in this case.

Consider now the case \( k \geq 1 \), arbitrary. We have \( A = A^* \cup A^{**} \) where \( A^* = \{ x \in A \mid \exists g: U \to \mathbb{R}, g|_A \equiv 0, \exists v \in \mathbb{R}_n \times \{0\}, \ dg(x) \cdot v \neq 0 \} \), \( A^{**} = A \setminus A^* \). If \( (x_0, y_0) \in A^* \) there is \( g \) as above, so there is \( \varepsilon > 0 \) such that \( g^{-1}(0) \cap B_\varepsilon(x_0, y_0) \) is contained in the image of \( \psi: B \times V \to U \) where \( B \) is a ball in \( \mathbb{R}^{n-p} \), as in the statement, and \( A \subset g^{-1}(0) \). Taking a countable subcovering of \( A^* \) by these balls we reduce the proof in this case to a case with smaller \( n \). If \( k = 1 \), the result was yet proved for \( A^{**} \). If \( k > 1 \), and assuming by induction the result for \( k - 1 \), we have
\[
A^{**} = \bigcup_{i=1}^{\infty} A_i^{**}, \ A_i^{**} = (A_i^{**})' \cup (A_i^{**})'', \ A_i^{**} \subset \psi_i(B_i \times V_i), \ \psi_i \in C^1,
\]
In the statements of Theorem 2.1 and Corollary 2.2, let exactly as the case on $D$ and of the mean value theorem. If $f$ is a ball in some $\mathbb{R}^r$, $r_i \geq 0$ and $V_i$ is a ball in $\mathbb{R}^p$ such that $\psi_i(x, y) = (\tilde{\psi}_i(x, y), y)$, and $|\psi_i(x_1, y_1) - \psi_i(x_2, y_2)| \leq |(x_1, y_1) - (x_2, y_2)|$, $\forall (x_1, y_1), (x_2, y_2) \in B_i \times V_i$ and $A_1 \subset \psi_i(B_i \times V_i)$, with the following property: We can write $A_1 = A'_1 \cup A''_1$ so that $\psi_i^{-1}(A''_1)$ has measure zero in $B_i \times V_i$, and if $f: U \to \mathbb{R}$ is $C^{k+\alpha}$ at $A$ and $D_x f \equiv 0$ in $A$ we have:

- $\limsup_{(x,y_0) \to (x_0,y_0)} \frac{|f(\psi_i(x,y_0)) - f(\psi_i(x_0,y_0))|}{|x-x_0|^{k+\alpha}} < +\infty$, $\forall (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0) \in A_1$,

- $\limsup_{(x,y_0) \to (x_0,y_0)} \frac{|f(\psi_i(x,y_0)) - f(\psi_i(x_0,y_0))|}{|x-x_0|^{k+\alpha}} = 0$, $\forall (x_0, y_0) \in B_i \times V_i$ such that $\psi_i(x_0, y_0) \in A'_1$.

Proof: If $k \geq 2$ this is an immediate consequence of Theorem 2.1 applied to $D_x f$ and of the mean value theorem. If $k = 1$ this can be proved exactly as the case $k = 1$ of the Theorem 2.1.

Corollary 2.3. In the statements of Theorem 2.1 and Corollary 2.2, for any $x \in B_i$ s.t. $\psi(x) \in A_1$ there are $\varepsilon_x > 0$, $K_x > 0$ such that $\forall (x,y) \in \varepsilon_x \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq K_x |y-x|^{k+\alpha}$, and for any $\varepsilon > 0$ there is a $\delta > 0$ so that $\frac{\lambda(\psi_i^{-1}(A_1) \cap B_i(x))}{\lambda(B_i(x))} > 1 - \delta \Rightarrow |f(\psi_i(y)) - f(\psi_i(x))| \leq \varepsilon K_x r^{k+\alpha}$, if $r \leq \varepsilon$ and $|y-x| \leq r$ ($\delta$ depends only on $\varepsilon$ and $n$, but not on $f$ or on $x$).
Remark 2.1. For \( k = 0 \) we have the same results, except the statement \( \lim_{y \to x} \frac{f(y)}{|y - x|^{r+\alpha}} = 0 \), for each \( x \in B \) such that \( \psi_i(x) \in A' \).

