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# ON THE EXISTENCE OF CANARD SOLUTIONS

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*Abstract*

We study the existence of global canard surfaces for a wide class of real singular perturbation problems. These surfaces define families of solutions which remain near the slow curve as the singular parameter goes to zero.

## 1. Introduction

Let $S$ be a one-dimensional connected real analytic manifold and $\pi\colon E \to S$ an analytic line bundle over $S$. We shall say that an analytic vector field $X$ on $E$ *vanishes to the first order at the zero section* if there exists an open covering $\{U_\alpha\}$ $(\alpha \in A)$ of $S$ and a collection of local trivializing charts

$$\varphi_\alpha(U_\alpha \times \mathbb{R}) \approx \pi^{-1}(U_\alpha)$$

such that, on the coordinates $(x, y) \in U_\alpha \times \mathbb{R}$,

$$X = y f_\alpha(x, y) \frac{\partial}{\partial y},$$

for some analytic function $f_\alpha$ which is non divisible by $y$.

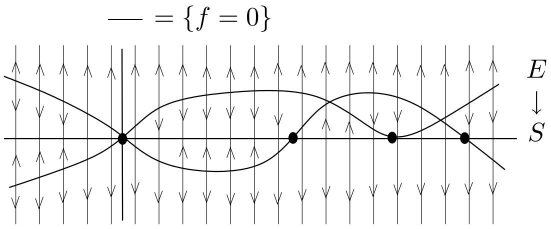


Figure 1. A vector field that vanishes to the first order at the zero section (the dots indicate the degenerate singularities on $S$).

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The zero section $\Gamma = \{y = 0\} \approx S$ of $E$ is a curve of singularities for $X$. These singularities are of two types:

- *Normally hyperbolic singular points*: points $x \in \Gamma$ where $B_x = f_\alpha(x, 0) \neq 0$, i.e. the jacobian matrix $DX(x, 0)$ has the eigenvector $(1, 0)$ associated to the non-zero eigenvalue $B_x$.
- *Degenerate singularities*: points $x \in \Gamma$ where $f_\alpha(x, 0)$ vanishes, i.e. the jacobian matrix is zero.

The set of degenerate singularities is a discrete subset of points $\mathrm{Deg} \subset \Gamma$. We shall say that a point $x \in \mathrm{Deg}$ is *degenerate of order* $p$ if, given some open set $U_\alpha$ such that $x \in U_\alpha$, the corresponding function $f_\alpha$ has a zero of multiplicity $p$ at $x$.

Let now $X_{\varepsilon,a}$ be a family of analytic vector fields on $E$, depending analytically on parameters $\varepsilon \in (\mathbb{R}^+, 0)$ and $\alpha \in (\mathbb{R}^n, 0)$ (where, as usual, $(\mathbb{R}^n, 0)$ denotes some neighborhood of the origin in $\mathbb{R}^n$). We shall say that $X_{\varepsilon,a}$ is a *singular perturbation of transition type* on $E \to S$ if

- there are charts $(U_\alpha, \varphi_\alpha)$ as above such that
$$X_{\varepsilon,a} = \varepsilon \frac{\partial}{\partial x} + F_\alpha(x, y, \varepsilon, a) \frac{\partial}{\partial y}$$
for some analytic function $F_\alpha(x, y, a, \varepsilon)$ and,
- $X_{0,0} = X$ vanishes to the first order at the zero section.

(This second condition being equivalent to require that $F_\alpha(x, y, 0, 0) = y f_\alpha(x, y)$ for some analytic function $f_\alpha$, non-divisible by $y$.) Notice that for $\varepsilon = 0$, the vector field $X_{0,a}$ is *vertical* (i.e. everywhere tangent to the fibration $\{d\pi = 0\}$).

The first component $\varepsilon \frac{\partial}{\partial x}$ of such family $X_{\varepsilon,a}$ naturally induces a global orientation on $\Gamma$. Given two points $x_0$, $x_1$ on $\Gamma$, we shall write $x_0 < x_1$ to indicate that there exists a positively oriented path on $\Gamma$ going from $x_0$ to $x_1$ (notice that if $\Gamma \approx \mathbb{S}^1$ we necessarily have $x_0 < x_1 < x_0$).

Qualitatively speaking, there are four basic types of degenerate points $x \in \mathrm{Deg}$:

(a) *Stable-stable transition points*: for each non-degenerate singularity $x' \in \Gamma$ in some small neighborhood of $x$, the nonzero eigenvalue $B_{x'}$ is strictly negative.
(b) *Unstable-unstable transition points*: for each non-degenerate singularity $x' \in \Gamma$ in some small neighborhood of $x$, $B_{x'}$ is strictly positive.
(c) The *unstable-stable transition points*, where for each point $x' < x$ (respect. $x' > x$) in some sufficiently small neighborhood of $x$, $B_{x'}$ is strictly positive (respect. negative).

(d) The *stable-unstable transition points*, where for each point $x' < x$ (respect. $x' > x$) in some sufficiently small neighborhood of $x$, $B_{x'}$ is strictly negative (respect. positive).

For shortness, we shall simply denote these four situations respectively by $(\mathbf{s}, \mathbf{s})$, $(\mathbf{u}, \mathbf{u})$, $(\mathbf{u}, \mathbf{s})$ and $(\mathbf{s}, \mathbf{u})$.

We shall say that a $C^\infty$ curve on the parameter space,

$$[0, \delta) \ni \rho \mapsto \gamma(\rho) = (\varepsilon(\rho), a(\rho)) \in (\mathbb{R}^+, 0) \times (\mathbb{R}^n, 0),$$

(for some $\delta > 0$) is a *control curve* for $X_{\varepsilon,a}$ if

- $\gamma(0) = 0$ and
- $\varepsilon(\rho) > 0$ for $\rho > 0$.

Given such a curve, one can consider the *one-parameter restriction*, $X_\rho^\gamma = X_{\varepsilon(\rho),a(\rho)}$ of the original family, which on each trivializing chart $(U_\alpha, \varphi_\alpha)$ is given by

$$X_\rho^\gamma = \varepsilon(\rho)\frac{\partial}{\partial x} + F_\alpha(x, y, \varepsilon(\rho), a(\rho))\frac{\partial}{\partial y}.$$

Such restricted family can be seen as a $C^\infty$ vector field on the 3-dimensional manifold (with boundary),

$$M = E \times (\mathbb{R}^+, 0),$$

and this vector field is everywhere tangent to the leaves of the foliation $\mathcal{F} = \{d\rho = 0\}$ (i.e. the function $\rho$ is a *first integral* of $X_\rho^\gamma$). The condition that $\varepsilon(\rho) > 0$ for $\rho > 0$ implies that $X_\rho^\gamma$ vanishes only on the curve

$$\Gamma = \{y = \rho = 0\}.$$

We shall say that a $C^0$ two-dimensional submanifold $W^\gamma \subset M$ is a *canard surface* for $X_\rho^\gamma$ if

- $W^\gamma$ is an invariant surface for $X_\rho^\gamma$ (seen as a vector field on $M$);
- $W^\gamma$ is the graph of a function

$$\begin{array}{rccc} w\colon & (\mathbb{R}^+, 0) \times S & \longrightarrow & E \\ & (\rho, x) & \longmapsto & w(\rho, x) \end{array}$$

  such that, for each fixed $\rho_0 \in (\mathbb{R}^+, 0)$, $w(\rho_0, \cdot)$ is a continuous section of the bundle $E \to S$, and

$$w(0, x) \equiv 0.$$

  Moreover, such function $w$ is $C^\infty$ on $M \setminus \mathrm{Deg}$ and has a $C^\infty$ blow-up extension (see definition at the next section) at each degenerate point $x \in \mathrm{Deg}$.

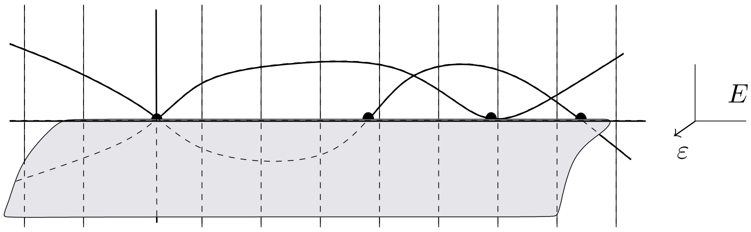



FIGURE 2. The canard surface.

The *existence problem* for canard surfaces can now be stated as follows:

**Canard Problem:** Given a singular perturbation of transition type $X_{\varepsilon,a}$ on a line bundle $E \to S$, can we find a control curve $\rho \mapsto \gamma(\rho)$ on the parameter space such that the restricted family $X^\gamma_\rho$ has a canard surface $W^\gamma$?

The main goal of this work is to consider the *local version* of this problem.

In the local formulation, we let $S = U_x \subset (\mathbb{R}, 0)$, $E = U_x \times \mathbb{R}$ and $\pi\colon E \to S$ be given by $\pi(x, y) = x$. Then, a singular perturbation problem of transition type is equivalent (up to a reparameterization of time) to a *first order singularly perturbed differential equation*,

$$\varepsilon \frac{dy}{dx} = F(x, y, \alpha, \varepsilon), \tag{1}$$

where $F(x, y, \alpha, \varepsilon)$ is an analytic function on $E$ such that

$$F(x, y, 0, 0) = yf(x, y),$$

for some $f$ non-divisible by $y$. In this case, if we allow $E$ and $S$ to shrink to smaller neighborhoods of the origin, we obtain the *Local Canard Problem*:

**Local Canard Problem:** Given a first order singularly perturbed differential equation as above, can we find a smaller neighborhood of the origin $U'_x \subset (\mathbb{R}, 0)$ such that, if we let $E' = U'_x \times \mathbb{R}$, the Canard Problem has a positive answer when restricted to $E' \to S'$?

Our main result is the following:

**Theorem 1.1.** *If the family $X_{\varepsilon,a}$ satisfies the Improved Transversality Hypothesis at $x = 0$ (see Section 11.2), then the Local Canard Problem has a positive answer.*

In fact, our results will provide a much more detailed answer to such problem. We will prove that, under the hypothesis of the theorem, there exists a non-empty region in the parameter space (so-called *canard region*)

$$\mathcal{O} \subset \{(\varepsilon, a) \in \mathbb{R}^+ \times \mathbb{R}^n \mid \varepsilon > 0\},$$

which contains the origin in its closure, and such that for each $C^\infty$ control curve $\gamma(\rho)$ on the space of parameters which verifies

$$\gamma((0, \delta)) \subset \mathcal{O},$$

the restricted family $X^\gamma_\rho$ has a canard surface. Such region $\mathcal{O}$ will be either *(i)* an open semi-analytic set or *(ii)* a $C^\infty$ graph over a semi-analytic codimension one hypersurface.

Our method is mainly geometric, and combines Desingularization Theory with the Center Manifold Theorem. Such geometric method has been firstly used in [**Du-R**].

In the last section, we shall give an example which shows how our construction can be used to attack the (much more difficult) global canard problem.

**Previous works.** We refer the reader to [**Be**] for an extensive bibliography of related works.

The canard problem has a counterpart in the complex setting (take $\mathbb{C}$ instead of $\mathbb{R}$ as the base field and let $\varepsilon \in (\mathbb{C}, 0)$ belong to some sector $V$ with vertex at the origin and angle less than $2\pi$). For a precise formulation, see [**W**], [**C-R-S-S**].

In the complex setting, the canard surface is usually called a (local) *overstable solution*. In [**C-R-S-S**], the authors provides a sufficient condition for the existence of local overstable solutions in a generalized setting, where $E = (\mathbb{C}^{k+1}, 0)$ and $E \to S$ is a $k$-dimensional vector bundle over $S$. This sufficient condition is equivalent to our Transversality Hypothesis (see Section 4).

### 1.1. Examples.

Most of the examples show that the extra parameters $a \in (\mathbb{R}^n, 0)$ are necessary to guarantee the the existence of a canard surface.

**Example 1.2.** Consider the simple linear family

$$X_\varepsilon = \varepsilon \frac{\partial}{\partial x} + (\varepsilon + xy)\frac{\partial}{\partial y},$$

which can be explicitly integrated on the region $\{\varepsilon > 0\}$. For any initial condition, say $y(-1) = y_0 \in \mathbb{R}$, we obtain the solution

$$y(x,\varepsilon) = e^{\frac{x^2}{2\varepsilon}} \left( y_0\, e^{\frac{-1}{2\varepsilon}} + \int_{-1}^{x} e^{-\frac{t^2}{2\varepsilon}}\, dt \right).$$

Let us suppose, by absurd, that the graph of $y(x,\varepsilon)$ defines a canard surface. Then, it follows that there exists some neighborhood of the origin $U_x = [x_0, x_1] \subset \mathbb{R}$ (for $x_0 < 0 < x_1$) and $U_\varepsilon = [0, \varepsilon_0] \subset \mathbb{R}^+$ such that $y(x,\varepsilon)$ is a continuous function on $U_x \times U_\varepsilon$; with $y(x,0) \equiv 0$.

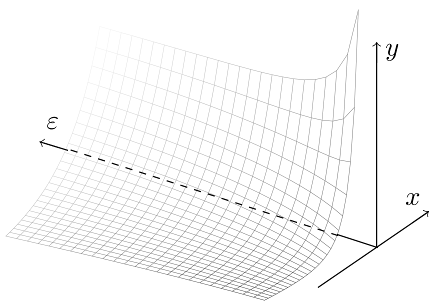



FIGURE 3. The graph of $y(x,\varepsilon)$ on Example 1.2.

In particular, if we restrict to the curve $\varepsilon = x^3$ (for $0 \leq x \leq x_1$), the function

$$h(x) := y(x, x^3) = e^{\frac{1}{2x}} \left( y_0 \, e^{\frac{-1}{2x^3}} + \int_{-1}^{x} e^{-\frac{t^2}{2x^3}} dt \right)$$

must be continuous at the origin. Notice that the first term

$$y_0 \, e^{\frac{1}{2x} - \frac{1}{2x^3}}$$

clearly goes asymptotically to zero as $x \to 0$. So, to get into an absurd, it suffices to prove that the term

$$H(x) := e^{\frac{1}{2x}} \int_{-1}^{x} e^{-\frac{t^2}{2x^3}} dt$$

goes to infinity. If we restrict $t$ to the interval $[\frac{x}{3}, \frac{x}{2}]$, it follows that

$$\frac{-1}{8x} \leq -\frac{t^2}{2x^3} \leq \frac{-1}{18x}$$

and so,

$$\int_{-1}^{x} e^{-\frac{t^2}{2x^3}} dt \geq \int_{x/3}^{x/2} e^{\frac{-1}{8x}} dt = \frac{1}{6} x \, e^{\frac{-1}{8x}}.$$

Substituting back into the expression of $H(x)$, we obtain

$$H(x) \geq e^{\frac{1}{2x}} \left( \frac{1}{6} x \, e^{\frac{-1}{8x}} \right)$$

which clearly implies that $H(x) \to \infty$ as $x \to 0$.

This example illustrates a typical *stable-unstable transition*. Namely, for each $x < 0$ (respect. $x > 0$) in $\Gamma$, the non-zero eigenvalue of $DX_x$ is strictly negative (respect. strictly positive).

Intuitively, all solutions $y(x)$ with an initial condition at the stable part of $\Gamma$ are initially attracted very rapidly to $\Gamma$ and stays very near such curve until $x$ crosses to the unstable part $x > 0$. At this point, $y(x)$ is immediately repelled.

Notice however that the existence of a canard surface is not guaranteed, even if we suppose that each non-degenerate singular point $x \in \Gamma \setminus \mathrm{Deg}$ has an strictly negative eigenvalue in the hyperbolic direction.

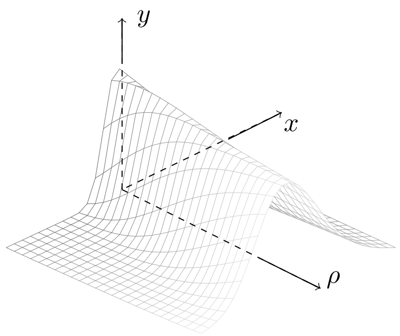



FIGURE 4. The graph of $y = h(x, \rho)$ on Example 1.3.

**Example 1.3.** Consider the family

$$X_{\varepsilon,a,b} = \varepsilon \frac{\partial}{\partial x} + (a + (b - x^2)y) \frac{\partial}{\partial y}.$$

For $\{\varepsilon = a = b = 0\}$, the line of singularities $\Gamma = \{y = 0\}$ presents a *stable-stable* transition, with a degenerate singularity at $x = 0$.

Let is suppose that one chooses two curves $\gamma_1$, $\gamma_2$ on the parameter space, given by

$$(\varepsilon, a, b) = \gamma_1(\rho) = (\rho, 0, 0) \quad \text{and} \quad (\varepsilon, a, b) = \gamma_2(\rho) = (\rho^4, \rho^5, \rho^2).$$

Then, we claim that the restricted family $X_\rho^{\gamma_1}$ has a canard surface, while $X_\rho^{\gamma_2}$ does not. For $\gamma_1$, this is obvious, since

$$X_\rho^{\gamma_1} = \rho \frac{\partial}{\partial x} - x^2 y \frac{\partial}{\partial y}$$

has $W = \{y = 0\}$ as an invariant surface. For $\gamma_2$, the restricted family is given by

$$X_\rho^{\gamma_2} = \rho^4 \frac{\partial}{\partial x} + (\rho^5 + (\rho^4 - x^2)y) \frac{\partial}{\partial y}.$$

Let us suppose by contradiction that there exists a canard surface for such family. Then, it can be defined as the graph of a continuous function $y = h(x, \rho)$. Let us see that such function can not be continuous at the origin. More precisely, let us see that if we restrict $h$ to the line $\{x = \rho\}$ then $h(\rho,\rho) \to \infty$ as $\rho \to 0$.

Restricting to the region $\rho > 0$, $h$ is the solution of the equation

$$\rho^5 \frac{\partial h}{\partial x} = \rho^4 \frac{\partial}{\partial x} + (\rho^5 + (\rho^4 - x^2)h)$$

with an initial condition $h(-x_0,\rho) = y_0$ (for some sufficiently small $x_0 > 0$). Explicitly, the solution is given by the analytic function

$$h(x,\rho) := \left( \int_{-x_0}^{x} \rho\, e^{\left(\frac{u(-\rho^2+1/3\,u^2)}{\rho^4}\right)}\, du + \frac{y_0}{e^{\left(-\frac{x_0(3\,\rho^2-x_0^2)}{3\rho^4}\right)}} \right) e^{\left(\frac{x(\rho^2-1/3\,x^2)}{\rho^4}\right)} \tag{2}$$

and, restricting to the line $x = \rho$,

$$h(\rho,\rho) := \left( \int_{-x_0}^{\rho} \rho\, e^{\left(\frac{u(-\rho^2+1/3\,u^2)}{\rho^4}\right)}\, du + \frac{y_0}{e^{\left(-\frac{x_0(3\,\rho^2-x_0^2)}{3\rho^4}\right)}} \right) e^{\left(\frac{2}{3\rho}\right)}. \tag{3}$$

Since the second term in the above sum goes to zero as $\rho \to 0$, to prove that $h(\rho,\rho)$ goes to infinity it suffices to study the integral

$$I(\rho) := \int_{-x_0}^{\rho} \rho\, e^{\left(\frac{u(-\rho^2+1/3\,u^2)}{\rho^4}\right)}\, du. \tag{4}$$

If we restrict $u$ to the interval $[-\rho, -\rho/2]$, then

$$\frac{u(-\rho^2+1/3\,u^2)}{\rho^4} \geq \frac{11}{24\rho}$$

and therefore,

$$I(\rho) \geq \int_{-\rho}^{-\rho/2} \rho\, e^{\left(\frac{11}{24\rho}\right)}\, du = \frac{\rho^2}{2}\, e^{\left(\frac{11}{24\rho}\right)} \tag{5}$$

which proves that $I(\rho) \to \infty$ as $\rho \to 0$.

### 1.2. Outline of the paper.

In the next section, we define some important classes of differentiable functions. In particular, these classes will contain all functions whose graph defines local center manifolds and canard surfaces.

In Section 3 we show how to define a smooth *dynamical center manifold* locally at a non-degenerate point $x \in \Gamma$. This manifold is uniquely determined by a function (the *initial condition function*).

In Section 4 we state the Transversality Hypothesis and describes the *blowing-up transformation*, which will be a fundamental tool to study the existence of canard surfaces near degenerate points $x \in \mathrm{Deg}$. The blowing-up transformation can be seen as a geometric reinterpretation of the *rescaling transformation*, a classical tool in the study of singular perturbation problems.

In Section 5 we describe how to define smooth dynamical center manifolds at the new non-degenerate singularities which will appear after the blowing-up transformation.

Section 6 describes how to *match together* the center manifolds which are described on the previous section. This matching is possible only over certain regions, and (wherever possible) defines global invariant surfaces over the *exceptional divisor* (i.e. counter-image of the degenerate point $x \in \mathrm{Deg}$ under blowing-up).

In Sections 7 and 8 we try to estimate the region where it is possible to perform the matching. This involves a detailed study of the asymptotic behavior of a certain family of Riccati differential equations.

Section 9 describes the *blowing-down* of the invariant manifold which are constructed in Section 6. We prove that these invariant manifolds can also be *matched* to the dynamical center manifolds which are described in Section 5.

Section 10 uses all the previous results to prove four theorems about the existence of invariant surfaces for singular perturbation families which satisfy the Transversality Hypothesis.

In Section 11 we show that the results of Section 10 remain valid under weaker transversality hypothesis.

Finally, the last section briefly presents two examples of the kind of study which can be made with the results presented in this work.

## 2. Some special classes of functions

Given an open subset $U \subset \mathbb{R}^n$, we shall denote by $C^k(U)$ the $\mathbb{R}$-algebra of $k$-times continuously differentiable functions on $U$. If $M \subset \mathbb{R}^n$ is a closed subset, we shall denote by $C^k(M)$ the class of $C^k$ functions on $M$ *in the sense of Whitney* (see, e.g. [**Wh**], [**Bi**]).

Given a $f \in C^k(U)$, we shall say that $f$ *has a $C^k$ extension to $\overline{U}$* (the closure of $U$) if $f$ and all its partial derivatives up to order $k$ can be continuously extended to $\overline{U}$, in such a way that $f \in C^k(\overline{U})$.

For the corresponding local notion, we shall say that $f \in C^k(U)$ has *a $C^k$ extension at a point $x \in \overline{U}$* if there exists a neighborhood $V \subset \mathbb{R}^n$ of $x$ such that $f \in C^k(\overline{U \cap V})$.

### 2.1. Blow-up extension of $C^k$ functions.

Let $U$ be an open subset in $\mathbb{R}^n$, and let

$$(x, y) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$$

be coordinates on $\mathbb{R}^n$. Given a function $f \in C^k(U)$, we shall say that $f$ has a *blow-up $C^k$ extension at $\{x = 0\}$* if there exist positive natural numbers

$$\alpha_1, \dots, \alpha_m \in \mathbb{N} \setminus \{0\}$$

such that if we consider the blowing-up map

$$\begin{array}{rccc} \phi\colon \mathbb{S}^{m-1} \times \mathbb{R}^+ \times \mathbb{R}^{n-m} & \longrightarrow & \mathbb{R}^n \\ (\bar{x}, \tau, y) & \longmapsto & (\tau^{\alpha_1}\bar{x}_1, \dots, \tau^{\alpha_n}\bar{x}_n, y), \end{array} \tag{6}$$

then the function $F\colon \mathcal{U} \to \mathbb{R}$, with domain $\mathcal{U} = \phi^{-1}(U)$, which is defined by

$$F(\bar{x}, \tau, y) = f \circ \phi(\bar{x}, \tau, y)$$

has a $C^k$ extension to $\overline{\mathcal{U}}$.

**Example 2.1.** The function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ is $C^\infty$ on $\mathbb{R}^2 \setminus \{0\}$. If we consider the blowing-up

$$(\bar{x}_1, \bar{x}_2, \tau) \overset{\phi}{\longmapsto} (x_1, x_2) = (\tau\,\bar{x}_1, \tau\,\bar{x}_2)$$

the function $F := f \circ \phi$ is defined on the domain $\{\tau > 0\}$ by

$$F(\bar{x}, \tau) = \sqrt{(\tau\bar{x}_1)^2 + (\tau\bar{x}_2)^2} = \tau.$$

Since $F$ clearly has $C^\infty$ extension to $\{\tau = 0\}$, $f$ has a blow-up $C^\infty$ extension to the origin.

For such class of functions, we have the following result:

**Lemma 2.2.** *Suppose that $f \in C^k(U)$ has a blow-up $C^k$ extension to $\{x = 0\}$, and let $\gamma\colon [0, \delta) \to \mathbb{R}^n$ be a $C^\infty$ curve (for some $\delta > 0$) such that*

$$\gamma((0, \delta)) \subset U \setminus \{x = 0\}.$$

*Then, there exists a natural number $p \in \mathbb{N} \setminus \{0\}$ such that the function*

$$\mathfrak{f}(\omega) := f \circ \gamma(\omega^p), \quad \textit{for } \omega \in [0, \delta^{1/p})$$

*belongs to $C^k([0, \delta^{1/p}))$.*

*Proof:* It suffices to consider the blowing-up of $\gamma$ under the map $\phi$. Since $\gamma((0,\delta)) \subset U \setminus \{x=0\}$, it is easy to prove that there exists a *blowing-up* on $\mathbb{R}^+$,

$$\varphi\colon \mathbb{R}^+ \ni \omega \longmapsto \omega^p \in \mathbb{R}^+$$

and a unique $C^\infty$ curve

$$\widetilde{\gamma}\colon [0,\delta^{1/p}) \to \mathbb{S}^{m-1} \times \mathbb{R}^+ \times \mathbb{R}^{n-m}$$

such that, $\gamma \circ \varphi(\omega) = \phi \circ \widetilde{\gamma}(\omega)$. Thus, by the definition of $F$,

$$f \circ \gamma(\omega^p) = F \circ \widetilde{\gamma}(\omega).$$

But since $F$ has a $C^k$ extension to $\overline{\mathcal{U}}$, $F \circ \widetilde{\gamma}$ has a $C^k$ extension to $\omega = 0$. This proves the lemma. $\square$

*Remark* 2.3. If we write the curve $\gamma$ in the form

$$\gamma(\rho) = \begin{cases} x_i = c_i \rho^{u_i} + \text{h.o.t} \\ y_j = d_j \rho^{v_j} + \text{h.o.t} \end{cases} \quad \text{where } c_i, d_j \neq 0,$$

the constant $p$ on the above lemma can be chosen as follows: If there exists one $u_i = 0$, then let $p := 1$. Otherwise, choose an index $1 \leq i \leq m$ such that $\frac{u_i}{\alpha_i} = \min\{\frac{u_1}{\alpha_1}, \dots, \frac{u_n}{\alpha_n}\}$, and let $p := \alpha_i$.

### 2.2. $\infty$-flat functions.

Let $U$ be an open subset of $\mathbb{R}^n$. Given a $f \in C^k(U)$, we shall say that $f$ in *infinitely flat* (or, shortly, $\infty$-flat) at the origin if for each compact subset $K \subset \mathbb{R}^n$ such that

$$K \setminus \{0\} \subset U, \tag{7}$$

and for each $n \in \mathbb{N}$, there exists a constant $C = C(K,n) > 0$ such that

$$|f(x)| \leq C\|x\|^n, \quad \text{for } x \in K,$$

(where we define $f(0) = 0$). We shall denote by $C^k_{\text{flat}}(U,0)$ the $\mathbb{R}$-subalgebra formed by all such functions.

*Remark* 2.4. Of course, if $\{0\} \notin \overline{U}$, the space $C^k_{\text{flat}}(U,0)$ simply coincides with $C^k(U)$.

**Example 2.5.** (i) The function $f(x) = \exp(-1/\|x\|^\alpha)$ is a $C^\infty$ function on $U = \mathbb{R}^n \setminus \{0\}$, which is $\infty$-flat at the origin, for any constant $\alpha \in \mathbb{R}^+ \setminus \{0\}$.

(ii) The function $f(x) = |y|\exp(-1/x)$ is a $C^0$ (but not $C^1$!) function on $\mathbb{R}^2 \cap \{x > 0\}$ which is $\infty$-flat at the origin.

Let us generalize the above definition as follows: Given a closed subset $M$ on $\mathbb{R}^n$, we shall say that a function $f \in C^k(U)$ is *$\infty$-flat at $M$* (or shortly, $f \in C^k_{\text{flat}}(U, M)$) if for each compact subset $K \subset \mathbb{R}^n$ such that

$$K \setminus M \subset U, \tag{8}$$

and each $n \in \mathbb{N}$ there exists a constant $C = C(K, n)$ such that

$$|f(x)| \leq C\, d(x, M \cap K)^n, \quad \text{for } x \in K,$$

where $d(x, K \cap M)$ is the distance between $x$ and the compact set $K \cap M$.

**Example 2.6.** Let $F \in C^k(\mathbb{R})$ be any $C^k$ function. Then, the function $f(x_1, x_2) = F(x_1) \exp(-1/\|x_2\|)$ belongs to $C^k_{\text{flat}}(U, \{x_2 = 0\})$, where $U = \mathbb{R}^2 \setminus \{x_2 = 0\}$.

More generally, suppose given a function $\widehat{\mathfrak{f}} \in C^\infty(M)$. Then, the Whitney's Extension Theorem (see, e.g. [**Bi**]) implies that there exists an extension map

$$E \colon C^\infty(M) \longrightarrow C^\infty(\mathbb{R}^n)$$

such that if $\mathfrak{f} = E(\widehat{\mathfrak{f}})$, then the Taylor expansion of $\mathfrak{f}$ at each point $p \in M$ coincides with $\widehat{\mathfrak{f}}$.

We shall say that a function $f$ is *infinitely flat to* $\widehat{f}$ (notation: $f \in C^k_{\text{flat}}(U, M, \widehat{\mathfrak{f}})$) if

$$(f - \mathfrak{f}) \in C^k_{\text{flat}}(U, M).$$

*Remark* 2.7. Notice that if the compact set $K$ is chosen according to (8), we can continuously extend all partial derivatives of $f$ to the set $K \cap M$ by defining

$$D^\alpha f(x) := D^\alpha \widehat{\mathfrak{f}}(x), \quad \text{for each } x \in K \cap M.$$

However, this *do not necessarily imply* that $f \in C^k(K)$. The following counter-example is taken from [**Bi**]: Let $U$ be the complement of the closed subset of $\mathbb{R}^2$ defined by $0 \leq x_2 \leq e^{-1/x_1^2}$, $x_1 \geq 0$. Let $f \in C^\infty(U)$ be defined by $f(x_1, x_2) = e^{-1/x_1^2}$ if $x_1 > 0$, $x_2 > e^{-1/x_1^2}$, and $f(x_1, x_2) = 0$ otherwise. Then, $f \in C^\infty_{\text{flat}}(U, 0)$ but $f \notin C^\infty(\overline{U})$. Indeed, if this were true, the Whitney Extension Theorem would imply that $f$ extends as a $C^\infty$ function to all $\mathbb{R}^2$. But this is impossible since,

$$\frac{f(x_1, e^{-1/x_1^2}) - f(x_1, 0)}{e^{-1/x_1^2} - 0} = 1$$

for $x_1 > 0$.

The reason for this phenomenon is that set $\overline{V}$ is not *regularly situated* (see [**Bi**], [**Wh**]). However, this cannot happen if we restrict to the particular class of subanalytic sets:

**Lemma 2.8.** *Suppose that $K \subset \mathbb{R}^n$ is a closed subanalytic set such that $K \setminus M$ is contained in $U$. Then, a function $f \in C^\infty_{\mathrm{flat}}(U, M, \widehat{\mathfrak{f}})$ necessarily belongs to $C^\infty(K)$.*

*Proof:* A closed subanalytic set is always regularly situated ([**Bi**, Theorem 6.17]). Thus, the result is a direct consequence of [**Bi**, Proposition 2.16]. ☐

### 2.3. Blowing-down of $\infty$-flat functions.

In the next sections, we will need the following easy result concerning the *blowing-down* of $\infty$-flat functions.

**Lemma 2.9.** *Consider a* blowing-up map

$$\begin{array}{rccc} \Phi\colon & \mathbb{S}^{n-1}\times\mathbb{R}^+ & \longrightarrow & \mathbb{R}^n \\ & ((\bar{x}_1,\dots,\bar{x}_n),\tau) & \longmapsto & x=(\tau^{\alpha_1}\bar{x}_1,\dots,\tau^{\alpha_n}\bar{x}_n), \end{array}$$

*with weights $(\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n \setminus \{0\}$. Let*

$$U \subset \{(\bar{x},\tau) \in \mathbb{S}^{n-1}\times\mathbb{R}^+ \mid \tau > 0\}$$

*be an open subset and $F \in C^k(U)$ be an arbitrary function on $U$. Since $\Phi$ is a diffeomorphism outside the* exceptional divisor $D := \{\tau = 0\}$*, we can define*

$$\mathfrak{F}(x) := F \circ \Phi^{-1}(x),$$

*which is a $C^k$ function on $\mathcal{U} := \Phi(U) \subset \mathbb{R}^n$. We affirm that*

$$F \in C^\infty_{\mathrm{flat}}(U, \{\tau = 0\}) \quad \text{if and only if } \mathfrak{F} \in C^\infty_{\mathrm{flat}}(\mathcal{U}, 0).$$

*Proof:* Suppose that $F \in C^\infty_{\mathrm{flat}}(U, \{\tau = 0\})$. By definition, for each compact subset $K \subset \mathbb{S}^{n-1}\times\mathbb{R}^+$ such that $K \setminus D \subset U$, and each $n \in \mathbb{N}$, there exists a constant $C(K,n) > 0$ such that

$$|F(\bar{x},\tau)| \le C(K,n)\,|\tau|^n.$$

Let now $\mathcal{K} \subset \mathbb{R}^n$ be a compact subset such that $\mathcal{K}\setminus\{0\} \subset \mathcal{U}$. Then, there exists a compact subset $K \subset \mathbb{S}^{n-1}\times\mathbb{R}^+$ as in the previous paragraph such that

$$\mathcal{K} = \Phi(K).$$

If a point $x \in \mathcal{K}$ is sufficiently near 0, it can be written in the form $x = \Phi(\bar{x},\tau)$, for some $0 \le \tau < 1$, and so

$$\|x\| = \|\tau^\alpha \bar{x}\| \ge |\tau|^a$$

where $a = \max\{\alpha_1, \dots, \alpha_n\}$. By compactness, there exists a constant $g(\mathcal{K}) > 0$ (depending only on $\mathcal{K}$) such that $\|x\| \geq g(\mathcal{K})|\tau|^a$, for any point $x \in \mathcal{K}$.

Therefore, given any $m \in \mathbb{N}$, if we define the constant

$$\mathfrak{C}(\mathcal{K}, m) := g(\mathcal{K})^{-ma} C(K, ma),$$

it is immediate to see that

$$|\mathfrak{F}(x)| = |F(\bar{x}, \tau)| \leq C(K, ma)|\tau|^{ma} \leq \mathfrak{C}(\mathcal{K}, m)\|x\|^m,$$

and so, $\mathfrak{F} \in C^\infty_{\text{flat}}(\mathcal{U}, 0)$.

The converse is proved in an analogous way. □

## 3. Center manifolds at non-degenerate points

Let $(x, y, \varepsilon, \mathcal{A})$ be coordinates in the open subset

$$U = U_x \times U_y \times U_\varepsilon \times U_{\mathcal{A}},$$

where $U_x \subset \mathbb{R}$ is an open connected set, $U_y = \mathbb{R}$, $U_{\mathcal{A}} \in (\mathbb{R}^n, 0)$ and $U_\varepsilon \in (\mathbb{R}^+, 0)$ (here, as usual, we denote by $(\mathbb{R}^k, 0)$ the set of all open neighborhoods of the origin in $\mathbb{R}^k$). Let $X$ be an analytic vector field on $U$, which has the form

$$X = \varepsilon \frac{\partial}{\partial x} + f(x, y, \varepsilon, \mathcal{A}) \frac{\partial}{\partial y},$$

for some function $f \in C^\omega(U)$ such that

- $f(x, 0, 0, 0) \equiv 0$, and
- $B_x = \frac{\partial f}{\partial y}(x, 0, 0, 0)$ is such that $|B_x| > \delta > 0$,

for some positive constant $\delta > 0$ and all $x \in U_x$. In the dynamical systems terminology, this is equivalent to say that the closed connected submanifold

$$\Gamma = \{y = \varepsilon = \mathcal{A} = 0\} \approx U_x$$

is a *normally hyperbolic* curve of singularities of $X$. Notice that since $\Gamma$ is connected, $B_x$ has a constant sign for all $x \in \Gamma$.

As a consequence of the Center Manifold Theorem (see e.g. [**C-L-W**]), we have the following result.

**Proposition 3.1.** *Let $x \in \Gamma$ be an arbitrary point. Then, for each $k \in \mathbb{N}$, there exists a neighborhood $V_{x,\varepsilon,\mathcal{A}} = V_x \times V_\varepsilon \times V_{\mathcal{A}}$ of $x$ in $U_{x,\varepsilon,\mathcal{A}} = U \cap \{y = 0\}$, and a $C^k$ function*

$$\begin{array}{rccc} w\colon & V_{x,\varepsilon,\mathcal{A}} & \longrightarrow & U_y \\ & (x,\varepsilon,\mathcal{A}) & \longmapsto & y = w(x,\varepsilon,\mathcal{A}) \end{array}$$

*defined on $V_{x,\varepsilon,\mathcal{A}}$, such that $w(x,0,0) = 0$ and*

$$W = \operatorname{graph}\{y = w(x,\varepsilon,\mathcal{A})\}$$

*is an invariant manifold for $X$.*

Such $W$ is called a *local center manifold at $x$*. As it is well-known, this local center manifold in not unique in general, and moreover the neighborhood $V$ may shrink to the point $x$ as the degree of differentiability $k$ goes to infinity.

In the rest of this section, our goal will be to show that the family $X$ has a particular class of $C^\infty$ center manifolds, which are *dynamically defined* by the flow of $X$.

Before defining such manifolds, let us turn ourselves to the issue of *formal expansion*:

**Lemma 3.2.** *Possibly restricting $U_{\mathcal{A}} \in (\mathbb{R}^n, 0)$ to some smaller neighborhood of the origin, there exists an unique formal series*

$$\widehat{W}(x,\varepsilon,\mathcal{A}) = \sum_{i=0}^{\infty} w_i(x,\mathcal{A})\,\varepsilon^i$$

*defined by a collection of analytic functions $w_i(x,\mathcal{A}) \in C^\omega(U_x \times U_{\mathcal{A}})$, such that for each point $x \in \Gamma$, and each local $C^k$ center manifold $W = \operatorname{graph}\{y = w(x,\varepsilon,\mathcal{A})\}$ defined in some neighborhood $U_0$ of $x$, we have $w \in C^k_{\text{flat}}(U_0, \{\varepsilon = 0\}, \widehat{W})$.*

*Proof:* It is well-known that all local center manifolds at a point $x$ in $\Gamma$ have the same Taylor expansion. Hence, it suffices to prove that such Taylor expansion is defined by the localization of the series $\widehat{W}$ at $x$.

The hypothesis $|\frac{\partial f}{\partial x}(x,0,0,0)| > \delta > 0$ implies that the set of singularities of the vector field $X$,

$$\mathcal{Z}(X) := \{(x,y,\mathcal{A},\varepsilon) \mid X(x,y,\mathcal{A},\varepsilon) = 0\}$$

is a smooth codimension 2 submanifold near $\Gamma$. Indeed, there exists an open connected neighborhood $U^0_{\mathcal{A}} \subset U_{\mathcal{A}}$ of $\{\mathcal{A} = 0\}$ such that

$$\frac{\partial f}{\partial y}(x,0,0,\mathcal{A}) \neq 0, \quad \text{for } x \in U_x,\ \mathcal{A} \in U^0_{\mathcal{A}}.$$

Thus, by the Implicit Function Theorem, there exists a unique analytic function $w_0(x,\mathcal{A})$ defined on $U_x \times U^0_{\mathcal{A}}$ such that $w_0(x,0) \equiv 0$ and, if we restrict the domain of $X$ to $U_x \times U_y \times U_\varepsilon \times U^0_{\mathcal{A}}$,

$$\mathcal{Z}(X) = \{(x,y,\mathcal{A},\varepsilon) \mid \varepsilon = 0,\, y = w_0(x,\mathcal{A})\}.$$

Let us make the analytic change of coordinates

$$y = y' - w_0(x,\mathcal{A}).$$

Then, if we write the new expression for $X$ in these new coordinates

$$X = \varepsilon\frac{\partial}{\partial x} + f'(x,y,\varepsilon,\mathcal{A})\frac{\partial}{\partial y'}$$

we necessarily obtain

$$f'(x,0,0,\mathcal{A}) \equiv 0 \quad \text{and} \quad \left|\frac{\partial f'}{\partial y'}(x,0,0,\mathcal{A})\right| > \delta > 0, \tag{9}$$

for all $(x,\mathcal{A}) \in U_x \times U^0_{\mathcal{A}}$.

Dropping again the primes, let us prove that (in these new coordinates) there exists a unique formal series of the form

$$\widehat{M}(x,\varepsilon,\mathcal{A}) = \sum_{i=1}^{\infty} w_i(x,\mathcal{A})\,\varepsilon^i$$

where $w_i \in C^\omega(U_x \times U^0_{\mathcal{A}})$, such that the vector field $X$, seen as a derivation on the space of formal series, maps the series

$$\widehat{K}(x,y,\varepsilon,\mathcal{A}) := y - \widehat{M}(x,\varepsilon,\mathcal{A})$$

into another series $X(\widehat{K})$ which is divisible by $\widehat{K}$. Once this is proved, the series on the enunciate of the lemma will be simply given by $\widehat{W} := w_0 + \widehat{M}$.

To say that $X(\widehat{K})$ is divisible by $\widehat{K}$ is equivalent to say that

$$X(\widehat{K})(x,y,\varepsilon,\mathcal{A})\big|_{y=\widehat{M}(x,\varepsilon,\mathcal{A})} \equiv 0. \tag{10}$$

If we write $f(x,y,\varepsilon,\mathcal{A}) = \sum_{i=0}^{\infty} f_i(x,\varepsilon,\mathcal{A})\,y^i$, the equation (10) gives

$$\varepsilon\frac{\partial}{\partial x}\widehat{M} = f_0(x,\varepsilon,\mathcal{A}) + \widehat{M}\cdot(f_1(x,\varepsilon,\mathcal{A}) + O(\widehat{M})).$$

Expanding both sides in powers of $\varepsilon$, and writing

$$f_i(x,\varepsilon,\mathcal{A}) = \sum_{j=1}^{\infty} f_{i,j}(x,\mathcal{A})\,\varepsilon^j,$$

we notice that $f_{0,0}(x,\mathcal{A}) \equiv 0$ by (9). Thus, the first term $w_1$ in the series $\widehat{M}$ must satisfy

$$0 = f_{0,1}(x,\mathcal{A}) + w_1 \cdot f_{1,0}(x,\mathcal{A}). \tag{11}$$

As $f_{1,0}(x,\mathcal{A}) = \frac{\partial f}{\partial y}(x,0,0,\mathcal{A}) > 0$, we can define $w_1 = -\frac{f_{0,1}}{f_{1,0}}$.