3. The main results

Lemma 3.1. Let \( A \subset \mathbb{R}^m \) with \( \lambda(A) < \infty \) and let \( \mathcal{U} \) be a family of balls \( B_r(x), x \in A \) such that for each \( x \in A \) there is an \( \varepsilon_x > 0 \) such that \( r \leq \varepsilon_x \Rightarrow B_r(x) \in \mathcal{U} \). Then for each \( \varepsilon > 0 \) there are \( x_n \in A, r_n > 0 \) with \( B_{r_n}(x_n) \in \mathcal{U} \) and \( A \subset \bigcup_{n=1}^{\infty} B_{r_n}(x_n) \) such that \( \sum_{n=1}^{\infty} \lambda(B_{r_n}(x_n)) < \lambda(A) + \varepsilon \).

Proof: This lemma is essentially the Vitali covering theorem from measure theory. Take \( U \supset A \) an open set with \( \lambda(U) < \lambda(A) + \varepsilon \). If we choose \( B_{\frac{r_n}{2}}(x_n), \ldots, B_{\frac{r_n}{2}}(x_n) \), define \( s_n = \sup \{ r > 0 \mid \exists x \in A \text{ s.t. } r < \frac{r_n}{5}, B_r(x) \subset U \text{ and } B_r(x) \cap (B_{\frac{r_n}{2}}(x_1) \cup \cdots \cup B_{\frac{r_n}{2}}(x_n)) = \emptyset \} \). Choose \( B_{\frac{r_n}{2}}(x_{n+1}) \) such that \( \bar{r}_{n+1} > \frac{s_n}{2}, \bar{r}_{n+1} < \frac{\varepsilon_{n+1}}{5}, B_{\bar{r}_{n+1}}(x_{n+1}) \subset U \) and \( B_{\bar{r}_{n+1}}(x_{n+1}) \cap (B_{\frac{r_n}{2}}(x_1) \cup \cdots \cup B_{\frac{r_n}{2}}(x_n)) = \emptyset \). Since the \( B_r(x) \) are disjoint and contained in \( U \) we have \( \sum_{i=1}^{\infty} \lambda(B_{\bar{r}_i}(x_i)) < \lambda(A) + \varepsilon \), and there is a \( n_0 \in \mathbb{N} \) such that \( \sum_{i=n_0}^{\infty} \lambda(B_{\bar{r}_i}(x_i)) < \varepsilon \). We take \( B_{\bar{r}_i}(x_i) = B_{\bar{r}_i}(x_i), i < n_0 \) and \( B_{\bar{r}_i}(x_i) = B_{\bar{r}_i}(x_i), i \geq n_0 \).

Clearly we have \( \sum_{i=1}^{\infty} \lambda(B_{\bar{r}_i}(x_i)) < \lambda(A) + \varepsilon \). To prove that \( A \subset \bigcup_{n=1}^{\infty} B_{r_n}(x_n) \), take \( x \in A \) and \( r = \min \{ r_n, \varepsilon_x / 5 \}, d(x, U^c \cup \bigcup_{i<n_0} B_{r_i}(x_i)) \). If \( r > 0 \), take \( n \geq n_0 \) such that \( s_n < r \leq s_{n-1} \) (we have \( r \leq \bar{r}_{n+1} \leq s_{n+1} \)), and note that \( s_n < r \Rightarrow B_r(x) \cap (B_{\frac{r_n}{2}}(x_1) \cup \cdots \cup B_{\frac{r_n}{2}}(x_n)) = \emptyset \Rightarrow \exists i \leq n \text{ such that } B_r(x) \cap B_{\frac{r_n}{2}}(x_i) \neq \emptyset \). We have \( n \geq n_0 \) since \( r \leq d(x, B_r(x)) \), and \( \bar{r}_i > \frac{s_n}{2} > \frac{r}{2} \), since \( i \leq n \). Therefore, we have \( x \in B_{\bar{r}_i}(x_i) \). If \( r = 0 \) then \( x \in B_{\bar{r}_i}(x_i) \) for some \( i < n_0 \). This proves that \( A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \).

Taking \( \bar{r}_n = \left( \frac{\lambda(A) + \varepsilon}{\sum_{i=1}^{\infty} \lambda(B_{\bar{r}_i}(x_i))} \right)^{1/2} \), we have \( A \subset \bigcup_{i=1}^{\infty} B_{\bar{r}_i}(x_i) \), with \( \sum_{i=1}^{\infty} \lambda(B_{\bar{r}_i}(x_i)) = (\lambda(A) + \varepsilon)^{1/2}(\sum_{i=1}^{\infty} \lambda(B_{\bar{r}_i}(x_i)))^{1/2} < \lambda(A) + \varepsilon \).