Similarly, we prove that each coefficient $w_i$ can be determined recursively from $f_j$ and $w_j$, for $0 \le j < i$ by applying successively the Implicit Function Theorem. This proves the result. □

We shall say that $\widehat{W}$ defines the *formal center manifold* over $\Gamma$.

*Remark* 3.3. Given an arbitrary closed set $M$ which is contained in $U_x \times U_{\mathcal{A}}$, it is clear that $\widehat{W}(x,\varepsilon,\mathcal{A})$ uniquely defines an element of $C^\infty(\{\varepsilon = 0\} \times M)$ (in the sense of Whitney).

For each point $x' \in \Gamma$, we define

$$\widehat{W}_{x'} = \widehat{W}(x',\varepsilon,\mathcal{A}) = \sum_{i=0}^{\infty} w_i(x',\mathcal{A})\,\varepsilon^i$$

to be the restriction of the formal center manifold to $\{x = x'\}$. We shall say that a $C^\infty$ function $i(\varepsilon,\mathcal{A})$ defined on some open subset

$$V \subset (U_\varepsilon \cap \{\varepsilon > 0\}) \times U_{\mathcal{A}}$$

is an *initial condition function* for $X$ at $x'$ if

$$i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x'})$$

(that is, $i$ is $\infty$-flat to the formal series $\widehat{W}_{x'}$ at $\{\varepsilon = 0\}$).

Given two distinct points $x_0, x_1 \in \Gamma$, let us define $\Gamma_{x_0,x_1}$ to be the segment of $\Gamma$ lying between these points. Thus,

$$\Gamma_{x_0,x_1} = \{x' \in \Gamma \mid x_0 \le x' \le x_1\} \quad \text{if } x_1 > x_0$$
$$\Gamma_{x_0,x_1} = \{x' \in \Gamma \mid x_1 \le x' \le x_0\} \quad \text{if } x_1 < x_0.$$

The next result shows that an initial condition function uniquely defines a $C^\infty$ invariant manifold.

**Proposition 3.4.** *Let $x_0 \in \Gamma$, and let $i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_0})$ be an arbitrary initial condition function for $X$ at $x_0$. Assume that $B_{x_0} < 0$. Then, for all $x_1 > x_0$, there exists a neighborhood $N \subset U_{\varepsilon,\mathcal{A}}$ of the origin $\{\varepsilon = \mathcal{A} = 0\}$, such that if we consider the restriction of $i$ to the set*

$$\mathcal{O} := V \cap N$$

*there exists an unique $C^\infty$ function $w(x, \varepsilon, \mathcal{A})$ defined on $\Gamma_{x_0,x_1} \times \mathcal{O}$ such that*

(i) $w(x_0, \varepsilon, \mathcal{A}) = i(\varepsilon, \mathcal{A})$,
(ii) $w \in C^\infty_{\text{flat}}(\Gamma_{x_0,x_1} \times \mathcal{O}, \{\varepsilon = 0\}, \widehat{W})$, *and*
(iii) $W = \text{graph}\{y = w(x, \varepsilon, \mathcal{A})\}$ *is an invariant manifold.*

*Proof:* First of all, let us prove that prove that $w$ is $C^\infty$ on the domain $\Gamma_{x_0,x_1} \times \mathcal{O}$.

Outside the set $\{\varepsilon = 0\}$, the vector field $X$ is non-singular, and equivalent to the first order differential equation

$$\frac{dy}{dx} = \frac{f(x, y, \varepsilon, \mathcal{A})}{\varepsilon}.$$

We consider the initial value problem associated to such equation, with initial condition

$$y_{\varepsilon,\mathcal{A}}(x_0) = i(\varepsilon, \mathcal{A})$$

at $x_0$. For $(\varepsilon, \mathcal{A}) \in V$ (and so, $\varepsilon \neq 0$), this initial condition uniquely defines a $C^\infty$ solution $y_{\varepsilon,\mathcal{A}}(x)$, over some maximal domain $x \in [x_0, \alpha)$, where $\alpha \subset U_x$ may depend on $\varepsilon$ and $\mathcal{A}$. Let us prove the following:

**Claim.** *There exists an open neighborhood of the origin $N \subset U_{\varepsilon,\mathcal{A}}$ such that $\alpha > x_1$ for each initial condition $(\varepsilon, \mathcal{A}) \in \mathcal{O} := V \cap N$.*

The condition $B_x < 0$ implies that each orbit of $X$ for $(\varepsilon, \mathcal{A}) = 0$ goes asymptotically to $\Gamma$ as $t \to \infty$. Indeed, the Stable Manifold Theorem assures that there exists an exponential contraction in the hyperbolic direction. Locally at a point $x \in \Gamma$, this contraction of order $e^{B_x}$. If we restrict to the compact segment $\Gamma_{x_0,x_1}$, there exists a constant $k > 0$ such that

$$B_x < -k < 0$$

for each $x \in \Gamma_{x_0,x_1}$. Therefore, since the contraction is uniform over this segment and the hypothesis $i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_0})$ implies that

$$\lim_{(\varepsilon,\mathcal{A})\to 0} i(\varepsilon, \mathcal{A}) = 0,$$

there exists a neighborhood $N$ of $(\varepsilon, \mathcal{A}) = (0,0)$ such that each orbit of $X$ starting at a point $(x,y) = (x_0, i(\varepsilon, \mathcal{A}))$ with $(\varepsilon, \mathcal{A}) \in V \cap N$ will cross the section $\{x = x_1\}$.

Therefore, if we define $w(x, \varepsilon, \mathcal{A}) = y_{\varepsilon,\mathcal{A}}(x)$, it is immediate to see that

- $w \in C^\infty(\Gamma_{x_0,x_1} \times \mathcal{O})$,
- $w|_{x=x_0} = i$, and
- $W = \mathrm{graph}\{y - w(x, \varepsilon, \mathcal{A})\}$ is an invariant manifold.

So, it remains to prove that $w$ is $\infty$-flat to $\widehat{W}$ on $\{\varepsilon = 0\}$.

Using Lemma 3.2 it suffices to prove that, given a point $x \in \Gamma_{x_0,x_1}$ and an arbitrary local $C^k$ center manifold $W' = \mathrm{graph}\{y - f'(x, \varepsilon, \mathcal{A})\}$ at $x$, the function $w' = w - f'$ belongs to $C^k_{\mathrm{flat}}(U', \{\varepsilon = 0\})$.

Consider first the point $x = x_0$. From Corollary 13.2, there exist local coordinates $(x', y', \varepsilon', \mathcal{A}')$ defined in a neighborhood $U'$ of $x$ such that $W' = \{y' = 0\}$ and $X$ is $C^k$-equivalent to

$$Y = \varepsilon' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'}.$$

On these coordinates, the initial condition function assumes the form

$$y'_0 = i'(\varepsilon', \mathcal{A}')$$

where $i' \in C^k_{\mathrm{flat}}(U', \{\varepsilon = 0\})$. The explicit solution of $Y$ from the initial point $(x,y) = (0, y'_0)$ is given by

$$\begin{cases} x'(t) = \varepsilon' t, \\ y'(t) = i'(\varepsilon', \mathcal{A}')\ e^{-t}. \end{cases}$$

Putting $t$ as a function of $x'$ in the first equation, we can write $w'$ as

$$w'(x', \varepsilon', \mathcal{A}') = \begin{cases} i'(\varepsilon', \mathcal{A}'), & \text{for } x = 0 \\ i'(\varepsilon', \mathcal{A}')\, e^{-\frac{x'}{\varepsilon'}}, & \text{for } x > 0. \end{cases}$$

The function $g(x', \varepsilon') = e^{-\frac{x'}{\varepsilon'}}$ is uniformly limited on the region $\{(\varepsilon', x') \mid \varepsilon' \geq 0, x' \geq 0\}$. Similarly, all its partial derivatives are uniformly limited by

$$\left| \frac{\partial^{s_1+s_2} g}{\partial^{s_1} x' \partial^{s_2} \varepsilon'} \right| \leq \frac{c}{\varepsilon'^{s_2}},$$

for some constant $c$ depending on $s_1$ and $s_2$. Thus, since

$$i' \in C^k_{\mathrm{flat}}(U', \{\varepsilon = 0\}),$$

it is easy to see that $w'$ is a $C^k$ function, $\infty$-flat on $\{\varepsilon' = 0\}$. The reasoning on the other points $x \in \Gamma_{x_0,x_1}$ is similar. $\square$

By reversing the sense of the flow, we clearly have an analogous result in the case where $B_{x_0} > 0$.

**Corollary 3.5.** *Let $x_0$ and $i$ be as in the above enunciate, and assume that $B_{x_0} < 0$. Then, for all $x_1 < x_0$, there exists a neighborhood $N \subset U_{\varepsilon,\mathcal{A}}$ of the origin such that if we define*

$$\mathcal{O} := V \cap N,$$

*there exists an unique $C^\infty$ function $w(x,\varepsilon,\mathcal{A})$ defined on $\Gamma_{x_0,x_1} \times \mathcal{O}$ such that the statements (i), (ii) and (iii) of the proposition holds.*

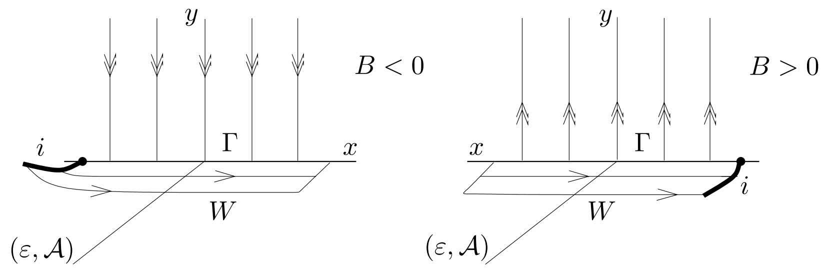



FIGURE 5. Dynamical center manifolds defined on Proposition 3.4 and its corollary.

*Remark* 3.6. Of course, although the function $w(x,\varepsilon,\mathcal{A})$ is independent of the endpoint $x_1$, its domain of definition $\mathcal{O}$ may shrink to zero as $x_1$ approaches the boundary of $U_x$. Clearly, if $x_1'$ lies between $x_0$ and $x_1$, we can choose a corresponding domain $\mathcal{O}'$ which contains $\mathcal{O}$.

Let us write $\widetilde{\Gamma} = \Gamma_{x_0,x_1}$. Then, the domain $\mathcal{O} = \mathcal{O}_{\widetilde{\Gamma}} \subset U_{\varepsilon,\mathcal{A}}$ defined on the above proposition will be called *canard region associated to the segment* $\widetilde{\Gamma}$. The invariant manifold $W_{\widetilde{\Gamma}}$ defined on item (iii) of Proposition 3.4 will be called a *dynamical center manifold* over $\widetilde{\Gamma}$.

## 4. Degenerate points with transversality hypothesis

Let us again consider $(x,y,\varepsilon,\mathcal{A})$ as coordinates on an open subset $U = U_x \times U_y \times U_\varepsilon \times U_{\mathcal{A}}$. Let

$$X = \varepsilon\frac{\partial}{\partial x} + F(x,y,\varepsilon,\mathcal{A})\frac{\partial}{\partial y}$$

be an analytic vector field on $U$, such that if we expand $F$ in powers of $y$,

$$F(x,y,\varepsilon,\mathcal{A}) = F_0(x,\varepsilon,\mathcal{A}) + yF_1(x,\varepsilon,\mathcal{A}) + y^2Q(x,y,\varepsilon,\mathcal{A})$$

the following conditions are verified

- $F_0(x,0,0) \equiv 0$,
- $F_1(0,0,0) = 0$
- $F_1(x,0,0) \neq 0$ for any $x \in U_x \setminus \{0\}$.

The first condition implies that

$$\Gamma = \{y = \varepsilon = \mathcal{A} = 0\} \approx U_x$$

is a curve of singularities. The second and third hypothesis implies that $x = 0$ is the only a degenerate singularity (not semi-hyperbolic) at $\Gamma$. Let $\mu(X) \geq 1$ denotes the multiplicity of the function $F_1(x,0,0)$ at $x = 0$. We shall say that $\mu(X)$ is the *multiplicity of the degenerate singularity at* $x = 0$. For future reference, we define the constants

$$\mathcal{B}_0 = \frac{1}{p!}\frac{\partial^p F_1}{\partial x^p}(0,0,0) \quad \text{and} \quad \mathcal{Q}_0 = Q(0,0,0) \tag{12}$$

where $p = \mu(X)$ (notice that $\mathcal{B}_0 \neq 0$).

In order to establish conditions for the existence of a local invariant surface it is essential look on how the functions $F_0$ and $F_1$ *unfold* as $\mathcal{A}$ and $\varepsilon$ varies. Let

$$F_0(x,a,\varepsilon) = \sum_{i=0}^{\infty} A_i(\varepsilon,\mathcal{A})\, x^i$$

be the Taylor expansion of $F_0$ in terms of the $x$-variable. Under the above hypothesis, we have $A_i(0,0) = 0$ for all $i \geq 0$. The function $F_1(x,\varepsilon,\mathcal{A})$ can also be expanded as

$$F_1(x,\varepsilon,\mathcal{A}) = \sum_{i=0}^{p-1} B_i(\varepsilon,\mathcal{A})x^i + x^p B(x,\varepsilon,\mathcal{A})$$

where $B_i(0,0) = 0$, for $0 \leq i \leq p-1$ and $B(0,0,0) = \mathcal{B}_0 \neq 0$.

Let us now state the following hypothesis:

**Transversality Hypothesis:** Suppose that $n \geq 3p$, and that, up to an analytic change of coordinates of the form

$$(\varepsilon, \mathcal{A}) = (\varepsilon, \Psi(\varepsilon, \mathcal{A})),$$

(with $\Psi(0,0) = 0$) we can write

$$\mathcal{A} = (a_0, \dots, a_{2p-1}, b_0, \dots, b_{p-1}, \mathcal{A}_r) \in (\mathbb{R}^n, 0),$$

where

$$\begin{aligned} A_i(\varepsilon, \mathcal{A}) &= a_i, \quad \text{for } 0 \leq i \leq 2p-1 \\ B_j(\varepsilon, \mathcal{A}) &= b_j, \quad \text{for } 0 \leq j \leq p-1. \end{aligned}$$

If a family $X$ satisfies this hypothesis, we shall say that it is a *transversal family* (unfolding $\Gamma$).

The parameters $\mathcal{A} = (a, b, \mathcal{A}_r)$ will be called *adapted parameters*, and $\mathcal{A}_r \in (\mathbb{R}^{n-3p}, 0)$ will be called *the inessential parameter* of the family.

*Remark* 4.1. Clearly, the Transversality Hypothesis is equivalent to require that the jacobian matrix

$$\frac{\partial(A_0, \dots, A_{2p-1}, B_0, \dots, B_{p-1})}{\partial(\mathcal{A}_0, \dots, \mathcal{A}_n)}$$

is non-vanishing at the origin.

### 4.1. The Blowing-up map.

On the next four subsections, we suppose that $X$ is a transversal unfolding of $\Gamma$ (i.e. satisfies the Transversality Hypothesis), and let $\mathcal{A} = (a, b, \mathcal{A}_r)$ be adapted parameters.

In order to study the existence of invariant surfaces over $\Gamma$, the essential step is desingularize the degenerate singularity at $x = 0$. For this, we make a *quasi-homogeneous blowing-up* of the submanifold $N = \{x = y = a = b = \varepsilon = 0\}$. This is defined by an analytic map

$$\Phi \colon \mathbb{R}^+ \times \mathbb{S}^{3p+2} \times \mathbb{R}^{n-3p} \longrightarrow \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{3p} \times \mathbb{R}^{n-3p} \tag{13}$$

$$(\tau, (\bar{x}, \bar{y}, \bar{\varepsilon}, \bar{a}, \bar{b}), \mathcal{A}_r) \longmapsto ((x, y), \varepsilon, a, b, \mathcal{A}_r) \tag{14}$$

which is given by

$$\Phi = \begin{cases} x = \tau\bar{x} \\ y = \tau^p\bar{y} \\ a_i = \tau^{2p-i}\bar{a}_i & \text{for } i = 0, \dots, 2p-1 \\ b_j = \tau^{p-j}\bar{b}_j & \text{for } j = 0, \dots, p-1 \\ \varepsilon = \tau^{p+1}\bar{\varepsilon} \\ \mathcal{A}_r = \mathcal{A}_r. \end{cases} \tag{15}$$

Let us study the pull-back of $X$ under this transformation. Looking at $X$ as a vector field on the *total space* $U = U_x \times U_y \times U_\varepsilon \times U_\mathcal{A}$ which vanishes identically on $N$, it can be proved that there exists an unique analytic vector field $\overline{X}$ defined on

$$\overline{U} = \Phi^{-1}(U) \in (\mathbb{R}^+, 0) \times \mathbb{S}^{3p+2} \times (\mathbb{R}^{n-3p}, 0)$$

such that $\overline{X}$ does not vanishes identically on the *exceptional divisor*

$$\mathcal{D} = \Phi^{-1}(N) = \{\tau = 0\} \cap \overline{U},$$

and such that $\Phi^*(\overline{X})$ is $C^\omega$*-equivalent* to $X$ on $U \setminus N$ (recall that two vector fields $Y_1$, $Y_2$ are $C^k$-equivalent if there exists a strictly positive $C^k$ function $f$ such that $Y_1 = f\ Y_2$). Indeed, $\overline{X}$ is given explicitly by

$$\overline{X} = (\tau^{-\rho})\ X \circ \Phi$$

for a suitable integer $\rho \in \mathbb{N}$.

As we mentioned above, we can look at $X$ as vector field on $U$ such that the coordinate functions $a_i$ (for $0 \le i \le 2p-1$), $b_j$ (for $0 \le i \le p-1$), $\mathcal{A}_r$ and $\varepsilon$ are *first integrals*, i.e. if we look at $X$ as a derivation on the module of differentiable functions on $U$, then

$$X(a_i) \equiv X(b_j) \equiv X(\varepsilon) \equiv X(\mathcal{A}_r) \equiv 0.$$

Therefore, if we define $\bar{g} = \varepsilon \circ \Phi$, $\bar{f}_i = a_i \circ \Phi$ and $\bar{g}_j = b_j \circ \Phi$, it is clear that $g$, $f_i$ and $g_j$ are first integrals of the new vector field $\overline{X}$, because

$$\overline{X}(\bar{g}) = (\tau^{-\rho})\ (X \circ \Phi)(\varepsilon \circ \Phi) = (\tau^{-\rho})\ (X(\varepsilon)) \circ \Phi \equiv 0,$$

and similarly for all $f_i$ and $h_j$.

This means that the vector field $\overline{X}$ is tangent to level sets of the $n+1$ analytic functions on $\overline{U}$ given by

$$\begin{aligned}
&\bar{g} = \tau^{p+1}\bar{\varepsilon} \\
&\bar{f}_i = \tau^{2p-i}\bar{a}_i \quad \text{for } i = 0, \dots, 2p-1 \\
&\bar{h}_j = \tau^{\gamma_i}\bar{b}_j \quad\quad \text{for } j = 0, \dots, p-1, \text{ and} \\
&\mathcal{A}_r.
\end{aligned}$$

In particular, notice that the exceptional divisor $\mathcal{D} = \{\tau = 0\}$ is an invariant manifold for $\overline{X}$ (i.e. for each point $P \in \mathcal{D}$, we have $\overline{X}(P) \in T_P\mathcal{D}$). Indeed, this is equivalent to say (looking again to $\overline{X}$ as a derivation) that

$$\overline{X}(\tau) = \tau \cdot H \tag{16}$$

for some analytic function $H$. To see this, it suffices to consider one of the first integrals given above, say $\bar{g}$, and expand

$$0 \equiv \overline{X}(\bar{g}) = (p+1)\tau^p\bar{\varepsilon}\,\overline{X}(\tau) + \tau^{p+1}\overline{X}(\bar{\varepsilon})$$

which gives the identity $\bar{\varepsilon}\overline{X}(\tau) = -\frac{1}{p+1}\tau\overline{X}(\bar{\varepsilon})$. As $\bar{\varepsilon}$ is not divisible by $\tau$, this clearly implies (16).

More precisely, it is easy to see that each *slice of the exceptional divisor*, which is obtained by fixing the inessential parameters $\mathcal{A}_r$,

$$\mathcal{D}_{\mathcal{A}_r^0} = \mathcal{D} \cap \{\mathcal{A}_r = \mathcal{A}_r^0\}, \quad \text{for some fixed } \mathcal{A}_r^0 \in (\mathbb{R}^{n-3p}, 0)$$

is an invariant submanifold.