Remark 3.1. In the Lemma 3.1 we can replace a family of balls \( B_r(x) \) by a family of cubes \( C_r(x) = \prod_{i=1}^{m} [x_i - r, x_i + r] \), where \( x = (x_1, \ldots, x_m) \), using the same proof.

Lemma 3.2. Let \( F: U \subset \mathbb{R}^n \to \mathbb{R}^m \) be a function, \( A \subset U \) and \( d > 0 \) such that for any \( x \in A \) there are \( \varepsilon_x > 0 \), \( K_x > 0 \) such that \( m_d(F(B_{\varepsilon_x}(x) \cap A)) \leq K_x \lambda(B_{\varepsilon_x}(x)), \forall \varepsilon < \varepsilon_x, \) where \( m_d \) is the Hausdorff measure of dimension \( d \), and there is \( A' \subset A \) such that \( \lambda(A \setminus A') = 0 \) and \( \lim_{\varepsilon \to 0} m_d(F(B_{\varepsilon}(x) \cap A')) = 0, \forall x \in A' \). Then \( m_d(F(A)) = 0 \).
Remark 3.2. The same result is true if we replace $B_x(x) \text{ by } C_x(x)$.

Remark 3.3. We can replace the condition

\[ m_d(F(B_x(x) \cap A)) \leq K_x \lambda(B_x(x)), \quad \forall \varepsilon < \varepsilon_x \]

by

\[ F(B_x(x) \cap A) \text{ can be covered by balls } B_{\delta_i}(y_i), \quad i \in \mathbb{N}, \]

with \( \sum_{i=1}^{\infty} \delta_i^d \leq K_x \lambda(B_x(x)), \quad \forall \varepsilon < \varepsilon_x \),

and the condition

\[ \lim_{\varepsilon \to 0} \frac{m_d(F(B_x(x) \cap A))}{\lambda(B_x(x))} = 0, \quad \forall x \in A' \]

by

\[ F(B_x(x) \cap A) \text{ can be covered by balls } B_{\delta^{(e)}_i}(y_i), \quad i \in \mathbb{N} \]

with \( \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{\infty} (\delta^{(e)}_i)^d}{\lambda(B_x(x))} = 0, \quad \forall x \in A' \).

The proof remains essentially the same, and Remark 3.2 is still valid.

Remark 3.4. If we replace the conditions of this lemma by "$F(B_x(x) \cap A)$ can be covered by balls $B_{\delta_i}(y_i), \quad i \in \mathbb{N}$, with \( \sum_{i=1}^{\infty} \delta_i^d \leq k \lambda(B_x(x)), \quad \forall \varepsilon < \varepsilon_x \) (note that here $k$ does not depend on $x$), and $\lambda(A) < \infty'$, then we can conclude, using the same proof, that $m_d(F(A)) \leq k \lambda(A)$.

Proof: We may suppose that $A$ has finite Lebesgue measure, since $A$ is a countable union of sets with finite measure, and a countable union of sets with Hausdorff $d$-measure zero has Hausdorff $d$-measure zero. Moreover, since $A = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{x \in A \mid K_x \leq k\}$, we may suppose $K_x \leq K, \forall x \in A$. Let $C$ be the Lebesgue measure of $A$.

Let $\varepsilon > 0$. For each $x \in A'$ take $\delta_x > 0$ such that $B_{\delta_x}(x) \subset U$ and

\[ r \leq \delta_x \Rightarrow \frac{m_d(F(B_{\delta_x}(x) \cap A))}{\lambda(B_{\delta_x}(x))} \leq \frac{\varepsilon}{2(2C+1)}. \]

By the Lemma 3.1 we can cover $A'$ by $\bigcup_{n=1}^{\infty} B_{r_n}(x_n)$ with

\[ \sum_{n=1}^{\infty} \lambda(B_{r_n}(x_n)) < C + 1 \]
Theorem 3.3. Let $r_0$ and $C$. G. T. de A. Moreira

Theorem 3.4. Let $t$ in the set of critical points under consideration. Here we do not suppose differentiability in every point, but only subspace by an interval of radius $r$ measure of dimension $0$ such that $\delta > R$ of $F$. It is a simple consequence of Lemma 3.2, since if $\lambda(B_{r_n}(\bar{x}_n)) < \frac{\varepsilon}{2K} \Rightarrow \sum_{n=1}^{\infty} m_d(F(B_{r_n}(\bar{x}_n))) \leq \frac{\varepsilon}{2K} \cdot K = \frac{\varepsilon}{2} \Rightarrow m_d(F(A')) \leq \frac{\varepsilon}{2} \Rightarrow m_d(F(A)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.