The blowing-up map is a diffeomorphism from $\overline{U} \setminus \mathcal{D}$ onto $U \setminus N$. So, the leaves of the two-dimensional foliation $\mathcal{F} = \{d\mathcal{A} = d\varepsilon = 0\}$, when restricted to $U \setminus N$, are mapped diffeomorphically onto two-dimensional submanifolds on $\overline{U} \setminus \mathcal{D}$, which correspond to regular leaves of the pull-backed foliation

$$\overline{\mathcal{F}} = \{d\bar{f} = d\bar{g} = d\bar{h} = d\mathcal{A}_r = 0\}.$$

This foliation extends to the exceptional divisor $\mathcal{D}$, in a way that we shall describe more precisely below, using the projective charts.

### 4.2. The $\bar{\varepsilon}$-chart and the Riccati family.

Consider the open subset of $\overline{U}$ given by $U_{\bar{\varepsilon}} = \{\bar{\varepsilon} > 0\}$. Then, there exists a diffeomorphism

$$\begin{array}{rrcl}
\varphi_{\bar{\varepsilon}}\colon & U_{\bar{\varepsilon}} & \longrightarrow & (\mathbb{R}^+, 0) \times \mathbb{R}^{3p+2} \times (\mathbb{R}^{n-3p}, 0) \\
& (\tau, (\bar{x}, \bar{y}, \bar{\varepsilon}, \bar{a}, \bar{b}), \mathcal{A}_r) & \longmapsto & (\tilde{\varepsilon}, (\tilde{x}, \tilde{y}, A, B), \mathcal{A}_r)
\end{array}$$

which maps $\mathcal{D} \cap U_{\bar{\varepsilon}}$ onto $\widetilde{\mathcal{D}} = \{\tilde{\varepsilon} = 0\} \approx \mathbb{R}^{3p+2} \times (\mathbb{R}^{n-3p}, 0)$, and is such that the composed map $\widehat{\Phi}_{\bar{\varepsilon}} = \Phi \circ \varphi_{\bar{\varepsilon}}$ takes the form

$$\widehat{\Phi}_{\bar{\varepsilon}} = \begin{cases} x = \tilde{\varepsilon}\tilde{x} \\ y = \tilde{\varepsilon}^p \tilde{y} \\ a_i = \tilde{\varepsilon}^{2p-i} A_i & \text{for } i = 0, \dots, 2p-1 \\ b_j = \tilde{\varepsilon}^{p-j} B_j & \text{for } j = p, \dots, p-1 \\ \varepsilon = \tilde{\varepsilon}^{p+1} \\ \mathcal{A}_r = \mathcal{A}_r. \end{cases} \tag{17}$$

Indeed, it is easy to compute the expression of $\varphi_{\bar{\varepsilon}}$. It is given by

$$\tilde{\varepsilon} = \tau \cdot \bar{\varepsilon}^{1/p+1}, \quad \tilde{x} = \frac{1}{\bar{\varepsilon}^{1/p+1}} \bar{x}, \dots, \quad A_j = \frac{1}{\bar{\varepsilon}^{2p-j/p+1}} \bar{a}_i, \dots$$

We shall say that the coordinates $(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, A, B, \mathcal{A}_r)$ given by such map are the $\bar{\varepsilon}$-*projective chart* of the blowing-up.

The main advantage of such coordinates is that the foliation $\bar{\mathcal{F}}$ becomes *rectified*. More precisely, we claim that, in these coordinates, $\overline{\mathcal{F}}$ is a regular two-dimensional foliation defined by

$$\overline{\mathcal{F}} = \{d\tilde{\varepsilon} = d\widetilde{\mathcal{A}} = 0\}, \tag{18}$$

where $\widetilde{\mathcal{A}} = (A, B, \mathcal{A}_r)$. Indeed, the pull-back of $\mathcal{F}$ under $\widehat{\Phi}_{\bar{\varepsilon}}$ is a foliation which satisfies the system of Pfaffian equations

$$\begin{aligned} &0 = d(\tilde{\varepsilon}^{p+1}) \\ &0 = d(\tilde{\varepsilon}^{2p-i} A_i) = (2p-i)\tilde{\varepsilon}^{2p-i-1} A_i \; d(\tilde{\varepsilon}) + \tilde{\varepsilon}^{2p-i} d(A_i), \quad && 0 \le i \le p-1 \\ &0 = d(\tilde{\varepsilon}^{p-j} B_j) = \gamma_j \tilde{\varepsilon}^{p-j-1} B_j d(\tilde{\varepsilon}) + \tilde{\varepsilon}^{p-j} d(B_j), && 0 \le j \le p-1 \\ &0 = d\mathcal{A}_r, \end{aligned}$$

which is easily seen to be equivalent to (18).

Therefore, in the domain $U_{\bar{\varepsilon}}$, the foliation $\overline{\mathcal{F}}$ extends in a regular way to the exceptional divisor $\mathcal{D}$. Moreover, from the fact that $\overline{X}$ is everywhere tangent to the foliation $\overline{\mathcal{F}}$, we conclude that the restriction of $\overline{X}$ to $U_{\bar{\varepsilon}}$ has again the structure of an analytic family, with phase space $(\tilde{x}, \tilde{y})$ and parameters $(\tilde{\varepsilon}, A, B, \mathcal{A}_r)$.

We can explicitly compute the expression $\overline{X}$ in such coordinates. To simplify our notation, let us drop the tildes, and denote the coordinates on the $\bar{\varepsilon}$-projective chart simply by $(x, y, \mathcal{A}, \varepsilon)$. Then, from the expression of $\widehat{\Phi}_{\bar{\varepsilon}}$ given above, easy computations give

$$\overline{X} = \frac{\partial}{\partial x} + \widetilde{F}(x, y, \mathcal{A}, \varepsilon) \frac{\partial}{\partial y} \tag{19}$$

where $\widetilde{F}(x,y,\mathcal{A},\varepsilon) = \widetilde{F}_0(x,\mathcal{A},\varepsilon) + y\widetilde{F}_1(x,\mathcal{A},\varepsilon) + y^2\widetilde{Q}(x,y,\mathcal{A},\varepsilon)$ is such that,

$$\widetilde{F}_0(x,\mathcal{A},\varepsilon) = \sum_{i=0}^{2p-1} A_i x^i + O(\varepsilon) \tag{20}$$

$$\widetilde{F}_1(x,\mathcal{A},\varepsilon) = \sum_{j=0}^{p-1} B_j x^j + \mathcal{B}(\mathcal{A}_r)x^p + O(\varepsilon) \tag{21}$$

$$\widetilde{Q}(x,y,\mathcal{A},\varepsilon) = \mathcal{Q}(\mathcal{A}_r)y^2 + O(\varepsilon) \tag{22}$$

where $\mathcal{B}(\mathcal{A}_r)$ and $\mathcal{Q}(\mathcal{A}_r)$ are analytic functions such that

$$\mathcal{B}(0) = \mathcal{B}_0 \quad \text{and} \quad \mathcal{Q}(0) = \mathcal{Q}_0 \tag{23}$$

are the constants given in (12). The symbol $O(\varepsilon)$ denote analytic functions which are divisible by $\varepsilon$.

Thus, $\overline{X}$ restricted to $U_{\bar{\varepsilon}}$ can be seen as a family of first order differential equations

$$\frac{dy}{dx} = \widetilde{F}(x,y,\mathcal{A},\varepsilon)$$

which depends *regularly* on the parameters $\varepsilon$ and $\mathcal{A}$. In particular, restricting to the slice of the exceptional divisor which is given by $\mathcal{D}_0 = \{\varepsilon = \mathcal{A}_r = 0\}$, we obtain

$$\begin{aligned}\frac{dy}{dx} &= A_0 + A_1 x + \cdots + A_{2p-1}x^{2p-1} \\ &\qquad + y\,(B_0 + \cdots + B_{p-1}x^{p-1} + \mathcal{B}_0 x^p) + \mathcal{Q}_0 y^2,\end{aligned} \tag{24}$$

which we shall call the *associated Riccati family* $R_{A,B}$. Notice that if $\mathcal{Q}_0 = 0$ this is simply a linear differential equation.

### 4.3. The $\bar{x}_\varepsilon$ and $-\bar{x}_\varepsilon$-charts.

Let us now consider the open subsets $U_{\bar{x}} = \{\bar{x} > 0\}$ and $U_{-\bar{x}} = \{-\bar{x} > 0\}$ in $\overline{U}$ and proceed as in the previous subsection (for briefness, we shall use the symbol $\pm$ to treat simultaneously the positive and negative cases).

On $U_{\pm\bar{x}}$ one can define as above a diffeomorphism

$$\begin{array}{rrcl}\varphi_{\pm\bar{x}}\colon & U_{\pm\bar{x}} & \longrightarrow & (\mathbb{R}^+,0)\times\mathbb{R}^{3p+2}\times(\mathbb{R}^{n-3p},0) \\ & (\tau,(\bar{x},\bar{y},\bar{\varepsilon},\bar{a},\bar{b}),\mathcal{A}_r) & \longmapsto & (\hat{x},(\hat{y},\hat{\varepsilon},\hat{a},\hat{b}),\mathcal{A}_r)\end{array}$$

which maps $\mathcal{D}\cap U_{\pm\bar{x}}$ onto $\widehat{\mathcal{D}}=\{\hat{x}=0\}$, and is such that $\widehat{\Phi}_{\pm\bar{x}}=\Phi\circ\varphi_{\pm\bar{x}}$ is given by

$$\widehat{\Phi}_{\pm\bar{x}}=\begin{cases} x=\pm\hat{x} \\ y=\hat{x}^p\hat{y} \\ a_i=\hat{x}^{2p-i}\hat{a}_i & \text{for } i=0,\dots,2p-1 \\ b_j=\hat{x}^{p-j}\hat{a}_j & \text{for } j=0,\dots,p-1 \\ \varepsilon=\hat{x}^{p+1}\hat{\varepsilon} \\ \mathcal{A}_r=\mathcal{A}_r. \end{cases}$$

In these coordinates, the foliation $\overline{\mathcal{F}}$ is described by the system of Pfaffian equations

$$\begin{aligned} &d(\hat{x}^{p+1}\hat{\varepsilon})=0 \\ &d(\hat{x}^{2p-i}\hat{a}_i)=0, \quad 0\le i\le 2p-1 \\ &d(\hat{x}^{p-j}\hat{b}_j)=0, \quad 0\le j\le p-1 \\ &d\mathcal{A}_r=0. \end{aligned}$$

To better describe its geometry, let us perform another local blowing-up along the submanifold

$$T=\{\hat{a}_i=\hat{b}_j=\hat{\varepsilon}=0,\, 0\le i\le 2p-1,\, 0\le j\le p-1\} \tag{25}$$

given by

$$\Psi=\begin{cases} (\hat{x},\hat{y},\mathcal{A}_r)=(\hat{x},\hat{y},\mathcal{A}_r), \\ \hat{\varepsilon}=R^{p+1}\check{\mathrm{e}}, \\ \hat{a}_i=R^{2p-i}\check{\alpha}_i, & 0\le i\le 2p-1, \\ \hat{b}_j=R^{p-j}\check{\beta}_j, & 0\le j\le p-1, \end{cases} \tag{26}$$

where $(R,(\check{\mathrm{e}},\check{\alpha}_i,\check{\beta}_j))\in(\mathbb{R}^+,0)\times\mathbb{S}^{3p}$.

Then, exactly as above, we can consider some special charts. Let us concentrate ourselves on the region $U_{\check{\mathrm{e}}}=\{\check{\mathrm{e}}>0\}$. In this region, there exists a diffeomorphism

$$\begin{array}{rrcl} \varphi_{\check{\varepsilon}}: & U_{\check{\mathrm{e}}} & \longrightarrow & (\mathbb{R}^+,0)\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^{3p+2}\times(\mathbb{R}^{n-3p},0) \\ & (\hat{x},\hat{y},(\check{\mathrm{e}},\check{\alpha}_i,\check{\beta}_j),\mathcal{A}_r) & \longmapsto & (\hat{x},\hat{y},\mathrm{e},(A,B),\mathcal{A}_r) \end{array}$$

such that $\Psi_{\tilde{\varepsilon}} = \Psi \circ \varphi_{\tilde{\varepsilon}}$ is given by

$$\begin{cases} (\hat{x}, \hat{y}, \mathcal{A}_r) = (\hat{x}, \hat{y}, \mathcal{A}_r), & \\ \hat{\varepsilon} = \mathrm{e}^{p+1}, & \\ \hat{a}_i = \mathrm{e}^{2p-i} A_i, & 0 \leq i \leq 2p-1, \\ \hat{b}_j = \mathrm{e}^{p-j} B_j, & 0 \leq j \leq p-1. \end{cases} \tag{27}$$

We shall call these coordinates $(\hat{x}, \hat{y}, \mathrm{e}, A, B, \mathcal{A}_r)$ the $\pm\bar{x}_\varepsilon$-*chart* of the blowing-up.

Easy computations show that the foliation $\overline{\mathcal{F}}$, when restricted to the domain of the $\pm\bar{x}_\varepsilon$-chart, is defined by the Pfaffian system

$$\begin{aligned} &d(\hat{x}\mathrm{e}) = 0 \\ &d(A_i) = 0, \quad 0 \leq i \leq 2p-1 \\ &d(B_j) = 0, \quad 0 \leq j \leq p-1 \\ &d\mathcal{A}_r = 0. \end{aligned}$$

This means that, for each fixed value of the regular parameters $\mathcal{A}^0 = (A^0, B^0, \mathcal{A}_r^0)$, $\overline{\mathcal{F}}$ is given by the level sets of the function $\hat{x}\mathrm{e} = \text{const}$. Taking const $\to 0$, we obtain the limit of the foliation $\overline{\mathcal{F}}$ as it approaches the exceptional divisor $\mathcal{D} = \{\hat{x} = 0\}$. This limit is a *singular leaf*, given by the union of the two-dimensional surfaces

$$\begin{aligned} L_0 &= \{\hat{x} = 0\} \cap \{(A, B, \mathcal{A}_r) = (A^0, B^0, \mathcal{A}_r^0)\} \quad \text{and} \\ L_1 &= \{\mathrm{e} = 0\} \cap \{(A, B, \mathcal{A}_r) = (A^0, B^0, \mathcal{A}_r^0)\}. \end{aligned}$$

The component $L_0 \subset \mathcal{D}$ is parameterized by the variables $(\hat{y}, \mathrm{e})$, and gives the extension of the foliation to the exceptional divisor.

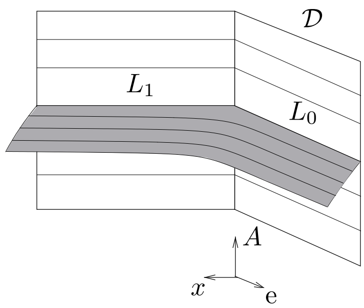



FIGURE 6. Extension of the foliation $\overline{\mathcal{F}}$ to $\mathcal{D}$.

Using the expression of $\widehat{\Phi}_{\pm\bar{x}}$ and the definition of $\Psi_{\bar{\varepsilon}}$ given on (27), we can compute the expression of $\overline{X}$ in such coordinates to be (dropping the hats for simplicity),

$$\overline{X}^{\pm} = \pm \mathrm{e}^{p+1}\left(x\frac{\partial}{\partial x} - \mathrm{e}\frac{\partial}{\partial \mathrm{e}}\right) + \widehat{F}^{\pm}(x,y,\mathcal{A},\mathrm{e})\frac{\partial}{\partial y}, \tag{28}$$

with $\widehat{F}^{\pm}(x,y,\mathcal{A},\mathrm{e}) = \widehat{F}_0^{\pm}(x,\mathcal{A},\mathrm{e}) + y\widehat{F}_1^{\pm}(x,\mathcal{A},\mathrm{e}) + y^2\widehat{Q}^{\pm}(x,y,\mathcal{A},\mathrm{e})$ given by

$$\begin{aligned}
\widehat{F}_0^{\pm}(x,\mathcal{A},\mathrm{e}) &= \sum_{i=0}^{2p-1} (\pm 1)^i\ A_i\,\mathrm{e}^{2p-i} + O(x)\\
\widehat{F}_1^{\pm}(x,\mathcal{A},\mathrm{e}) &= \mp p\,\mathrm{e}^{p+1} + \sum_{j=0}^{p-1} (\pm 1)^j\ B_j\,\mathrm{e}^{p-j} + (\pm 1)^p\mathcal{B}(\mathcal{A}_r) + O(x)\\
\widehat{Q}^{\pm}(x,y,\mathcal{A},\mathrm{e}) &= \mathcal{Q}(\mathcal{A}_r) + O(x)
\end{aligned}$$

where $\mathcal{B}(\mathcal{A}_r)$ and $\mathcal{Q}_0(\mathcal{A}_r)$ are as in (23) and $O(x)$ denotes analytic functions which are divisible by $x$.

*Remark* 4.2. Notice that $\overline{X}^{\pm}$ can be seen as an analytic family of *three-dimensional* vector fields, with phase space $(x,y,\mathrm{e}) \in (\mathbb{R}^+,0)\times(\mathbb{R}^2,0)$ and parameters $\mathcal{A} = (A,B,\mathcal{A}_r)$. Moreover, each vector field in such family has the function $f(x,\mathrm{e}) = x\mathrm{e}$ as a first integral.

Restricting $\overline{X}^{\pm}$ to the slice of the exceptional divisor $\mathcal{D}_0 = \{x = \mathcal{A}_r = 0\}$, we obtain the two-dimensional analytic family

$$\begin{aligned}
\mp\,\mathrm{e}^{p+2}\frac{\partial}{\partial \mathrm{e}} + \Bigg(&\sum_{i=0}^{2p-1} (\pm 1)^i\,A_i\,\mathrm{e}^{2p-i} + y(\mp p\ \mathrm{e}^{p+1}) + \\
&+ y\Big(\sum_{j=0}^{p-1} (\pm 1)^j\,B_j\,\mathrm{e}^{p-j} + (\pm 1)^p\mathcal{B}_0\Big) + y^2\mathcal{Q}_0\Bigg)\frac{\partial}{\partial y}
\end{aligned} \tag{29}$$

which can be seen as a *weighted compactification* of the Riccati family $R_{A,B}$ given on (24) at $x = \pm\infty$. Indeed, if we let $(\tilde{x},\tilde{y},\tilde{\varepsilon},A,B,\mathcal{A}_r)$ and $(\hat{x},\hat{y},\mathrm{e},A,B,\mathcal{A}_r)$ denote, respectively, the coordinates at $\mathcal{D}$ on the $\pm\bar{x}_\varepsilon$-chart and $\bar{\varepsilon}$-chart, the relations

$$\hat{x} = \pm\tilde{\varepsilon}\,\tilde{x}, \quad \mathrm{e} = \frac{\pm 1}{\tilde{x}} \quad \text{and} \quad \hat{y} = (\pm 1)^p\,\frac{\tilde{y}}{\tilde{x}^p}, \tag{30}$$

can be easily verified from the expressions of $\widehat{\Phi}_{\pm\bar{x}}$ and $\widehat{\Phi}_{\bar{\varepsilon}}$ (notice that the parameters $(A,B,\mathcal{A}_r)$ on the Riccati family and the parameters $(A,B,\mathcal{A}_r)$ on the family (29) will correspond to each other).

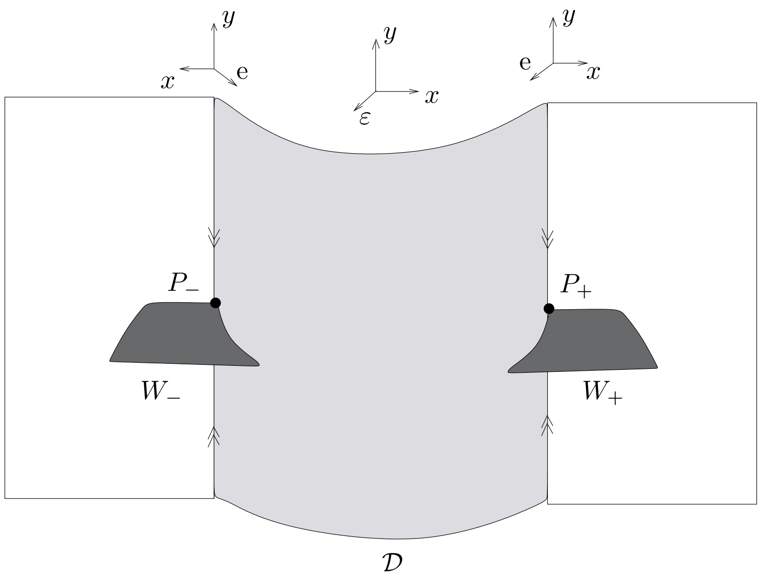



Figure 7. The compactification of the Riccati family.

*Remark* 4.3. Under the blowing-up, the curve

$$\Gamma_0 := \Gamma \setminus \{0\} \subset U$$

of non-degenerate singular points is mapped diffeomorphically to some open curve $\overline{\Gamma}_0 \subset \overline{U}$ (because $\Gamma_0$ does not intersect the blowing-up center $N \subset U$). On the coordinates of the $\pm\bar{x}$-chart, it is easy to see that such curve is given by

$$\overline{\Gamma}_0 = \{(\hat{x}, \hat{y}, \mathrm{e}, \hat{a}, \hat{b}, \mathcal{A}_r) \mid \hat{y} = \mathrm{e} = \hat{a} = \hat{b} = \mathcal{A}_r = 0\}.$$

Notice however that the blowing-up $\Psi$ which is defined on (26) has its center on the manifold $T = \{\mathrm{e} = \hat{a} = \hat{b} = \mathcal{A}_r = 0\}$, which contains $\overline{\Gamma}_0$. Thus, $\overline{\Gamma}_0$ can have several counter-images under $\Psi$. In fact, the counter-image of $\overline{\Gamma}_0$ on the $\pm\bar{x}_\varepsilon$-chart is explicitly given by the codimension $2 + (n - 3p)$ submanifold

$$\Psi^{-1}(\overline{\Gamma}_0) = \{(\hat{x}, \hat{y}, \mathrm{e}, A, B, \mathcal{A}_r) \mid \hat{y} = \mathrm{e} = \mathcal{A}_r = 0\}.$$

## 5. Local center manifolds at $P_+$ and $P_-$

To simplify the notation, we shall continue to write the coordinates at the $\pm\bar{x}_\varepsilon$-chart simply as $(x, y, \mathrm{e}, A, B, \mathcal{A}_r) \in U_{\hat{x}} \times U_{\hat{y}} \times U_{\mathrm{e}} \times U_{A,B} \times U_{\mathcal{A}_r}$, where

$$U_{\hat{x}} \in (\mathbb{R}^+, 0),\ U_{\hat{y}} = \mathbb{R},\ U_{\mathrm{e}} = \mathbb{R}^+,\ U_{A,B} = \mathbb{R}^{3p} \text{ and } U_{\mathcal{A}_r} \in (\mathbb{R}^{n-3p}, 0).$$

Observe that, from the expression in (28), it follows that for each fixed parameter $\mathcal{A}_0 = (A, B, \mathcal{A}_r)$, the vector field $\overline{X}^+$ (respect. $\overline{X}^-$) has a singularity at the point

$$p_+^{\mathcal{A}_0} = \{x = \mathrm{e} = y = 0,\ \mathcal{A} = \mathcal{A}_0\}$$

(respect. $p_-^{\mathcal{A}_0} = \{x = \mathrm{e} = y = 0,\ \mathcal{A} = \mathcal{A}_0\}$).

Possibly restricting the inessential parameters $\mathcal{A}_r$ to some smaller connected neighborhood $U_{\mathcal{A}_r} \in (\mathbb{R}^{n-3p}, 0)$ of the origin, we can suppose that the function $\mathcal{B}(\mathcal{A}_r)$ is non-vanishing and has a constant sign. This implies that all singular points $p_-^{\mathcal{A}_0}$ and $p_+^{\mathcal{A}_0}$ are of *semi-hyperbolic type*. The jacobian matrix $D\overline{X}^+(p_+^{\mathcal{A}_0})$ (respect. $D\overline{X}^+(p_-^{\mathcal{A}_0})$) has a single non-zero eigenvalue

$$\mathcal{B}_+(\mathcal{A}_r) := \mathcal{B}(\mathcal{A}_r) \quad (\text{respect. } \mathcal{B}_-(\mathcal{A}_r) := (\pm 1)^p \mathcal{B}(\mathcal{A}_r))$$

associated to the eigenvector $(x, y, \mathrm{e}) = (0, 1, 0)$. The $y$-axis is an invariant one-dimensional hyperbolic manifold. This manifold will be an unstable manifold for $p_+^{\mathcal{A}_0}$ (respect. $p_-^{\mathcal{A}_0}$) whenever $\mathcal{B}_+(\mathcal{A}_r) > 0$ (respect. $\mathcal{B}_-(\mathcal{A}_r) > 0$), otherwise it is a stable manifold. For briefness, we shall define the non-zero constants $\mathcal{B}_+ := \mathcal{B}_+(0)$ and $\mathcal{B}_- := \mathcal{B}_-(0)$.

The union of all such points $p_+^{\mathcal{A}}$ (respect. $p_-^{\mathcal{A}}$) for $\mathcal{A} \in U_{A,B,\mathcal{A}_r}$, defines a codimension 3 submanifold of *normally hyperbolic singularities*

$$P_+ := \{(x, y, \mathrm{e}, A, B, \mathcal{A}_r) \mid x = y = \mathrm{e} = 0\} \approx U_{A,B} \times U_{\mathcal{A}_r} \tag{31}$$

(respect. $P_- := \{(x, y, \mathrm{e}, A, B, \mathcal{A}_r) \mid x = y = \mathrm{e} = 0\}$).

The Center Manifold Theorem immediately implies the existence of local center manifolds at each point of $P_+$ and $P_-$:

**Proposition 5.1.** *For each natural number $k \in \mathbb{N}$, and each fixed $p_+ \in P_+$ and $p_- \in P_-$, there exist neighborhoods $V_+, V_- \subset U_{\hat{x},\mathrm{e},A,B,\mathcal{A}_r}$ of these points and $C^k$-functions*

$$\begin{array}{rccc} w_+ \colon & V_+ & \longrightarrow & \mathbb{R} \\ & (x,\mathrm{e},\mathcal{A}) & \longmapsto & y = w_+(x,\mathrm{e},\mathcal{A}) \end{array}$$

*(and similarly $w_- \colon V_- \to \mathbb{R}$), such that $w_+(x,0,0) = w_-(x,0,0) = 0$ and*

$$W_+ = \mathrm{graph}\{y = w_+(x,\mathrm{e},\mathcal{A})\}, \quad W_- = \mathrm{graph}\{y = w_-(x,\mathrm{e},\mathcal{A})\}$$

*are local invariant manifolds for $\overline{X}$.*

Although such center manifolds are not unique in general, we have the following uniqueness result, which will be useful later:

**Lemma 5.2.** *Suppose that $\mathcal{B}_+ > 0$ (respect. $\mathcal{B}_- < 0$). Then, the intersection of any $C^k$ local center manifold with the exceptional divisor $\mathcal{D} = \{x = 0\}$,*

$$W_+ \cap \mathcal{D} = \mathrm{graph}\{y = w_+(0,\mathrm{e},\mathcal{A})\},$$

*(respect. $W_- \cap \mathcal{D} = \mathrm{graph}\{y = w_-(0,\mathrm{e},\mathcal{A})\}$) is unique. That is, if $W'_\pm$ is another local $C^{k'}$ center manifolds, then $W_\pm \cap \mathcal{D}$ necessarily coincides with $W'_\pm \cap \mathcal{D}$ on their common domain of definition). Moreover, the function $w_+(0,\mathrm{e},\mathcal{A})$ (respect. $w_-(0,\mathrm{e},\mathcal{A})$) is $C^\infty$.*

*Proof:* The manifold $c := W_+ \cap \mathcal{D}$ is a $C^k$ local center manifold for the restricted two-dimensional analytic family $\overline{X}^+|_{\mathcal{D}}$, whose expression is given in (29).

Now, since $\mathcal{B}_+ > 0$, the hypothesis of Theorem 3.2 in [**Si**] are fulfilled. Thus, this local center manifold $c$ is necessarily unique.

Finally, the fact that $c \in C^\infty$ is an immediate consequence of [**Si**, Theorem 5.1].

The argument for $W_-$ is analogous. □

We can also consider the issue of *formal expansion* of such center manifolds:

**Lemma 5.3.** *Given an arbitrary open subset $V_{\mathcal{A}} = V_{A,B} \times V_{\mathcal{A}_r} \subset U_{A,B} \times U_{\mathcal{A}_r}$ such that $V_{\mathcal{A}_r} \in (\mathbb{R}^{n-3p}, 0)$ and $V_{A,B} \subset \mathbb{R}^{3p}$ has a compact closure, there exist open intervals in $\mathbb{R}^+$,*

$$V_x^+ = [0, x_+) \quad \textit{and} \quad V_x^- = [0, x_-)$$

*(where $x_+, x_- > 0 \in U_{\hat{x}}$ depend on $V_{\mathcal{A}}$) and there exist unique formal series*

$$\widehat{W}_+(x, \mathrm{e}, \mathcal{A}) = \sum_{i=0}^{\infty} w_i^+(x, \mathcal{A}) \mathrm{e}^i, \quad \widehat{W}_-(x, \mathrm{e}, \mathcal{A}) = \sum_{i=0}^{\infty} w_i^-(x, \mathcal{A})\, \mathrm{e}^i$$

*defined by a collection of analytic functions*

$$w_i^+ \in C^\omega(V_x^+ \times V_{\mathcal{A}}),\ w_i^- \in C^\omega(V_x^- \times V_{\mathcal{A}}) \quad \textit{for } i \in \mathbb{N},$$

*such that for each point $p_+ \in P_+ \cap V_{\mathcal{A}}$ (respect. $p_- \in P_- \cap V_{\mathcal{A}}$), and each local $C^k$-center manifold $W_+ = \operatorname{graph}\{y = w_+(x, \varepsilon, \mathcal{A})\}$ (respect. $W_- = \operatorname{graph}\{y = w_-(x, \varepsilon, \mathcal{A})\}$) defined in a neighborhood $U$ of such point, it follows that*

$$w_+ \in C^k_{\mathrm{flat}}(U, \{\mathrm{e} = 0\}, \widehat{W}_+)$$

*(respect. $w_- \in C^k_{\mathrm{flat}}(U, \{\mathrm{e} = 0\}, \widehat{W}_-)$).*

*Proof:* Let us prove the statement just for $P_+$, since the argument at $P_-$ is completely analogous.

We proceed as in the proof of Lemma 3.2. Using the expression for $\overline{X}$ which is given by (28), one sees that, possibly restricting $x$ to some smaller interval $V_x^+ = [0, x_+) \subset U_{\hat{x}}$, we can use the Implicit Function Theorem to define an initial translation of the form

$$y = y' - w_0^+(x, \mathcal{A})$$

in such a way that the set of singularities for $\overline{X}^+$ on $V_x^+ \times U_y \times U_{\mathrm{e}} \times V_{\mathcal{A}}$ becomes $\mathcal{Z}(\overline{X}^+) = \{\mathrm{e} = y' = 0\}$.

Thus (dropping again the primes), it suffices to prove that there exists a unique formal series

$$\widehat{M}_+(x, \mathrm{e}, \mathcal{A}) = \sum_{i=1}^{\infty} w_i^+(x, \mathcal{A}) \mathrm{e}^i$$

such that (in these new coordinates) the vector field $\overline{X}$, seen as a derivation on the space of formal series, maps the series

$$\widehat{K}(x, y, \mathrm{e}, \mathcal{A}) := y - \widehat{M}_+(x, \mathrm{e}, \mathcal{A})$$

into some series $\overline{X}(\widehat{K})$ such that

$$\overline{X}(\widehat{K})|_{y=\widehat{M}_+} \equiv 0.$$

Using the expression in (28), this equation becomes

$$\mathrm{e}^{p+1}\left(x\frac{\partial}{\partial x} - \mathrm{e}\frac{\partial}{\partial \mathrm{e}}\right)\widehat{M}_+ = \widehat{F}^+(x, \widehat{M}_+, \mathcal{A}, \mathrm{e}).$$

If we expand each side of this equality in powers of e, and use the fact that $\mathcal{B}_+(\mathcal{A}_r) \neq 0$, it can be seen that the coefficients $w_i^+ \in C^\omega(V_x^+ \times V_\mathcal{A})$ are uniquely determined in a recursive way by the Implicit Function Theorem. □

*Remark* 5.4. Notice that if we consider two domains $V'_\mathcal{A}$ and $V''_\mathcal{A}$ as above, the corresponding functions $w_i^{\pm\prime}$ and $w_i^{\pm\prime\prime}$ will necessarily coincide in their common domain of definition.

We shall say that $\widehat{W}_+$ and $\widehat{W}_-$ defines the *formal center manifolds* over $P_+ \cap V_\mathcal{A}$ and $P_- \cap V_\mathcal{A}$, respectively.

*Remark* 5.5. For an arbitrary closed subset $M \subset V_x^+ \times V_\mathcal{A}$ (respect. $M \subset V_x^- \times V_\mathcal{A}$), it is clear that $\widehat{W}_+$ (respect. $\widehat{W}_-$) uniquely defines an element of $C^\infty(M \times \{\mathrm{e} = 0\})$ (in the sense of Whitney).

Given an arbitrary point $x_0 \in V_x^+$ (respect. $x_0 \in V_x^-$) let us define

$$\widehat{W}_{+,x_0} := \sum_{i=0}^{\infty} w_i^+(x_0, \mathcal{A}) \quad (\text{respect. } \widehat{W}_{-,x_0} := \sum_{i=0}^{\infty} w_i^-(x_0, \mathcal{A}))$$

to be the restriction of the formal center manifold to $x_0$.

In analogy with the previous subsection, let us now describe how one construct a local dynamical center manifold over $P_+$ and $P_-$. For shortness, we shall only describe the details for $P_+$, since the construction at $P_-$ is completely analogous.

Let us suppose firstly that $\mathcal{B}_+ > 0$. Fixing an arbitrary open subset $V_\mathcal{A} \subset U_\mathcal{A}$ as in the enunciate of Lemma 5.3, we choose a point $x_0 > 0 \in V_x^+$, and let

$$V_\mathrm{e} = (0, \mathrm{e}_0) \quad \text{for some } \mathrm{e}_0 > 0 \in U_\mathrm{e}.$$

Let us now consider

$$\begin{array}{rccl} i\colon V_\mathrm{e} \times V_\mathcal{A} & \longrightarrow & \mathbb{R}, \\ (\mathrm{e}, \mathcal{A}) & \longmapsto & y = i(\mathrm{e}, \mathcal{A}) \end{array} \tag{32}$$

to be an arbitrary $C^\infty$ function, such that

$$i \in C^\infty_{\text{flat}}(V_\mathrm{e} \times V_\mathcal{A}, \{\mathrm{e} = 0\}, \widehat{W}_{+,x_0}).$$

Then, we shall say that $i$ is an *initial condition function* for $\overline{X}$ at $P_+$.

**Proposition 5.6.** *Suppose that $\mathcal{B}_+ > 0$ and let $i$ be an arbitrary initial condition function like in (32). Let $N_x := [0, x_0] \subset \mathbb{R}^+$. Then, there exists an interval $N_{\mathrm{e}} = (0, \mathrm{e}_1) \subset V_{\mathrm{e}}$ $(0 < \mathrm{e}_1 \leq \mathrm{e}_0)$ and there exists an unique $C^\infty$ function $w_+(x, \mathrm{e}, \mathcal{A})$ defined on the open set $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$ which verifies the following conditions:*

(i) $w_+(x_0, \mathrm{e}, \mathcal{A}) = i(\mathrm{e}, \mathcal{A})$, *for each* $(\mathrm{e}, \mathcal{A}) \in N_{\mathrm{e}} \times V_{\mathcal{A}}$;

(ii) $w_+ \in C^\infty_{\mathrm{flat}}(N^+, \{\mathrm{e} = 0\}, \widehat{W}_+)$, *and*

(iii) $W_+ = \mathrm{graph}\{y = w_+(x, \mathrm{e}, \mathcal{A})\}$ *is an invariant manifold.*

*Proof:* First of all, if we consider the saturate $\mathcal{W}$ of the set

$$\xi = \bigcup_{(\mathrm{e},\mathcal{A}) \in V_{\mathrm{e}} \times V_{\mathcal{A}}} \{(x, y, \mathrm{e}, \mathcal{A}) \mid x = x_0, y = i(\mathrm{e}, \mathcal{A})\} \tag{33}$$

under the flow of $-\overline{X}$, it is easy to see (from the condition $\mathcal{B}_+ > 0$) that there exist domains $N_x$, $N_{\mathrm{e}}$ as above such that the restriction of $\mathcal{W}$ to $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$ is defined as the graph of a function $w_+ \in C^\infty(N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}})$.

Therefore, it remains to show that $w^+$ is $\infty$-flat to the formal series $\widehat{W}_+$ on $\{\mathrm{e} = 0\}$. Adopting the same strategy used on Proposition 3.4, we shall prove that for any point $p \in P_+$ and any $C^k$ local center manifold $W' = \mathrm{graph}\{y = w'(x, \mathrm{e}, \mathcal{A})\}$ defined on some neighborhood $U'$ of $p$, the function $w - w'$ belongs to $C^k_{\mathrm{flat}}(U', \{\mathrm{e} = 0\})$.

From Corollary 13.3, there exists $C^k$ local fiber-preserving change of coordinates

$$\psi \colon (x, y, \mathrm{e}, \mathcal{A}) \to (x', y', \mathrm{e}', \mathcal{A}') \tag{34}$$

defined on some neighborhood of $p$ such that $W' = \{y' = 0\}$ and $-\overline{X}$ is equivalent to (dropping again the primes to simplify the notation)

$$Y = -y\frac{\partial}{\partial y} + \mathrm{e}^{p+1}\, G(x, \mathrm{e}, \mathcal{A}) \left( x\frac{\partial}{\partial x} - \mathrm{e}\frac{\partial}{\partial \mathrm{e}} \right), \tag{35}$$

for some strictly positive $C^k$ function $G(x, \mathrm{e}, \mathcal{A})$.

Let us define $\tilde{i} := i \circ \psi^{-1}$. Then, $\tilde{i}$ is a $C^k$ function defined on some open subset of $\{\mathrm{e} > 0\}$, $\infty$-flat at $\{\mathrm{e} = 0\}$. If we consider the initial condition (which corresponds to (33))

$$\tilde{\xi} = \{(x, y, \mathrm{e}, \mathcal{A}) = (x_0, \tilde{i}(\mathrm{e}, \mathcal{A}), \mathrm{e}, \mathcal{A})\},$$

and integrate the vector field

$$-\left( \frac{1}{\mathrm{e}^{p+1} G} \right) y \frac{\partial}{\partial y} + \left( x\frac{\partial}{\partial x} - \mathrm{e}\frac{\partial}{\partial \mathrm{e}} \right)$$

(which is equivalent to $Y$ on the set $\{\mathrm{e} > 0\}$), the explicit solution is given by

$$\begin{cases} \mathrm{e}(t) = \mathrm{e}\exp(-t), \\ x(t) = x_0 \exp(t), \\ y(t) = \tilde{i}(\mathrm{e}(t), \mathcal{A}) \exp\left(-\int_0^t \frac{1}{x(\tau)^{p+1} G(x(t), \mathrm{e}(t), \mathcal{A})} d\tau\right). \end{cases}$$

Based on these equations, we can eliminate the variable $t$ from the last expression by taking $t = -\ln(x/x_0)$. We refer the reader to [**Du-R**, Lemma 8] for the proof that the resulting function $y$ is $\infty$-flat at $\{\mathrm{e} = 0\}$. □

Let us now suppose that $\mathcal{B}_+ < 0$. Here, in order to define the initial condition, we let

$$(36) \qquad \begin{array}{rcl} i\colon V_x \times V_{\mathcal{A}} & \longrightarrow & \mathbb{R} \\ (x, \mathcal{A}) & \longmapsto & y = i(x, \mathcal{A}) \end{array}$$

be a $C^\infty$ function defined on an open subset $V_x \times V_{\mathcal{A}}$, where $V_{\mathcal{A}} \subset U_{\mathcal{A}}$ is like in Lemma 5.3 and $V_x := [0, x_0]$ for some $x_0 > 0 \in V_x^+$.

We also need to fix a *base* point $\mathrm{e}_0 > 0$. Here, one must be careful to choose $i(0, \mathcal{A})$ and $\mathrm{e}_0$ sufficiently small, in order to remain in the *attracting region* of the semi-hyperbolic set $\mathcal{P}_+$. This region is defined by the following lemma:

**Lemma 5.7.** *On the above notations, there exist constants* $\mathrm{e}_+, y_+ > 0$*, depending only on the subset* $V_{\mathcal{A}}$ *such that for each given point*

$$q_0 = \{(x, y, \mathrm{e}, \mathcal{A}) = (0, y_0, \mathrm{e}_0, \mathcal{A})\} \in \mathcal{D},$$

*where* $0 < \mathrm{e}_0 < \mathrm{e}_+$*,* $|y_0| < y_+$ *and* $\mathcal{A} \in V_{\mathcal{A}}$*, the orbit of* $\overline{X}|_{\mathcal{D}}$ *starting at* $q_0$ *has its* $\omega$*-limit in* $\mathcal{P}_+$*.*

*Proof:* There are several ways to prove such result. Here, we shall use the Normal Form Theorem given on the Appendix.

For each point $p \in \mathcal{P}_+$, Corollary 13.3 shows that there exists a $C^1$ change of coordinates defined on a neighborhood $U_p$ of $p$ such that $\overline{X}|_{\mathcal{D}}$ is $C^1$ equivalent to

$$-y\frac{\partial}{\partial y} - \mathrm{e}^{p+2}\, G(\mathrm{e}, \mathcal{A})\frac{\partial}{\partial \mathrm{e}},$$

for some strictly positive $C^1$ function $G(\mathrm{e}, \mathcal{A})$. For this vector field, it is easy to see that each orbit on the region $\{\mathrm{e} > 0\}$ has the origin as $\omega$-limit.

Since $V_{\mathcal{A}}$ has a compact closure, we can choose a finite number of points $p \in \mathcal{P}_+$ such that the union of the domains $U_p$ covers $V_{\mathcal{A}}$. The constants $\mathrm{e}_+$, $y_+$ can now be chosen in such a way that the region

$$\{(\mathrm{e}, y) \mid 0 \le \mathrm{e} < \mathrm{e}_+,\ |y| < y_+\} \tag{37}$$

lies in intersection of all domains $U_p \cap \{\mathcal{A} = \text{const}\}$. This proves the lemma. □

We shall say that the open set defined on (37) is the *attracting region* for the restricted vector field $\overline{X}|_{\mathcal{D}}$, relatively to the set $V_{\mathcal{A}}$.