Since $\varepsilon > 0$ is arbitrary we have $m_d(F(A)) = 0$. \hfill \Box

We first use Lemma 3.2 to prove the following strong version of Constantin's result ([Co]), that does not suppose continuity of the derivatives. Here we do not suppose differentiability in every point, but only in the set of critical points under consideration.

**Theorem 3.3.** Let $F: X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, and let $A = \{ x \in X \mid DF(x) exists and is not surjective \}$. Then $\lambda(F(A)) = 0$.

**Proof:** It is a simple consequence of Lemma 3.2, since if $x \in A$ then $\lim_{r \to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. Indeed, $x \in A \Rightarrow F(x + h) = F(x) + DF(x).h + r(h)$, where $\lim_{h \to 0} \frac{\|h\|}{m} = 0$. Let $K = ||DF(x)||$, and let $\varepsilon \in (0,1)$. Let $\delta > 0$ such that $|h| \leq \delta \Rightarrow \frac{|r(h)|}{|h|} < \frac{\varepsilon}{2(K + 1)^{\frac{n-1}{2}}}$. Then, if $|h| \leq \delta$, $F(x + h) - F(x)$ belongs to an $\frac{\varepsilon|\|h\||}{2(K + 1)^{\frac{n-1}{2}}}$ neighbourhood of a ball of radius $K|\|h\||$ in a subspace of $\mathbb{R}^n$ of dimension $n - 1$ (a fixed subspace of $\mathbb{R}^n$ of dimension $n - 1$ which contains the image of $DF$), and thus belongs to the orthogonal product of a ball of radius $(K + 1)|\|h\||$ in this subspace by an interval of radius $\frac{\varepsilon|\|h\||}{2(K + 1)^{\frac{n-1}{2}}}$. Therefore, $\lambda(F(B_r(x))) \leq \varepsilon \cdot \frac{\varepsilon|\|h\||}{2(K + 1)^{\frac{n-1}{2}}} \cdot \pi^{n-1} = \varepsilon r^n v_{n-1},$ where $v_{n-1}$ is the volume of the unitary ball in $\mathbb{R}^{n-1}$, and, since $\varepsilon > 0$ is arbitrary, $\lim_{r \to 0} \frac{\lambda(F(B_r(x)))}{\lambda(B_r(x))} = 0$. \hfill \Box

**Theorem 3.4.** Let $F: U \subset \mathbb{R}^n \overset{C^k}{\longrightarrow} \mathbb{R}^m$ be a function of class $C^{k+(\alpha)}(\alpha \in (0,1])$ at $C(F) := \{ x \in U \mid \text{rank}(DF(x)) \leq p \}$. Then the Hausdorff measure of dimension $d = p + \frac{n-p}{k+\alpha}$ of $F(C_p(F))$ is zero, $\forall \ p < \min\{m,n\}$.
Proof: Since $C_p(F) = \{ x \in U \mid \text{rank}(DF(x)) = r \}$, and $r + \frac{2}{k+\alpha} \leq p + \frac{2}{k+\alpha}$ for $0 < r \leq p$, we may restrict our attention to $\tilde{C}_p(F) = \{ x \in U \mid \text{rank}(DF(x)) = p \}$. If $x_0 \in C_p(F)$, we have, after a change of coordinates of class $C^\infty$, $F(z, y) = (z, G(z, y))$, with $(z, y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $G(z, y) \in \mathbb{R}^{n-p}$, in a neighbourhood of $x_0 = (z_0, y_0)$. We shall restrict our attention to this neighbourhood. We have $x = (z, y) \in C_p(F) \Leftrightarrow D_y G(z, y) = 0$. We can apply the results of Theorem 2.1, Corollary 2.3 and Remark 2.1 to the function $D_y G$, and obtain the decomposition $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \psi_i(V_i \times B_i)$, where $\lambda(A_i) = 0$. Let us fix such an $A_i$.

Suppose $\varepsilon < \delta(\varepsilon)$, $\delta(\varepsilon) \leq \delta(\varepsilon)$. Let us fix such an $A_i$. Since $\psi_i^{-1}(A_i) = \bigcup_{m \in \mathbb{N}} \{ x \in \psi_i^{-1}(A_i) \mid \varepsilon_x \geq \frac{1}{m}, K_x \leq M \}$, we may suppose $\varepsilon_x \geq \frac{1}{n}$, $K_x \leq M$, $\forall x \in \psi_i^{-1}(A_i)$, for some fixed $M$ and also that $V$ has finite Lebesgue measure $\lambda(V)$.