A function $i(x, \mathcal{A})$ as in (36) will be called an *initial condition function* (for the case $\mathcal{B}_+ < 0$) if

$$\sup_{\mathcal{A} \in V_{\mathcal{A}}} \|i(0, \mathcal{A})\| < y_+.$$

Finally, we can prove the following result, which defines the dynamical center manifold.

**Proposition 5.8.** *Suppose that $\mathcal{B}_+ < 0$. Choose an arbitrary point $\mathrm{e}_0 \in (0, \mathrm{e}_+)$ (where $\mathrm{e}_+$ is defined on the previous lemma), and let $N_{\mathrm{e}} := (0, \mathrm{e}_0] \subset \mathbb{R}$. Then, for any initial condition function $i(x, \mathcal{A})$, there exists an interval $N_x = [0, x_1) \subset V_x$ $(0 < x_1 \le x_0)$, and an unique $C^\infty$ function $w_+(x, \mathrm{e}, \mathcal{A})$ defined on the open set $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$ such that*

(i) $w_+(x, \mathrm{e}_0, \mathcal{A}) = i(x, \mathcal{A})$, *for each* $(x, \mathcal{A}) \in N_x \times V_{\mathcal{A}}$;
(ii) $w_+ \in C^\infty_{\mathrm{flat}}(N^+, \{\mathrm{e} = 0\}, \widehat{W}_+)$, *and*
(iii) $W_+ = \mathrm{graph}\{y = w_+(x, \mathrm{e}, \mathcal{A})\}$ *is an invariant manifold.*

*Proof:* The proof can be obtained by easy modifications on the proof of Proposition 5.6. □

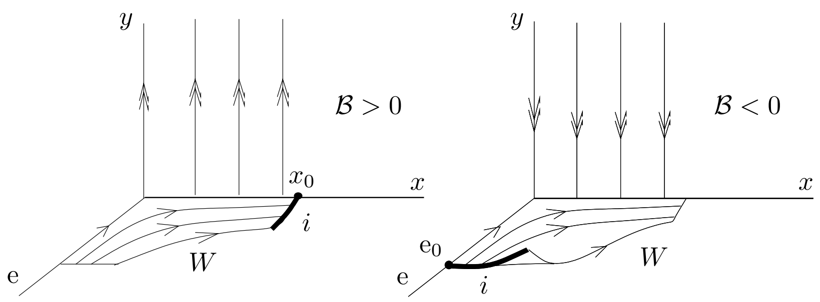



FIGURE 8. Dynamical center manifolds of Propositions 5.6 and 5.8.

As we have said on the beginning of this section, the above two propositions have an exact analogous at $P_-$. Let us denote by $W_-$ the corresponding center manifold (depending on the initial function $i$), and let $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$ be its domain of definition.

We shall say that the manifolds $W^+$ and $W_-$ which are obtained by these constructions are *local dynamical center manifolds* at $P_+$ and $P_-$.

For future reference, let us establish the following classification of the degenerate singularity $x = 0 \in \Gamma$, with respect to the stability of the hyperbolic directions at the sets $P_-$ and $P_+$:

- *the degenerate point* $0 \in \Gamma$ *is in the* $(\mathbf{u}, \mathbf{u})$*-case* if $\mathcal{B}_+, \mathcal{B}_- > 0$;
- *the degenerate point* $0 \in \Gamma$ *is in the* $(\mathbf{s}, \mathbf{u})$*-case* if $\mathcal{B}_- < 0$, $\mathcal{B}_+ > 0$;
- *the degenerate point* $0 \in \Gamma$ *is in the* $(\mathbf{u}, \mathbf{s})$*-case* if $\mathcal{B}_- > 0$, $\mathcal{B}_+ < 0$;
- *the degenerate point* $0 \in \Gamma$ *is in the* $(\mathbf{s}, \mathbf{s})$*-case* if $\mathcal{B}_+, \mathcal{B}_- < 0$.

*Remark* 5.9. Notice that the sets $P_+$ and $P_-$ are on the boundary of the set $\Psi^{-1}(\overline{\Gamma})$, defined on Remark 4.3. By continuity, the sign of $\mathcal{B}_-$ (respect. $\mathcal{B}_+$) corresponds to the the sign of the nonzero eigenvalue at each one of the non-degenerate singular points in $\Gamma \cap \{x < 0\}$ (respect. $x \in \Gamma \cap \{x > 0\}$). Thus, the above classification corresponds precisely to the classification made on the introduction.

## 6. Extension of center manifolds and the matching region

### 6.1. The distance function $\boldsymbol{\Delta}(\varepsilon, \mathcal{A})$.

Let us consider arbitrary local center manifolds $W_-$ and $W_+$ at $P_-$ and $P_+$, which are defined on domains $N^-, N^+ = N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$ by

$$W^- = \mathrm{graph}(y = w_-(x, \mathrm{e}, \mathcal{A})) \quad \text{and} \quad W^+ = \mathrm{graph}(w = w_+(x, \mathrm{e}, \mathcal{A})),$$

respectively. Our goal in this subsection is to describe how one can *dynamically extend* the domains of definition for such invariant manifolds.

Let us just discuss the extension of $W_-$, since the description for $W^+$ is analogous. First of all, if we fix an arbitrary point $\mathrm{e}_1 > 0 \in N_{\mathrm{e}}$, the intersection of the manifold $W_-$ with the set $\{\mathrm{e} = \mathrm{e}_1\}$ is given by the codimension 2 submanifold

$$\xi := \bigcup_{(x,\mathcal{A}) \in N_x \times V_{\mathcal{A}}} (x, y, \mathrm{e}, \mathcal{A}) = (x, w_+(x, \mathrm{e}_0, \mathcal{A}), \mathrm{e}_0, \mathcal{A}).$$

Since $\mathrm{e}_1 > 0$, such submanifold is also contained in the domain $U_{\bar{\varepsilon}}$ of the $\bar{\varepsilon}$-chart which is described on Subsection 4.2.

If we denote by $(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, \mathcal{A})$ the coordinates of the $\bar{\varepsilon}$-chart, we have the following relations

$$\tilde{x} = \frac{1}{\mathrm{e}}, \quad \tilde{y} = (-1)^p \frac{y}{\mathrm{e}^p}, \quad \tilde{\varepsilon} = x\,\mathrm{e} \tag{38}$$

between such coordinates and the coordinates of the $-\bar{x}$-chart. Thus, the set $\xi$ can be expressed in these coordinates by

$$\xi := \bigcup_{(\tilde{\varepsilon},\mathcal{A}) \in N_{\tilde{\varepsilon}} \times V_{\mathcal{A}}} (\tilde{x}, \tilde{y}, \tilde{\varepsilon}, \mathcal{A}) = \left(\frac{1}{\mathrm{e}_0}, H(\tilde{\varepsilon}, \mathcal{A}), \tilde{\varepsilon}, \mathcal{A}\right),$$

where we define the domain

$$N_{\tilde{\varepsilon}} := \psi_{\mathrm{e}_1}(N_x) = \left\{\tilde{\varepsilon} \in \mathbb{R}^+ \mid \frac{\tilde{\varepsilon}}{\mathrm{e}_1} \in N_x\right\} \tag{39}$$

and the corresponding function

$$H(\tilde{\varepsilon}, \mathcal{A}) := (-1)^p \mathrm{e}_0^{-p} \cdot w_+ \left(\frac{\tilde{\varepsilon}}{\mathrm{e}_0}, \mathcal{A}\right).$$

On the $\bar{\varepsilon}$-projective chart, the blowed-up vector field $\overline{X}$ is given by the expression on (19), which can be seen as a family of analytic first order differential equations

$$d\tilde{y}/d\tilde{x} = \widetilde{F}(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, \mathcal{A}). \tag{40}$$

Now, for each point $q \in \xi$, we can consider the solution of the initial value problem associated to such point. Let us write this solution as a function $\tilde{y} = \phi(q, \tilde{x})$, which depends analytically on $q$ and $\tilde{x}$, and let $I(q) = (-\infty, \tilde{x}(q))$ be the maximal interval of definition of such solution (which depends continuously on the point $q$).

Then, we can define the *saturate of* $\xi$ under the flow of $\overline{X}$ to be the open subset in $U_{\bar{\varepsilon}}$ given by

$$\mathcal{W}^- := \bigcup_{q \in \xi,\, \tilde{x} \in I(q)} \tilde{y} = \phi(q, \tilde{x}).$$

We shall say that $\mathcal{W}^-$ is the *extension of the local center manifold* $W^-$.

We are particularly interested at those points $q \in \xi$ such that the corresponding maximal interval of definition $I(q)$ is sufficiently large.

**Proposition 6.1.** *There exists an open (possibly empty) subset $\mathcal{O}^- \subset N_{\tilde{\varepsilon}} \times V_{\mathcal{A}}$ such that for each point $q$ belonging to the* restricted submanifold

$$\xi|_{\mathcal{O}^-} := \bigcup_{(\tilde{\varepsilon},\mathcal{A})\in\mathcal{O}^-} (\tilde{x},\tilde{y},\tilde{\varepsilon},\mathcal{A}) = \left(\frac{1}{\mathrm{e}_0}, H(\tilde{\varepsilon},\mathcal{A}), \tilde{\varepsilon}, \mathcal{A}\right)$$

*the corresponding maximal interval of definition $I(q)$ contains the origin $\{\tilde{x} = 0\}$.*

*Proof:* The result is a trivial consequence of the continuous dependence of the solutions on the initial condition and the fact that $\overline{X}|_{U_{\tilde{\varepsilon}}}$ defines a $C^\infty$ (even $C^\omega$) flow on $U_{\tilde{\varepsilon}}$. □

Let us call $\mathcal{O}^-$ the *maximal domain of extension* of the local center manifold $W_-$. Geometrically, the meaning of $\mathcal{O}^-$ is that each solution curve starting at a point $q \in \xi|_{\mathcal{O}^-}$ cuts the transversal section

$$\Sigma := \{\tilde{x} = 0\}$$

in a unique point $q' \in \Sigma$. The correspondence $q \mapsto q'$ defines a $C^\infty$ map which is described in the next result:

**Corollary 6.2.** *Let $q$ be a point in $\xi|_{\mathcal{O}^-}$, with coordinates*

$$q = \left\{(\tilde{x},\tilde{y},\tilde{\varepsilon},\mathcal{A}) = \left(\frac{1}{\mathrm{e}_0}, H(\tilde{\varepsilon},\mathcal{A}), \tilde{\varepsilon}, \mathcal{A}\right)\right\},$$

*and let*

$$q' = \phi(0,q) = \{(\tilde{x},\tilde{y},\tilde{\varepsilon},\mathcal{A}) = (0, y', \tilde{\varepsilon}, \mathcal{A})\}$$

*be the unique point where the orbit issuing from $q$ cuts $\Sigma$. Then, the map*

$$\mathcal{O}^- \ni (\tilde{\varepsilon},\mathcal{A}) \longmapsto y' \in \mathbb{R}$$

*is a $C^\infty$ function $Y_-(\tilde{\varepsilon},\mathcal{A})$ on $\mathcal{O}^-$.*

*Proof:* $Y_-(\tilde{\varepsilon},\mathcal{A})$ is obtained by composing the $C^\omega$ flow map $\phi(\cdot,0)$ with the $C^\infty$ map $(w,\mathcal{A}) \mapsto (\frac{1}{\mathrm{e}_0}, H(\tilde{\varepsilon},\mathcal{A}), \tilde{\varepsilon}, \mathcal{A})$, and so it is clearly $C^\infty$. □

Of course, all the above arguments can be easily adapted to define the extension $\mathcal{W}^+$ of the local center manifold $W^+$. Similarly, we can define the *maximal domain of extension* $\mathcal{O}^+ \subset N_{\tilde{\varepsilon}} \times V_{\mathcal{A}}$ of $W^+$ and the corresponding *transport function*

$$Y_+ : \mathcal{O}^+ \to \mathbb{R}.$$

*Remark* 6.3. As pictured on Figure 9, the graph of the functions

$$y = Y_-(\hat{\varepsilon},\mathcal{A}) \quad \text{and} \quad y = Y_+(\hat{\varepsilon},\mathcal{A})$$

describes the intersections $\mathcal{W}^-$ and $\mathcal{W}^+$, with the transversal section $\Sigma$.

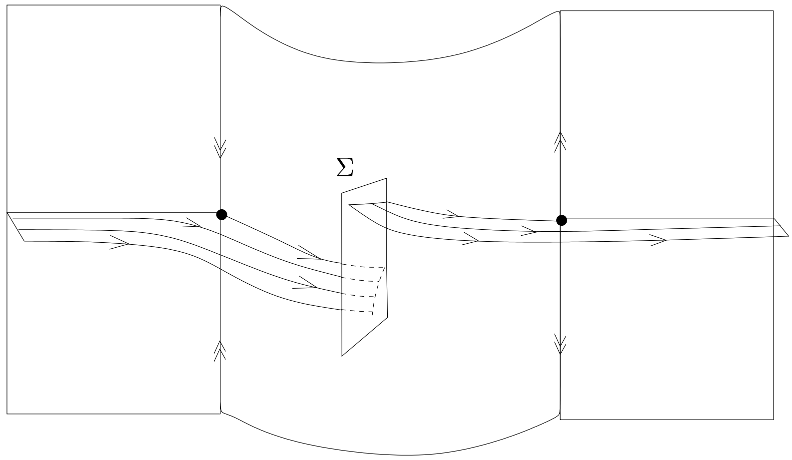



FIGURE 9. The transversal section Σ.

The set $\mathcal{O}^- \cap \mathcal{O}^+$ is the common domain of definition of $Y_-$ and $Y_+$. On this domain, one can define the *distance function*

$$\begin{array}{rcl}\Delta\colon \mathcal{O}^- \cap \mathcal{O}^+ & \longrightarrow & \mathbb{R} \\ (\tilde{\varepsilon}, \mathcal{A}) & \longmapsto & \Delta(\tilde{\varepsilon}, \mathcal{A}) := Y_+(\tilde{\varepsilon}, \mathcal{A}) - Y_-(\tilde{\varepsilon}, \mathcal{A}).\end{array}$$

Let us define the *matching region* (associated to the center manifolds $W_+$ and $W_-$) to be the closed subset in $\mathcal{O}^- \cap \mathcal{O}^+$ where such distance function vanishes

$$\mathcal{O}(W_-, W_+) := \{(\tilde{\varepsilon}, \mathcal{A}) \in \mathcal{O}^- \cap \mathcal{O}^+ \mid \Delta(\tilde{\varepsilon}, \mathcal{A}) = 0\}. \tag{41}$$

*Remark* 6.4. The description of the sets $\mathcal{O}^-$, $\mathcal{O}^+$ and $\mathcal{O}(W_-, W_+)$ seems to be a hard problem. On the other hand, we shall see in the next section that it is easier to describe the intersections of $\mathcal{O}^+$, $\mathcal{O}^-$ with the set $\{\tilde{\varepsilon} = 0\}$ by studying the asymptotic behavior of the Riccati family $R_{A,B}$ which is given on (24).

**6.2. Matching dynamical center manifolds.**

Instead of fixing both dynamical center manifolds $W_-$ and $W_+$, there are some situations where it is better to define one of them from the other, in such a way that they *match* in their common domain of definition.

**Proposition 6.5.** *Suppose that $\mathcal{B}_+ < 0$. Let $W_-$ be an arbitrary local dynamical center manifold at $P_-$, defined on some domain $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$. Then, there exists an open (possibly empty) subset*

$$N'_x \times V'_{\mathcal{A}} \subset N_x \times V_{\mathcal{A}}$$

*where $N'_x = [0, x'_0)$, and a unique local dynamical center manifold $W_+$ defined on the open domain*

$$N'_x \times N'_{\mathrm{e}} \times V'_{\mathcal{A}},$$

*(where $\widetilde{N}_{\mathrm{e}} = (0, \mathrm{e}'_0)$ for some $\mathrm{e}'_0 > 0$), such that, if we consider the maximal domains of extension $\mathcal{O}^-$, $\mathcal{O}^+$ of $W_-$ and $W_+$ and the distance function*

$$\Delta \colon \mathcal{O}^+ \cap \mathcal{O}^- \to \mathbb{R},$$

*then $\Delta \equiv 0$ (or, in other words, the matching region $\mathcal{O}(W_-, W^+)$ is the whole set $\mathcal{O}^+ \cap \mathcal{O}^-$).*

*Proof:* Let $R_+ := \{(x, \mathrm{e}, y) \mid x = 0, 0 \le \mathrm{e} < \mathrm{e}_+, |y| < y_+\}$ be the attracting region associated to $P_+$ (relatively to $V_{\mathcal{A}}$), which is defined on Lemma 5.7, and let

$$c_- := W_- \cap \mathcal{D}$$

denotes the intersection of the center manifold $W_-$ with the exceptional divisor. In the coordinates of the $\bar{\varepsilon}$-chart, this curve is given by the graph of the $C^\infty$ function

$$c_- := \operatorname{graph}\{\tilde{y} = y_-(\tilde{x}, \mathcal{A})\}$$

which is defined for $\mathcal{A} \in V_{\mathcal{A}}$ and $x$ in some neighborhood of $-\infty$.

It is easy to prove (exactly as in the proof of Proposition 6.1) that there exists an open (possibly empty) subset $V'_{\mathcal{A}} \subset V_{\mathcal{A}}$ such that for each $\mathcal{A} \in V'_{\mathcal{A}}$, the curve $c_-$ can be extended to a solution of the vector field $\overline{X}|_{\mathcal{D}}$ which enters into the attracting region $R_+$.

Therefore, by the continuity of the flow, we can choose a small constant $x'_0 > 0 \in N_x$ such that, restricting the domain of definition of this manifold $W_-$ to

$$[0, x'_0] \times N_{\bar{\varepsilon}} \times V'_{\mathcal{A}},$$

it can be extended to a larger center manifold $\mathcal{W}_{-,\infty}$, which cuts the transversal section $\Sigma_{\mathrm{e}_+} = \{\mathrm{e} = \mathrm{e}_+\}$.

Possibly taking some smaller $x'_0 > 0$, we can also guarantee that the intersection $\mathcal{W}_{-,\infty} \cap \Sigma_{\mathrm{e}_+}$ is the graph of some *initial condition function* $i$ which verifies the requirements of Proposition 5.8. Therefore, it uniquely defines a local center manifold $W_+$ at $P_+$. This proves the result. $\square$

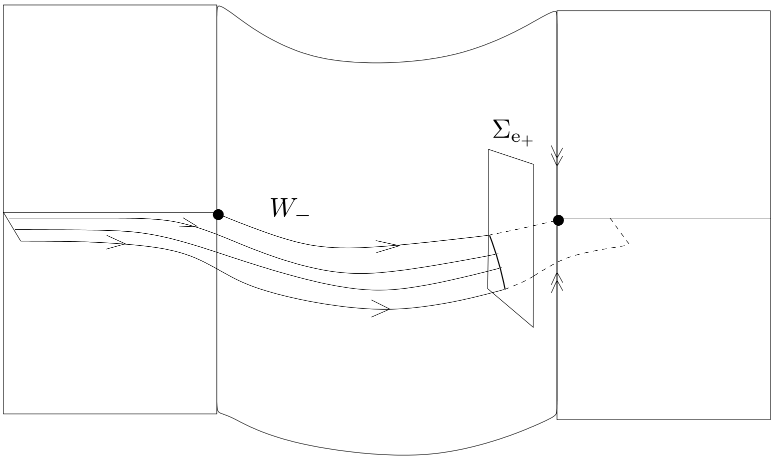



FIGURE 10. The matching of center manifolds.

Of course, we have an analogous result for $\mathcal{B}_- > 0$. In this case, given the set $V_{\mathcal{A}}$, we can define an *attracting region* $R_-$ for $P_-$ using the inverted vector field $-\overline{X}|_{\mathcal{D}}$.

**Proposition 6.6.** *Suppose that $\mathcal{B}_- > 0$. Let $W_+$ be an arbitrary local dynamical center manifold at $P_+$, defined on some domain $N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}$. Then, there exists an open (possibly empty) subset*

$$N'_x \times V'_{\mathcal{A}} \subset N_x \times V_{\mathcal{A}}$$

*where $N'_x = [0, x'_0)$, $N'_{\mathrm{e}} = (0, \mathrm{e}'_0)$, and a unique local dynamical center manifold $W_-$ defined on the domain $N'_x \times N'_{\mathrm{e}} \times V'_{\mathcal{A}}$, such that $\mathcal{O}(W_-, W_+) = \mathcal{O}_- \cap \mathcal{O}_+$.*

## 7. Asymptotic behavior of the Riccati family

In this section, our main goal is to characterize solutions of the associated Riccati family

$$\frac{dy}{dx} = \sum_{i=0}^{2p-1} A_i x^i + y \left( \sum_{j=0}^{p-1} B_j x^j + \mathcal{B}_0 x^p \right) + \mathcal{Q}_0\, y^2, \tag{42}$$

with $\mathcal{B}_0, \mathcal{Q}_0 \in \mathbb{R}$, which are *asymptotic to zero* as $x \to \pm\infty$, with respect to the *weighted compactification* at $x = \pm\infty$,

$$X = \frac{\pm 1}{x}, \quad Y = \frac{(-1)^p y}{x^p}.$$

Thus, putting in more precise terms, we want to obtain conditions on the values of the parameters $A$, $B$ which guarantee the existence of solutions $y_-(x)$ and $y_+(x)$ such that $y_-(0), y_+(0) \in \mathbb{R}$ are well-defined and

$$\lim_{X\to 0} X^p \cdot y_- \left(\frac{-1}{X}\right) = \lim_{X\to 0} X^p \cdot y_+ \left(\frac{1}{X}\right) = 0.$$

**7.1. The linear case.**

To simplify the exposition, we will start by studying a simpler linear family

$$\frac{dy}{dx} = A_0 + A_1 x + \cdots + A_{2p-1} x^{2p-1} + y x^p, \tag{43}$$

whose general solution from an initial point $y(0) = y_0$ can be written as:

$$y(x) = e^{\frac{x^{p+1}}{p+1}} \left( y_0 + A_0 \int_0^x e^{-\frac{t^{p+1}}{p+1}} dt + \cdots + A_{2p-1} \int_0^x t^{2p-1} e^{-\frac{t^{p+1}}{p+1}} dt \right).$$

Let us calculate the asymptotic expansion of the integral

$$J_k(x) := \int_0^x t^k e^{-\frac{t^{p+1}}{p+1}} dt$$

when $x \to \infty$. Since this integral is convergent for any $k$, we can write

$$J_k(x) = \int_0^\infty t^k e^{-\frac{t^{p+1}}{p+1}} dt - \int_x^\infty t^k e^{-\frac{t^{p+1}}{p+1}} dt.$$

Now, we estimate each integral separately. From ([**Di**, IV.3.3.1]),

$$\int_0^\infty t^k e^{-\frac{t^{p+1}}{p+1}} dt = (p+1)^{(k+1)/(p+1)-1} \Gamma\left(\frac{k+1}{p+1}\right)$$

(where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function) and from ([**Di**, 10.7.2]),

$$\int_x^\infty t^k e^{-\frac{t^{p+1}}{p+1}} dt = x^{k-p} e^{-\frac{x^{p+1}}{p+1}} + o(x^{k-p} e^{-\frac{x^{p+1}}{p+1}}),$$

(where $o(f)$ means, as usual, some function $g$ such that $g/f$ tends to 0 as $x \to \infty$). Putting this together, we get

$$J_k(x) = (p+1)^{(k+1)/(p+1)-1} \Gamma\left(\frac{k+1}{p+1}\right) - x^{k-p} e^{-\frac{x^{p+1}}{p+1}} + o(x^{k-p} e^{-\frac{x^{p+1}}{p+1}}).$$

For shortness, let us denote the constant appearing in the above expression by

$$c_k := (p+1)^{(k+1)/(p+1)-1}\Gamma\left(\frac{k+1}{p+1}\right). \tag{44}$$

Inserting these asymptotic estimates back into the expression of $y(x)$, we conclude that the necessary and sufficient condition to have a solution which does not grow exponentially for $x \to \infty$ is

$$y_0 + A_0c_0 + A_1c_1 + \cdots + A_{2p-1}c_{2p-1} = 0.$$

For each fixed value of $(A_0, \ldots, A_{2p-1})$, let us note by $y_+(x)$ the unique solution with such initial condition $y_+(0) = y_0$. From the expression of $J_k(x)$, such solution has the asymptotic expansion

$$y_+(x) \sim -(A_0x^{-p} + A_1x^{-p+1} + \cdots + A_{2p-1}x^{p-1}) \tag{45}$$

which implies that

$$X^p \cdot y(1/X) \to 0, \quad \text{for } X \to 0,$$

as we wanted.

We now determine solutions $y(x)$ which goes to zero as $x$ tends to minus infinity. First of all, if $p+1$ is an even number, it is easy to see that

$$J_k(-x) = (-1)^{k+1}J_k(x).$$

So a necessary and sufficient condition to have

$$X^p \cdot y(-1/X) \to 0, \quad \text{for } X \to 0$$

is that

$$y_0 - A_0c_0 + \cdots + (-1)^{2p}A_{2p-1}c_{2p-1} = 0.$$

Let us note by $y_-(x)$ the unique solution such that $y_-(0) = y_0$.

On the other hand, if $p+1$ is odd, we obtain

$$J_k(-x) = (-1)^{k+1}\int_0^x t^k e^{\frac{t^{p+1}}{p+1}}\,dt.$$

The principal term of the asymptotic expansion for this integral (according to [**Di**, III.10.7.1]) is

$$J_k(-x) \sim (-1)^{k+1}e^{\frac{x^{p+1}}{p+1}}x^{k-p}.$$

Inserting into the general expression of the solution $y(x)$, this gives

$$y(-x) \sim y_0 e^{-\frac{x^{p+1}}{p+1}} + A_0 x^{-p} + \cdots + A_{2p-1} x^{p-1},$$

and we conclude that

$$X^p \cdot y(-1/X) \to 0$$

for an arbitrary choice of $(y_0, A_0, \ldots, A_{2p-1})$. Thus, we have proved the following result:

**Lemma 7.1.** (1) *Given the linear equation (43), a solution $y_+(x)$ is asymptotic to zero at infinity if and only if its initial condition $y_0 := y_+(0)$ satisfies the linear equation*

$$y_0 + A_0 c_0 + A_1 c_1 + \cdots + A_{2p-1} c_{2p-1} = 0.$$

(2) *If $p+1$ is even, a solution $y_-(x)$ is asymptotic to zero at minus infinity if and only if $y_0 := y_-(0)$ satisfies*

$$y_0 - A_0 c_0 + \cdots + (-1)^{2p} A_{2p-1} c_{2p-1} = 0.$$

(3) *If $p+1$ is odd, any solution $y(x)$ is asymptotic to zero at minus infinity.*

Let us now study the more general equation

$$\frac{dy}{dx} = \sum_{i=0}^{2p-1} A_i x^i + y \left( \sum_{j=0}^{p-1} B_j x^j + \mathcal{B}_0 x^p \right), \quad \mathcal{B}_0 \neq 0, \tag{46}$$

whose general solution can be written as

$$y(x) = e^{P(x)} \left( y_0 + \int_0^x Q(t) e^{-P(t)} dt \right)$$

where $Q(x) = A_0 + A_1 x + \cdots + A_{2p-1} x^{2p-1}$ and

$$P(x) = \int_0^x \left( \sum_{j=0}^{p-1} B_j t^j + \mathcal{B}_0 t^p \right) dt = B_0 x + \cdots + \frac{B_{p-1} x^p}{p} + \frac{\mathcal{B}_0 x^{p+1}}{p+1}.$$

Proceeding as above, we must calculate the asymptotic expansion of the integral

$$J_k^P(x) := \int_0^x t^k e^{-P(x)} dt$$

when $x \to \infty$. Suppose firstly that $\mathcal{B}_0 > 0$. Then, this integral is convergent for any $k$, and we can write

$$J_k^P(x) = \int_0^\infty t^k e^{-P(t)} dt - \int_x^\infty t^k e^{-P(t)} dt.$$

The first term of this expansion,

$$C_k(B) := \int_0^\infty t^k e^{-P(t)} dt \tag{47}$$

is $C^\infty$ function of the parameters $(B_0, \dots, B_{p-1}) \in \mathbb{R}^p$, and is such that $C_k(0) = c_k/\mathcal{B}_0^{(k+1)/(p+1)}$. From ([**Di**, III.10.7.2]),

$$\begin{aligned}\int_x^\infty t^k e^{-P(t)} dt &= \frac{e^{-P(x)}}{P'(x)x^{-k}} + o\left(\frac{e^{-P(x)}}{P'(x)x^{-k}}\right)\\ &= \mathcal{B}_0 x^{k-p} e^{-P(x)} + o(x^{k-p} e^{-P(x)}),\end{aligned}$$

and so,

$$J_k^P(x) = C_k(b) + \mathcal{B}_0 x^{k-p} e^{-P(x)} + o(x^{k-p} e^{-P(x)}).$$

On the other hand, supposing that $\mathcal{B}_0 < 0$, we get ([**Di**, III.10.7.1])

$$\begin{aligned}J_k^P(x) &= \frac{e^{-P(x)}}{-P'(x)x^{-k}} + o\left(\frac{e^{-P(x)}}{-P'(x)x^{-k}}\right)\\ &= (-\mathcal{B}_0) x^{k-p} e^{-P(x)} + o(x^{k-p} e^{-P(x)}).\end{aligned}$$

Thus, as in the previous lemma, we proved the following:

**Lemma 7.2.** *Given the linear equation (46), the necessary and sufficient conditions to have a solution $y_+(x)$ such that*

$$\lim_{X \to 0} X^p \cdot y_+(1/X) = 0$$

*are the following:*

(i) *If $\mathcal{B}_0 > 0$, we shall have*

$$y_0 + C_0 A_0 + C_1 A_1 + \cdots + C_{2p-1} A_{2p-1} = 0, \tag{48}$$

*where $y_0 = y_+(0)$ and the $C_k = C_k(B)$ is the function on (47).*

(ii) *If $\mathcal{B}_0 < 0$, any solution is asymptotic to zero.*

Making the obvious modifications to study the behavior at $x = -\infty$, we obtain the following result:

**Lemma 7.3.** *Given the linear equation (46), the necessary and sufficient conditions to have a solution $y_-(x)$ such that*

$$\lim_{X\to 0} X^p \cdot y_-(-1/X) = 0$$

*are the following:*

(i) *If $\mathcal{B}_0 > 0$,*

(a) *If $p+1$ is even, we shall have*

$$y_0 + C_0A_0 + \cdots + (-1)^{k+1}C_kA_k + \cdots + C_{2p-1}A_{2p-1} = 0, \tag{49}$$

*where $y_0 = y_+(0)$.*

(b) *If $p+1$ is odd, any solution is asymptotic to zero.*

(ii) *If $\mathcal{B}_0 < 0$,*

(a) *If $p+1$ is even, any solution is asymptotic to zero.*

(b) *If $p+1$ is odd, we shall have*

$$y_0 + C_0A_0 + \cdots + (-1)^{k+1}C_kA_k + \cdots + C_{2p-1}A_{2p-1} = 0.$$

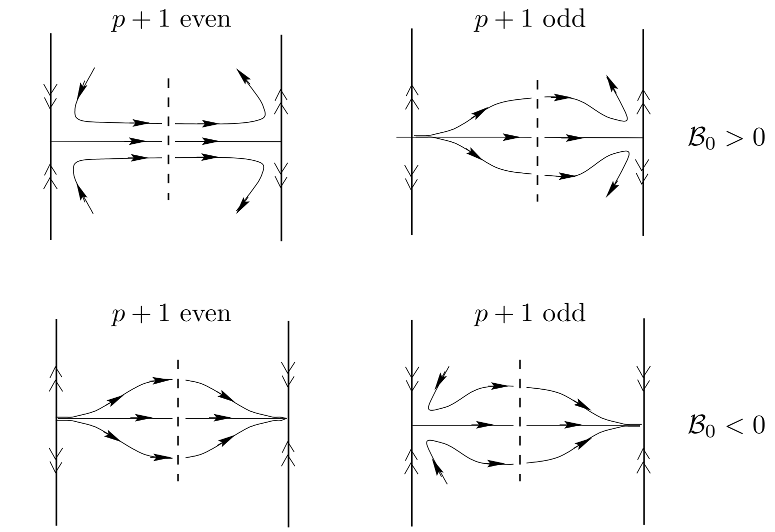



FIGURE 11. The four possible situations with their compactifications at $\pm\infty$.

*Remark* 7.4. (1) Geometrically, the solutions $y_-$ and $y_+$ can be interpreted as *traces of center manifolds $W_-$ and $W_+$ on the exceptional divisor*. Indeed, from the change of coordinates given on (38), it is immediate to see that $y_-$ (respect. $y_+$) is a local center manifold for the restricted vector field $\overline{X}|_{\mathcal{D}}$, on a neighborhood of $P_-$ (respect. $P_+$).

(2) The above results show the uniqueness of $y_-$ when $(-1)^{p+1}\mathcal{B}_0 < 0$ and the uniqueness of $y_+$ when $B_0 > 0$. This provides an alternative proof of the uniqueness part of Lemma 5.2.

(3) Also, the lemmas above distinguish four possible situations, which can be easily related to the classification of the degenerate singularity made on the previous section:
   - $\mathcal{B}_0 > 0$ and $p+1$ even corresponds to the $(\mathbf{s},\mathbf{u})$-case;
   - $\mathcal{B}_0 > 0$ and $p+1$ odd corresponds to the $(\mathbf{u},\mathbf{u})$-case;
   - $\mathcal{B}_0 < 0$ and $p+1$ even corresponds to the $(\mathbf{u},\mathbf{s})$-case;
   - $\mathcal{B}_0 < 0$ and $p+1$ odd corresponds to the $(\mathbf{s},\mathbf{s})$-case.

**The distance function on the $(\mathbf{s},\mathbf{u})$ case.** Notice that the $(\mathbf{s},\mathbf{u})$ case presents the following particular feature: For each $(A,B) \in \mathbb{R}^{3p}$ there exist unique solutions $y_+$ and $y_-$ which are asymptotic to zero at $x \to +\infty$ and $x \to -\infty$, respectively. Such solutions are determined by their initial points,

$$y_+(0) := y_+ \cap \{x=0\}, \quad \text{and} \quad y_-(0) = y_- \cap \{x=0\},$$

which must satisfy the equations (48) and (49), respectively.

Based on this remark, there exists a well-defined function on the $(\mathbf{s},\mathbf{u})$ case,

$$\delta(a,b) := y_+(0) - y_-(0),$$

which measures the distance between the points where the solutions $y_+$ and $y_-$ intersect the line $\{x=0\}$ . From (48) and (49), it is easy to see that $\delta(A,B)$ is the $C^\infty$ function on $\mathbb{R}^{3p}$ given by

$$\delta(A,B) = -2\,C_0\,A_0 - 2\,C_2\,A_2 - \cdots - 2\,C_{2p-2}\,A_{2p-2} \tag{50}$$

(the sum taken over all even terms).

### 7.2. The nonlinear case.

We study now the general Riccati family (42) for $\mathcal{Q}_0 \neq 0$. Here, we will not be able to be so explicit as in the linear case. Indeed, due to the appearance of the non-linear term $\mathcal{Q}_0 y^2$, we can have solutions which *escape to infinity at finite time.*

**Example 7.5.** For instance, the unique solution of (42) which is asymptotic to zero for $\mathcal{B}_0 = \mathcal{Q}_0 = 1$ and the parameter values

$$\begin{aligned} A_{p-1} &= 1, \\ A_i &= 0, \quad (\text{for } 0 \le i \ne p-1 \le 2p-1), \\ B_j &= 0, \quad (\text{for } 0 \le j \le p-1) \end{aligned}$$

is given by $y(x) = -1/x$. Such solution *escapes to infinity at the time* $x = 0$.

In the rest of this subsection, we prove that such kind of phenomena can be *controlled* if we restrict ourselves to a sufficiently small neighborhood of the parameters $\{A = B = 0\}$.

Let $U \in (\mathbb{R}^{3p}, 0)$ be an open neighborhood of $\{A = B = 0\}$, and let us suppose that, for each $(A, B) \in U$, one associates two solutions of the Riccati family,

$$\mathbf{y}_-^{(A,B)} \colon U_{-\infty} \to \mathbb{R}, \quad \text{and} \quad \mathbf{y}_+^{(A,B)} \colon U_\infty \to \mathbb{R}, \tag{51}$$

such that:

- $U_{-\infty}, U_\infty \subset \mathbb{R}$ are neighborhoods of $-\infty$ and $+\infty$, respectively;
- $\mathbf{y}_+^{(A,B)}(x)$ and $\mathbf{y}_-^{(A,B)}(x)$ depend continuously on $(A, B) \in U$;
- these solutions have the asymptotic behavior

$$\lim_{X\to 0} X^p \cdot \mathbf{y}_+^{(A,B)}\left(\frac{1}{X}\right) = \lim_{X\to 0} X^p \cdot \mathbf{y}_-^{(A,B)}\left(\frac{-1}{X}\right) = 0 \tag{52}$$

  for each $(A, B) \in U$.

Our goal is to control the escape time of such solutions.

**Lemma 7.6.** *Let us suppose that $\mathcal{B}_- < 0$ (respect. $\mathcal{B}_+ > 0$). Then, given any arbitrarily large constant $K > 0$, there exists a smaller neighborhood $N = N(K) \subset U$ of $\{A = B = 0\}$ such that for each $(A, B) \in N$, the solution $\mathbf{y}_-^{(A,B)}$ (respect. $\mathbf{y}_+^{(A,B)}$) can be extended to the interval $x \in (-\infty, K]$ (respect. $x \in [-K, \infty)$).*

*Proof:* If we suppose that $\mathcal{B}_- < 0$, the solution $\mathbf{y}_-^{(A,B)}$ is necessarily *unique* for each $(A, B) \in U$. Indeed, as we have already mentioned, solutions of $R_{A,B}$ which have the asymptotic behavior indicated on (52) can be seen as intersections of local a center manifold $W_-$ at $P_-$ with the exceptional divisor. Thus,

$$\mathbf{y}_-^{(A,B)} = W_- \cap \mathcal{D},$$

which implies, from Lemma 5.2, that $\mathbf{y}_-^{(A,B)}$ is unique.

For $(A,B) = (0,0)$, $y(x) \equiv 0$ is a solution of $R_{A,B}$. So, by the uniqueness

$$\mathbf{y}_-^{(0,0)}(x) \equiv 0.$$

Since this solution is defined for all $x \in \mathbb{R}$, the result follows by the continuous dependence of the center manifold on the parameters $A$, $B$. The argument for $\mathbf{y}_+^{(A,B)}$ is analogous. □

In the cases where it is not possible use this uniqueness argument, we have to be more careful in studying the *escape time* of a solution of the Riccati differential equation $R_{0,0}$.

**The escape time.** If we let $A = B = 0$, the Riccati family takes the form

$$\frac{dy}{dx} = \mathcal{B}_0 x^p\, y + \mathcal{Q}_0\, y^2. \tag{53}$$

Fix an arbitrary point $x_0 \in \mathbb{R}$. Then, for the initial condition $y(x_0) = 0$ this equation has the trivial solution $y(x) \equiv 0$. For $y(x_0) = y_0 \neq 0$, we can obtain the explicit solution

$$y(x) = \frac{e^{\mathcal{B}_0 \frac{x^{p+1}}{p+1}}}{\dfrac{1}{y_0} - \mathcal{Q}_0 \displaystyle\int_{x_0}^{x} e^{\mathcal{B}_0 \frac{x^{p+1}}{p+1}}\, dt}.$$

Thus, it is easy to see that such solution $y(x)$ escapes to infinity at a *finite time* $x_1 \in \mathbb{R}$ if and only if

$$\mathcal{Q}_0 \int_{x_0}^{x_1} e^{\mathcal{B}_0 \frac{t^{p+1}}{p+1}}\, dt = \frac{1}{y_0}. \tag{54}$$

Such point $x_1$ is a function of $x_0$, $y_0$, $\mathcal{B}_0$ and $\mathcal{Q}_0$.

In the next two lemmas, we consider the asymptotic behavior of the analytic function

$$L(x, x_0) := \mathcal{Q}_0 \int_{x_0}^{x} e^{\mathcal{B}_0 \frac{t^{p+1}}{p+1}}\, dt.$$

**Lemma 7.7.** *Suppose that $p+1$ is even and $\mathcal{B}_0 < 0$. Then,*

(i) *If $x_0 \leq 0$,*

$$-\lambda\, e^{\mathcal{B}_0 \frac{|x_0|^{p+1}}{p+1}} < L(x, x_0) < 2\,\lambda, \qquad \textit{if } \mathcal{Q}_0 > 0$$
$$\lambda\, e^{\mathcal{B}_0 \frac{|x_0|^{p+1}}{p+1}} > L(x, x_0) > -2\,\lambda, \quad \textit{if } \mathcal{Q}_0 < 0.$$

(ii) *If* $x_0 \geq 0$,

$$-2\,\lambda < L(x,x_0) < \lambda\, e^{\mathcal{B}_0 \frac{|x_0|^{p+1}}{p+1}}, \qquad \textit{if } \mathcal{Q}_0 > 0$$
$$2\,\lambda > L(x,x_0) > -\lambda\, e^{\mathcal{B}_0 \frac{|x_0|^{p+1}}{p+1}}, \quad \textit{if } \mathcal{Q}_0 < 0,$$

*where* $\lambda := c_0|\mathcal{Q}_0| \cdot |\mathcal{B}_0|^{-\frac{1}{p+1}}$, *for the constant* $c_0$ *defined in (44).*

*Proof:* Let us suppose that $x_0 \geq 0$ and that $\mathcal{Q}_0 > 0$. Then, for all $-\infty < x < x_0$, we have

$$\mathcal{Q}_0 \int_{x_0}^{x} e^{\mathcal{B}_0 \frac{t^{p+1}}{p+1}}\, dt \geq -\mathcal{Q}_0 \int_{-\infty}^{\infty} e^{\mathcal{B}_0 \frac{t^{p+1}}{p+1}}\, dt = -2c_0|\mathcal{Q}_0| \cdot |\mathcal{B}_0|^{-\frac{1}{p+1}},$$

(by the definition of $c_0$ and the fact that $p+1$ is even).