With these assumptions, we shall prove that there is a constant $K_0$ such that for any $X \subset V$, $\nu > 0$, we can cover $F(A_i \cap X)$ by balls $B_{\delta}(p_i)$ so that $\sum_{i=1}^{\infty} \delta_i^2 \leq K_0(\lambda(X) + \nu)$. For this, given a point $x \in A_i \cap X$ and an $\varepsilon < \frac{1}{\sqrt{n}M}$, we can divide the cube $C_\varepsilon(x) = C_\varepsilon(z) \times C_\varepsilon(y)$ into $[\varepsilon^{1-(k+\alpha)} + 1]^p$ boxes $C_{\delta}(z_i) \times C_\varepsilon(y_i)$, $\delta < \varepsilon^{k+\alpha}$. If there is some point $(z_i, y_i)$ in $C_{\delta}(z_i) \times C_\varepsilon(y_i) \cap (A_i \cap X)$, then for any point $(z_i', y_i')$ in $C_{\delta}(z_i) \times C_\varepsilon(y_i) \cap (A_i \cap X)$, we have $|F(z_i', y_i') - F(z_i, y_i)| \leq K_0|F(z_i, y_i) - F(z_i', y_i')|$. This is $C_{\alpha}$ times a Lipschitz constant of $F|_V$, which we may suppose to exist.

Observe now that $(z_i, y_i) = (z_i, \tilde{\psi}_i(p_1))$ and $(z_i', y_i') = (z_i, \tilde{\psi}_i(p_2))$, for some $p_1, p_2 \in \{ z_i \} \times B_i$ with $|p_1 - p_2| \leq |y_i - y_i'| \leq 2\varepsilon\sqrt{n}$. Let $\gamma : [0, 1] \to V_i \times B_i$ be a straight path joining $p_1$ and $p_2$. Then $G(z_i, y_i) - G(z_i', y_i') = \int_0^1 \frac{\partial G}{\partial y}(\gamma(t)) \cdot \gamma'(t) \, dt$, where $\gamma := \psi_i \circ \gamma$. We have $\frac{\partial G}{\partial y}(\gamma(0)) = 0$, so

$$\left\| \frac{\partial G}{\partial y}(\gamma(t)) \right\| \leq M|p_1 - p_2|^{k+\alpha-1} \leq M(2\varepsilon\sqrt{n})^{k+\alpha-1} \Rightarrow \left\| \frac{\partial G}{\partial y}(\gamma(t)) \right\| \| \gamma'(t) \| \leq K'' \varepsilon^{k+\alpha},$$

for some constant $K''$. Indeed, $|\gamma'(t)|$ is limited by a constant multiple of $|p_1 - p_2| \leq 2\sqrt{n}\varepsilon$. So

$$|G(z_1, y_2) - G(z_i, y_i)| \leq \int_0^1 \left| \frac{\partial G}{\partial y}(\gamma(t)) \circ \gamma(t) \right| \, dt \leq K'' \varepsilon^{k+\alpha}.$$
and

\[ |F(z_i', y') - F(z_i, y_i)| \leq K' \delta + K'' \varepsilon^{k+\alpha} \leq K_0' \cdot \varepsilon^{k+\alpha}, \]

where \( K_0' = K' + K'' \).

Therefore, \( F(C_{\delta}(z_i) \times C_{\varepsilon}(y)) \) is contained in some ball \( B_{\delta_i}(q_i) \), with \( \delta_i \leq K_0' \cdot \varepsilon^{k+\alpha} \), so

\[
\sum_i \delta_i^d \leq ((\varepsilon^{1-(k+\alpha)}) + 1)^p(K_0' \varepsilon^{k+\alpha})^d
\]

\[
= (K_0')^d((\varepsilon^{1-(k+\alpha)}) + 1)^p(\varepsilon^{k+\alpha})^p \frac{\nu - \nu}{\nu - \alpha}
\]

\[
\leq K_0 \varepsilon^n
\]

for some constant \( K_0 \). So, \( F(C_{\varepsilon}(x)) \) can be covered by balls \( B_{\delta_i}(q_i) \) with \( \sum_i \delta_i^d \leq K_0 \lambda(C_{\varepsilon}(x)) \), and by the Lemma 3.2, Remarks 3.2 and 3.4, we can conclude that \( m_d(F(A_i \cap X)) \leq K_0 \lambda(X) \) where \( K_0 = 2^n K_0 \), and so we can cover \( F(A_i \cap X) \) by balls \( B_{\delta_i}(p_i) \) so that \( \sum_{i=1}^{\infty} \delta_i^d \leq K_0 \lambda(X + \nu) \), as we stated.

We shall prove now that there is an \( A'_i \subset A_i \subset V \) with \( \lambda(A_i \setminus A'_i) = 0 \) such that \( F(C_{\varepsilon}(x) \cap A_i) \) can be covered by balls \( B_{\delta'_i}(W_i) \), \( i \in \mathbb{N} \) with \( \lim \varepsilon \to 0, \sum \frac{\delta_i^{i(p+q)}}{\lambda(C_{\varepsilon}(x))} = 0, \forall x \in A'_i \). This will imply our theorem, by the Lemma 3.2, Remarks 3.2 and 3.3, since we have proved above that \( m_d(F(C_{\varepsilon}(x) \cap A_i)) \leq K_0 \lambda(C_{\varepsilon}(x)) \), \( \forall \varepsilon < \frac{1}{2\sqrt{n}M} \). For this, since \( A_i \subset \psi_i(V_i \times B_i) \), \( B_i \subset \mathbb{R}^+, r_i \leq n - p \), we may suppose \( r_i = n - p \) and \( \psi_i = \text{identity} \), because \( r_i < n - p \Rightarrow \lambda(A_i) = 0 \) and we can take \( A'_i = A_i \). Let us take \( A'_i \) equal to the set of the density points of \( A_i \). Given a point \( x \in A'_i \), and an \( \eta' > 0 \), we want to find an \( \varepsilon_0 > 0 \) such that \( \varepsilon < \varepsilon_0 \Rightarrow F(C_{\varepsilon}(x) \cap A_i) \) can be covered by balls \( B_{\delta'_i}(W_i) \), \( i \in \mathbb{N} \) such that \( \sum_i \frac{\delta_i^{i(p+q)}}{\lambda(C_{\varepsilon}(x))} \leq \eta' \lambda(C_{\varepsilon}(x)) \). Let \( \eta, \tilde{\eta} > 0, \tilde{\varepsilon} < \frac{1}{2\sqrt{n}M} \) such that \( \frac{\lambda(C_{\varepsilon}(x) \cap A_i)}{\lambda(C_{\varepsilon}(x))} > 1 - \tilde{\eta}^2, \forall \varepsilon \leq \tilde{\varepsilon} \). Divide the cube \( C_{\varepsilon}(x) = C_{\varepsilon}(z) \times C_{\varepsilon}(\tilde{y}) \), \( \varepsilon < \tilde{\varepsilon} \) into \( N = (\frac{1}{1-\eta} \eta^{-1} + 1)^p \) boxes \( C_{\delta}(z_i) \times C_{\varepsilon}(\tilde{y}), \delta < \eta \varepsilon^{k+\alpha}, 1 \leq i \leq N \). Then for at least \( (1 - \tilde{\eta})N \) values of \( i \), there is a \( z_i \in C_{\delta}(z_i) \) such that \( \lambda((y \in C_{\varepsilon}(\tilde{y}) \mid (y, z_i) \in A_i)/\lambda(C_{\varepsilon}(\tilde{y})) > 1 - \tilde{\eta} \) (here \( \lambda \) is the Lebesgue measure in \( \mathbb{R}^{n-p} \), because \( \lambda(C_{\varepsilon}(x) \cap A_i) > (1 - \tilde{\eta}^2) \lambda(C_{\varepsilon}(x)) \).
For such an $i$, take an $y_i$ such that $(z_i, y_i) \in A_i$. Then, applying Theorem 2.1, Corollaries 2.2 and 2.3, given $\eta$ we can choose $\eta_i$ so that $|F(z_i, y) - F(z_i, y_i)| < \eta \cdot \varepsilon^{k+\alpha} \Rightarrow |F(z, y) - F(z_i, y_i)| \leq 2K' \sqrt{n} \eta \varepsilon^{k+\alpha} + \eta \varepsilon^{K+\alpha}$, for some constant $K''$ and for any $z \in C_{\varepsilon}(\tilde{z}_i)$, where $K'$ is a Lipschitz constant for $F \Rightarrow F(C_{\varepsilon}(\tilde{z}_i) \times C_{\varepsilon}(\tilde{y}))$ is contained in a ball $B_{\delta_i}(Q_i)$, with $\sum_i \delta_i$ over these values of $i$ less than

$$(|\varepsilon^{-1(k+\alpha)}\eta^{-1}| + 1)^p \varepsilon^{k+\alpha}\eta^n = \varepsilon^{k+\alpha}\eta^n \leq \tilde{K}_0 \eta^{\frac{n}{k+\alpha}} \varepsilon^n$$

for some constant $\tilde{K}_0$. The union of the remaining (at most $\tilde{N}$) boxes has volume at most $\tilde{N} \eta^n \Rightarrow$ the union of the image of the intersection of $A_i$ with the union of these boxes by $F$ is contained in a union of balls $B_{\delta_i}(Q_i)$ with $\sum_i \delta_i \leq 2\tilde{K}_0 \eta^n$, by the statement proved before, and so $F(C_{\varepsilon}(x))$ can be covered by balls $B_{\delta_i}(\tilde{Q}_i)$ with $\sum \delta_i \leq (\tilde{K}_0 \eta^{\frac{n}{k+\alpha}} + 2\tilde{K}_0 \eta) \varepsilon^n$. Choosing $\eta$, $\tilde{N}$ so small that $\tilde{K}_0 \eta^{\frac{n}{k+\alpha}} + 2\tilde{K}_0 \eta \leq \eta'$, we obtain the desired result with $\varepsilon_0 = \tilde{\varepsilon}$.

**Remark 3.5.** In the cases of functions of class $C^k$ ($C^{k+\alpha}$ with $\alpha = 0$) we have the same result. If $k \geq 2$, it follows from the case of class $C^{k-1+1(\varepsilon)}$ of the theorem. If $k = 1$, $p + \frac{n}{k+\alpha} = n$, and the proof of the Theorem 3.3 shows that if $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function and $C(F) = \{x \in U \mid DF(x) \text{ exists and rank } DF(x) < n\}$ then $m_n(F(C(F))) = 0$, where $m_n$ is the Hausdorff measure of dimension $n$.

### 4. Examples

In this section we give some examples which show that the previous results are quite sharp. In all these examples we shall use a certain kind of functions of the real line that we shall describe below.

**Definition 4.1.** Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$, $\forall i \in \mathbb{N}$. The central Cantor set $K_{\lambda}$ is the Cantor set constructed as follows: We remove from the interval $[0, 1]$ the central open interval $U_{1,1}$ of proportion $1 - 2\lambda_1$, then we remove from the two remaining intervals the central open intervals $U_{2,1}$ and $U_{2,2}$ of proportion $1 - 2\lambda_2$, and so on. After the $r$-th step of the construction there will remain $2^r$ intervals of length $\lambda_1 \lambda_2 \ldots \lambda_r$. The intersection of all these sets is the central Cantor set $K_{\lambda}$. The open intervals removed in the $r$-th step of the construction have length $\lambda_1 \lambda_2 \ldots \lambda_{r-1}(1 - 2\lambda_r)$. 
Let \( \psi: \mathbb{R} \to \mathbb{R} \) be a fixed function such that \( \psi([0,1]) \subseteq [0,1], \psi(x) = 0, \forall x \leq 0, \psi(x) = 1, \forall x \geq 1 \). Given two central Cantor sets \( K_\lambda \) and \( K_\mu \), we construct the function \( f_{\lambda,\mu}: \mathbb{R} \to \mathbb{R} \) as follows: \( f_{\lambda,\mu}(x) = 0, \forall x \leq 0, f_{\lambda,\mu}(x) = 1, \forall x \geq 1 \), and if \( U_{i,j} = (a,b) \) and \( \tilde{v}_{i,j} = (c,d) \) are corresponding removed intervals in the constructions of \( K_\lambda \) and \( K_\mu \), respectively, we define \( f_{\lambda,\mu}(x) = c + (d-c)\psi(\frac{x-a}{b-a}) \), \( \forall x \in (a,b) \). We extend \( f_{\lambda,\mu} \) to \( K_\lambda \) by continuity, obtaining \( f_{\lambda,\mu}(K_\lambda) = K_\mu \). It is easy to check that if \( g_r := \lambda_1 \lambda_2 \ldots \lambda_{r-1}(1-2\lambda_r) \) and \( \tilde{g}_r := \mu_1 \mu_2 \ldots \mu_{r-1}(1-2\mu_r) \) satisfy \( \lim_{r \to \infty} \tilde{g}_r = 0 \) then \( f_{\lambda,\mu} \) is \( C^k \) (if \( k \geq 1 \) is an integer). Moreover, if \( q > 1 \), and \( \sup_{r} \frac{\tilde{g}_r}{g_r} < \infty \) then \( f_{\lambda,\mu} \) is \( C^{q-1,1} \) if \( q \) is integer and is \( C^q \) (i.e., it is \( C^{q+(q)} \), where \( \{q\} = q-[q] \in (0,1) \)) otherwise. See [BMPV] for more details.

**Example 4.1.** Let \( \lambda_n = \frac{1}{2} - \frac{1}{2^n}, \mu_n = a \). Then \( \lim_{n \to \infty} \frac{\tilde{g}_n}{g_n} = 0, \forall q < -\frac{\log a}{\log 2} \), and so \( f_{\lambda,\mu} \) is \( C^q \), \( \forall q < -\frac{\log a}{\log 2} \). On the other hand, \( m_d(K_\mu) = 1 \) where \( d = -\frac{\log 2}{\log a} \) (see [PT]). Moreover, since \( a \in (0, \frac{1}{2}) \), \( \lim_{n \to \infty} \frac{\tilde{g}_n}{g_n} = 0 \), and so \( f_{\lambda,\mu}'(x) = 0, \forall x \in K_\lambda \). If \( F: \mathbb{R}^{n+p} \to \mathbb{R}^{n+p} \) is given by

\[
F(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+p}) = (f_{\lambda,\mu}(x_1), f_{\lambda,\mu}(x_2), \ldots, f_{\lambda,\mu}(x_n), x_{n+1}, \ldots, x_{n+p}),
\]

then

\[
F(C_p(F)) = F(K_\lambda \times K_\lambda \times \cdots \times K_\lambda \times \mathbb{R}^p) = K_\mu \times K_\mu \times \cdots \times K_\mu \times \mathbb{R}^p
\]

that is a set with positive \((nd+p)\)-measure. This shows that given \( q > 1 \), \( p > 0 \) and \( n > p \) there is a map \( F: \mathbb{R}^n \to \mathbb{R}^n \) such that \( m_d(F(C_p(F))) > 0 \), where \( d = p + \frac{n-p}{q} \), and \( F \) is of class \( C^q \) for each \( q' < q \).

**Remark 4.1.** If \( a = \frac{1}{2^n} \), \( F(x_1, x_2, \ldots, x_n) = f_{\lambda,\mu}(x_1) + 2f_{\lambda,\mu}(x_2) + \cdots + 2^{n-1}f_{\lambda,\mu}(x_n) \) gives an example of a function \( \tilde{F}: \mathbb{R}^n \to \mathbb{R} \) which is of class \( C^q \), \( \forall q < n \ (q \in \mathbb{R}) \) such that \( F(\mathbb{R}^n) \) contains an open set, since \( K_\lambda + 2K_\lambda + \cdots + 2^{n-1}K_\lambda = [0, 2^n - 1] \), which can be proved easily using representation in basis \( 2^n \).

**Example 4.2.** Let \( \lambda_n = \frac{1}{2} - \frac{1}{2^n}, \mu_n = a - \frac{\theta_n}{2^n}, a \in (0, 1/2) \). Then \( \lim_{n \to \infty} \frac{\tilde{g}_n}{\tilde{g}_n} = 0, \forall q = -\frac{\log a}{\log 2} \), and so \( f_{\lambda,\mu} \) is \( C^q \). On the other hand we have \( HD(K_\mu) \geq -\frac{\log a}{\log 2} \). Indeed, if \( b < a \) and \( \theta_n \equiv b, f_{\lambda,\mu}(b) \) is clearly \( C^1 \), and \( f_{\lambda,\mu}(b) = K_\mu \Rightarrow HD(K_\mu) \geq HD(K_\mu) = -\frac{\log a}{\log 2} \), \forall b < a. If
$F: \mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ is given by

$$F(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+p}) = (f_{\lambda, \mu}(x_1), \ldots, f_{\lambda, \mu}(x_n), x_{n+1}, \ldots, x_{n+p})$$

then

$$F(C_p(F)) = K_\mu \times \cdots \times K_\mu \times \mathbb{R}^p,$$

that is a set with Hausdorff dimension $nd + p$, where $d = -\log a / \log 2$. This shows that given $q \geq 1$, $p > 0$ and $n > p$ there is a map $F: \mathbb{R}^n \to \mathbb{R}^n$ such that $HD(F(C_p(F))) = p + \frac{n-p}{q}$, and $F$ is of class $C^q$.

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