On the other hand, if $x > x_0$, we make the change of variable $u = t - x_0$, and get

$$L(x,x_0) = \mathcal{Q}_0 \int_0^{x-x_0} e^{\mathcal{B}_0 \frac{(u+x_0)^{p+1}}{p+1}}\, du.$$

Now, since $u \geq 0$ and $x_0 \geq 0$,

$$(u+x_0)^{p+1} \geq u^{p+1} + x_0^{p+1},$$

and, using the fact that $\mathcal{B}_0 < 0$,

$$\mathcal{Q}_0 \int_0^{x-x_0} e^{\mathcal{B}_0 \frac{(u+x_0)^{p+1}}{p+1}}\, du \leq \mathcal{Q}_0\, e^{\mathcal{B}_0 \frac{x_0^{p+1}}{p+1}} \int_0^{\infty} e^{\mathcal{B}_0 \frac{u^{p+1}}{p+1}}\, du = \lambda\, e^{\mathcal{B}_0 \frac{x_0^{p+1}}{p+1}}.$$

This proves (ii). The proof of (i) is completely analogous. ☐

In a similar way, we can prove the following estimates:

**Lemma 7.8.** *Suppose that* $p+1$ *is odd. Then,*

(i) *If* $[\mathcal{Q}_0 > 0,\, \mathcal{B}_0 > 0$ *and* $x_0 \leq 0]$ *or* $[\mathcal{Q}_0 < 0,\, \mathcal{B}_0 < 0$ *and* $x_0 \geq 0]$,

$$L(x,x_0) \geq -\lambda\, e^{\mathcal{B}_0 \frac{x_0^{p+1}}{p+1}}.$$

(ii) *If* $[\mathcal{Q}_0 < 0,\, \mathcal{B}_0 > 0$ *and* $x_0 \leq 0]$ *or* $[\mathcal{Q}_0 > 0,\, \mathcal{B}_0 < 0$ *and* $x_0 \geq 0]$,

$$L(x,x_0) \leq \lambda\, e^{\mathcal{B}_0 \frac{x_0^{p+1}}{p+1}},$$

*where* $\lambda$ *is the constant defined in the previous lemma.*

*Proof:* It is completely analogous to the proof of Lemma 7.7. ☐

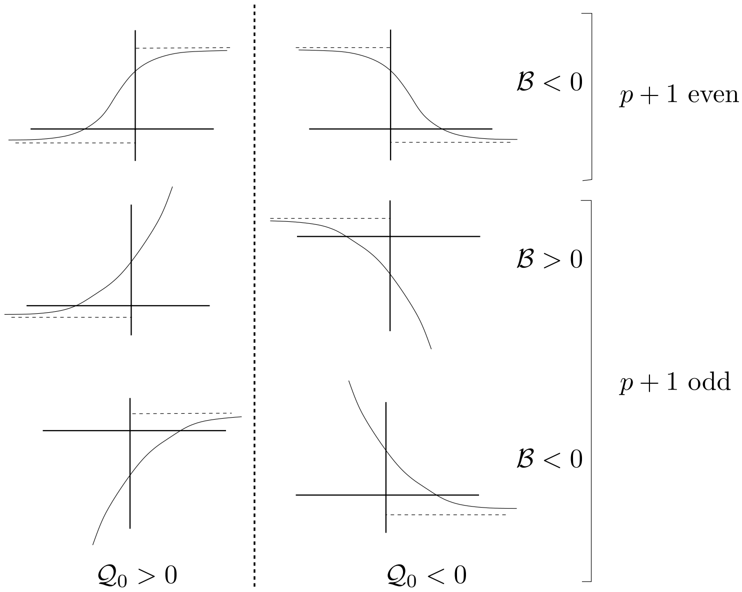



FIGURE 12. The six possibilities where we can bound $L(x)$.

As a consequence of these estimates, we have the following result:

**Corollary 7.9.** *Let $y(x)$ be a solution of (53). Suppose that there exists a point $x_0 \in \mathbb{R}$ in the interval of definition of $y(x)$ such that the data $[\mathcal{Q}_0, \mathcal{B}_0, x_0, y_0]$ satisfies one of the following conditions:*

(i) *If $p+1$ is even, $\mathcal{B}_0 < 0$, and*
   (a) $[\mathcal{Q}_0 > 0,\, x_0 \leq 0]$ *or* $[\mathcal{Q}_0 < 0,\, x_0 \geq 0]$, *and*
$$\frac{1}{2\lambda} > y(x_0) > -\frac{1}{\lambda}\; e^{|\mathcal{B}_0|\frac{|x_0|^{p+1}}{p+1}},$$
   (b) $[\mathcal{Q}_0 < 0,\, x_0 \leq 0]$ *or* $[\mathcal{Q}_0 > 0,\, x_0 \geq 0]$, *and*
$$-\frac{1}{2\lambda} < y(x_0) < \frac{1}{\lambda}\; e^{|\mathcal{B}_0|\frac{x_0^{p+1}}{p+1}}.$$
(ii) *If $p+1$ is odd, and*
   (a) $[\mathcal{Q}_0 > 0,\, \mathcal{B}_0 > 0,\, x_0 \leq 0]$ *or* $[\mathcal{Q}_0 < 0,\, \mathcal{B}_0 < 0,\, x_0 \geq 0]$, *and*
$$0 \geq y(x_0) > -\frac{1}{\lambda} e^{|\mathcal{B}_0|\frac{|x_0|^{p+1}}{p+1}},$$

(b) $[\mathcal{Q}_0 < 0,\, \mathcal{B}_0 > 0,\, x_0 \leq 0]$ *or* $[\mathcal{Q}_0 > 0,\, \mathcal{B}_0 < 0,\, x_0 \geq 0]$, *and*

$$0 \leq y(x_0) < \frac{1}{\lambda} e^{|\mathcal{B}_0| \frac{|x_0|^{p+1}}{p+1}}.$$

*Then, the solution $y(x)$ can not escape to infinity at finite time.*

*Proof:* Indeed, in these three cases, the above two lemmas show that there can be no $x_1 \in \mathbb{R}$ which solves the equation (54). □

Finally, we can extend the result of Lemma 7.6 as follows:

**Lemma 7.10.** *Let $\mathbf{y}_-^{(A,B)}$ and $\mathbf{y}_+^{(A,B)}$ be as in (51). Suppose that there exists a point $x_0 \in U_{-\infty}$ (respect. $x_0 \in U_\infty$) such that the value of $\mathbf{y}_-^{(A,B)}$ (respect. $\mathbf{y}_+^{(A,B)}$) for $x = x_0$ and $A = B = 0$,*

$$y(x_0) := \mathbf{y}_-^{(0,0)}(x_0) \quad (\textit{respect. } y(x_0) := \mathbf{y}_+^{(0,0)}(x_0))$$

*verifies one of the conditions of Corollary 7.9. Then, for each arbitrarily large constant $K > 0$, there exists some smaller neighborhood $N = N(K) \subset U$ of $\{A = B = 0\}$ such that, for each $(A,B) \in N$, the solution $\mathbf{y}_-^{(A,B)}(x)$ (respect. $\mathbf{y}_+^{(A,B)}(x)$) can be extended to a solution of $R_{A,B}$ which is defined for all $x \in (-\infty, K]$ (respect. $x \in [-K, \infty)$).*

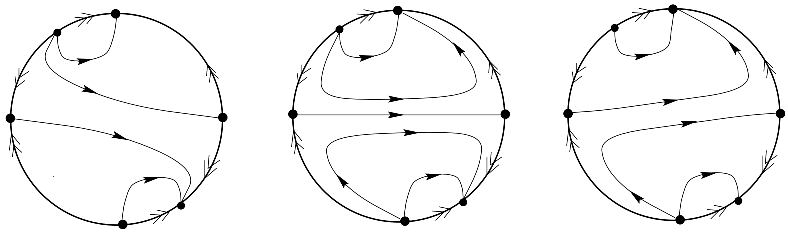
*Proof:* This is an immediate consequence of the above results. Indeed, if we suppose that no such neighborhood $N$ exists, the continuity of $y_\pm^{(A,B)}(x)$ with respect to $(A,B)$ would imply that $y_\pm^{(0,0)}$ escapes to infinity at finite time. This would contradict Corollary 7.9. □

**The distance function in the $(\mathbf{s},\mathbf{u})$ case.** By variation of parameters, the general solution of (42) can be (implicitly) written as

$$y(x) = e^{P(x)} \left( y_0 + \int_0^x (Q(t) + \mathcal{Q}_0 y(t)^2) e^{-P(t)} dt \right), \tag{55}$$

where $Q(x) = A_0 + A_1 x + \cdots + A_{2p-1} x^{2p-1}$ and

$$P(x) = B_0 x + \cdots + \frac{B_{p-1} x^p}{p} + \frac{\mathcal{B}_0 x^{p+1}}{p+1}.$$



$$\frac{dy}{dx} = a_0 + a_1 x + a_2 x^2 + x^3 y + y^2$$

FIGURE 13. Possible phase portraits as $a_0$ varies, with $a_1 = a_2 = 0$ keep fixed.

Let us now study the $(\mathbf{s},\mathbf{u})$ case (i.e. $\mathcal{B}_- < 0$ and $\mathcal{B}_+ > 0$). If we let $K = 0$ on Lemma 7.6 we know that there exists a neighborhood $N \subset \mathbb{R}^{3p}$ of $\{A = B = 0\}$ such that, for each $(A, B) \in N$, there exist unique solutions

$$y_-^{(A,B)} \quad \text{and} \quad y_+^{(A,B)}$$

which have the asymptotic behavior (52), and which are defined for all $x \in (-\infty, 0]$ and $x \in [0, \infty)$, respectively. Thus, we can consider the *distance function*

$$\begin{array}{rccc} \delta\colon & N & \longrightarrow & \mathbb{R} \\ & (A,B) & \longmapsto & y_+^{(A,B)}(0) - y_-^{(A,B)}(0) \end{array} \tag{56}$$

as we have done in the study of the linear case.

Let us study the behavior of such function near $\{A = B = 0\}$ by considering the quantities

$$I_+(A,B) := \int_0^{\infty} e^{-P(x)} y_+^{(A,B)}(x)^2 dx \quad \text{and}$$
$$I_-(A,B) := \int_0^{-\infty} e^{-P(x)} y_-^{(A,B)}(x)^2 dx.$$

From Lemma 5.2, we know that the curves $y_+^{(A,B)}$ and $y_-^{(A,B)}$ depend $C^\infty$-ly on $(A, B)$, and so, $I_+$ and $I_-$ belong to $C^\infty(N)$.

If $(0,B)$ is a point in $M=\{A=0\}$, then obviously $y_+^{(0,B)}(x) = y_-^{(0,B)}(x) \equiv 0$ and $I_+(0,B) = I_-(0,B) = 0$. Since $I_+$, $I_- \geq 0$ are positive functions, this implies that

$$\nabla I_+(0,B) = \left(\frac{\partial I_+}{\partial A_0}, \dots, \frac{\partial I_+}{\partial A_{2p-1}}, \frac{\partial I_+}{\partial B_0}, \dots, \frac{\partial I_+}{\partial B_{p-1}}\right)_{(0,B)} = (0, \dots, 0),$$

(and similarly $\nabla I_-(0,B) = 0$).

Choose now a parameter $(A,B) \in N$. Then, according to the integral equation (55), since $y_+^{(A,B)}(x)$ exists, we necessarily have (making the compactification $x = 1/X$, $y = Y/X^p$ and letting $X \to 0$),

$$0 = y_+^{(A,B)}(0) + C_0 A_0 + \cdots + C_{2p-1}A_{2p-1} + \mathcal{Q}_0 \cdot I_+(A,B), \tag{57}$$

where $C_i = C_i(B)$ are the functions defined on (47). In a similar way,

$$\begin{aligned} 0 = y_-^{(A,B)}(0) - C_0 a_0 + \cdots + (-1)^{k+1} C_k a_k + \cdots + C_{2p-1}a_{2p-1} \\ + \mathcal{Q}_0 \cdot I_-(A,B). \end{aligned} \tag{58}$$

Subtracting (58) from (57) we conclude that the separation function is given by

$$\delta(A,B) = -2C_0A_0 - \cdots - 2C_{2p-2}A_{2p-2} + \mathcal{Q}_0 \cdot [-I_+(A,B) + I_-(A,B)],$$

and this proves the following result.

**Lemma 7.11.** *On the* $(\mathbf{s},\mathbf{u})$ *case, the distance function* $\delta$ *is defined on a sufficiently small neighborhood* $N \subset \mathbb{R}^{3p}$ *of* $\{A = B = 0\}$. *Moreover,*

$$\frac{\partial \delta}{\partial A_s} = -2C_s + O(A) \quad \text{on } N,$$

*for each even index* $s = 0, 2, \dots, 2p-2$.

*Proof:* This is trivial, since $\nabla I_+|_{\{A=0\}} = \nabla I_-|_{\{A=0\}} = (0, \dots, 0)$. $\square$

## 8. Estimating the matching regions

Let us now use the asymptotic analysis of the Riccati family in order to describe the regions $\mathcal{O}^+$, $\mathcal{O}^-$ and $\mathcal{O}(W_-, W_+)$ which have been defined on Section 6.

**8.1. The $(\mathrm{s},\mathrm{s})$, $(\mathrm{u},\mathrm{s})$ and $(\mathrm{u},\mathrm{u})$-cases: Open matching regions.**

**Lemma 8.1.** *Assume that $\mathcal{B}_+ < 0$, and let $W_-$ be local center manifold at $P_-$, defined on a domain $N_x \times N_\mathrm{e} \times V_\mathcal{A}$. Suppose that $\mathcal{A} = 0$ belongs to $V_\mathcal{A}$ and that*

$$y_-^{(0,0)} := W_- \cap \mathcal{D}|_{\{\mathcal{A}=0\}}$$

*is a solution of the Riccati family $R_{A,B}$ which does not escape to infinity at finite time. Then, there exists a matching center manifold $W_+$, defined on a domain $N'_x \times N'_\mathrm{e} \times V'_\mathcal{A}$ according to Proposition 6.5, such that the matching region $O(W_-, W_+)$ contains an open neighborhood $U$ of the origin $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$ in $\mathbb{R}^+ \times \mathbb{R}^n$.*

*Proof:* Using the notation of the proof of Proposition 6.5, it suffices to guarantee that $W_- \cap \mathcal{D}$ cuts the transversal section $\Sigma_{\mathrm{e}_+}$.

From Lemmas 7.6 and 7.10 this is clearly true if $\mathcal{A} = (A, B, \mathcal{A}_r)$ is taken in a sufficiently small neighborhood of $\{A = B = \mathcal{A}_r = 0\}$. □

Of course, we have the similar matching result in the case $\mathcal{B}_- < 0$.

**Lemma 8.2.** *Assume that $\mathcal{B}_- > 0$, and let $W_+$ be local center manifold at $P_+$, defined on a domain $N_x \times N_\mathrm{e} \times V_\mathcal{A}$. Suppose that $\mathcal{A} = 0$ belongs to $V_\mathcal{A}$ and that*

$$y_+^{(0,0)} := W_+ \cap \mathcal{D}|_{\{\mathcal{A}=0\}}$$

*is a solution of the Riccati family $R_{A,B}$ which does not escape to infinity at finite time. Then, there exists a matching center manifold $W_-$, defined on a domain $N'_x \times N'_\mathrm{e} \times V'_\mathcal{A}$ according to Proposition 6.6, such that the matching region $O(W_-, W_+)$ contains an open neighborhood $U$ of the origin $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$ in $\mathbb{R}^+ \times \mathbb{R}^n$.*

*Remark* 8.3. On the special case where $R_\mathcal{A}$ is a *linear* differential equation (i.e. if $\mathcal{Q}_0 = 0$), no solution of $R_\mathcal{A}$ escapes to infinity at finite time. This implies that we do not need to restrict ourselves to a neighborhood of $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$. In fact, from Subsection 7.1, it is easy to prove the following: *On the hypothesis of Lemma 8.1 (respect. Lemma 8.2), suppose that $\mathcal{Q}_0 = 0$. Then, for an arbitrary open set $V'_{A,B} \subset \mathbb{R}^{3p}$ whose closure is contained in $V_\mathcal{A} \cap \{\mathcal{A}_r = 0\}$, there exist sufficiently small domains*

$$N'_x, N'_\mathrm{e} \subset \mathbb{R}^+, \quad \textit{and} \quad V'_{\mathcal{A}_r} \in (\mathbb{R}^{n-3p}, 0),$$

*such that a matching center manifold $W_+$ (respect. $W_-$) can be defined on the domain $N'_x \times N'_{\mathrm{e}} \times (V'_{A,B} \times V'_{\mathcal{A}_r})$. In particular, the matching region has the form*

$$\mathcal{O}(W_-, W_+) = N_{\tilde{\varepsilon}} \times (V'_{A,B} \times V'_{\mathcal{A}_r})$$

*for a sufficiently small domain $N_{\tilde{\varepsilon}} = [0, \tilde{\varepsilon}_0) \subset \mathbb{R}^+$.*

Thus, combining Propositions 6.5 and 6.6 with the above results, it is immediate to conclude the following

- In the $(\mathbf{u}, \mathbf{s})$ or $(\mathbf{s}, \mathbf{s})$ cases: given any center manifold $W_-$ at $P_-$, which verifies the hypothesis of Lemma 8.1, one can uniquely define a center manifold $W_+$ at $P_+$ such that the matching region $\mathcal{O}(W_-, W_+)$ is an open neighborhood of $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$.
- In the $(\mathbf{u}, \mathbf{u})$ or $(\mathbf{u}, \mathbf{s})$ cases: given any center manifold $W_+$ at $P_+$, which verifies the hypothesis of Lemma 8.2, one can uniquely define a center manifold $W_-$ at $P_-$ such that the matching region $\mathcal{O}(W_-, W_+)$ is an open neighborhood of $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$.

For shortness, we shall use the notation

$$W_- \to W_+$$

to indicate that the center manifold $W_-$ at $P_-$ *induces a matching center manifold* $W_+$ *at* $P_+$. Correspondingly, we shall denote by $W_+ \to W_-$ the matching of a center manifold $W_-$ at $P_-$ to a given center manifold $W_+$ at $P_+$. Thus, we can shortly summarize the above results by the following schema:

- On the $(\mathbf{s}, \mathbf{s})$ case: $W_- \to W_+$;
- On the $(\mathbf{u}, \mathbf{u})$ case: $W_+ \to W_-$;
- On the $(\mathbf{s}, \mathbf{u})$ case: $W_- \to W_+$ and $W_+ \to W_-$.

### 8.2. The $(\mathbf{s}, \mathbf{u})$-case: Matching region as a graph.

In contrast to the previous cases, in the $(\mathbf{s}, \mathbf{u})$-case the matching region $\mathcal{O}(W_-, W_+)$ is a *thin set* (i.e. it has empty interior). Indeed, let us see that, in a sufficiently small neighborhood of $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$ it is the graph of a $C^\infty$ function.

First of all, we prove that the distance function $\Delta$ can be defined in a sufficiently small neighborhood of $\{\tilde{\varepsilon} = \mathcal{A} = 0\}$.

**Lemma 8.4.** *Suppose that we are on the* $(\mathbf{s},\mathbf{u})$*-case. Let* $W_-$ *and* $W_+$ *be local center manifolds at* $P_-$ *and* $P_+$*, respectively, which are defined on a common open domain*

$$N_x \times N_{\mathrm{e}} \times V_{\mathcal{A}}.$$

*Suppose that* $\mathcal{A}=0$ *belongs to* $V_{\mathcal{A}}$*. Then, the common domain of maximal extension of these center manifolds*

$$\mathcal{O}^- \cap \mathcal{O}^+ \subset N_{\tilde{\varepsilon}} \times V_{\mathcal{A}}$$

*(where* $N_{\tilde{\varepsilon}} := \psi_{\mathrm{e}_0}(N_x)$ *is defined as in (39)) contains an open neighborhood* $U$ *of the origin* $\{\tilde{\varepsilon}=\mathcal{A}=0\}$*.*

*Proof:* It follows directly from Proposition 7.6 if we let $K=0$. Indeed, for $(A,B,\mathcal{A}_r)$ in some open subset $N\subset V_{\mathcal{A}}$, suppose that $y_-=W_-\cap\mathcal{D}$ and $y_+=W_+\cap\mathcal{D}$ are solutions of the Riccati family $R_{A,B}$ which cut the transversal section $\Sigma=\{x=0\}$. Then, the center manifolds $W_-$ and $W_+$ also cut transversely the section $\Sigma$, provided that we restrict $\tilde{\varepsilon}$ to some sufficiently small neighborhood of zero. □

Thus, the distance function $\Delta\colon \mathcal{O}^-\cap\mathcal{O}^+\to\mathbb{R}$ is defined on such neighborhood $U$ of $\{\tilde{\varepsilon}=\mathcal{A}=0\}$, and we can prove the following result:

**Proposition 8.5.** *Let us keep the notation of the previous lemma and fix an arbitrary even index* $0\le s\le 2p-2$*. Then, there exists some smaller neighborhood of the origin* $U_s\subset U$ *and a* $C^\infty$ *function*

$$\mathbf{a}_s\colon \widehat{U}_s\to\mathbb{R},\quad \mathbf{a}_s(0)=0$$

*defined on* $\widehat{U}_s := U_s\cap\{A_s=0\}$*, such that the restriction of the matching region* $\mathcal{O}(W_-,W_+)$ *to* $U_s$ *is given by the graph of* $\mathbf{a}_s$*,*

$$\mathcal{O}(W_-,W_+)\cap U_s = \mathrm{graph}\{A_s=\mathbf{a}_s(\tilde{\varepsilon},\widehat{A},B,\mathcal{A}_r)\}$$

*where* $\widehat{A} := (A_0,\dots,A_{s-1},A_{s+1},\dots,A_{2p-1})$*.*

*Proof:* Recall that the matching region is defined by

$$\mathcal{O}(W_-,W_+) := \{(\tilde{\varepsilon},\mathcal{A})\in\mathcal{O}^-\cap\mathcal{O}^+ \mid \Delta(\tilde{\varepsilon},\mathcal{A})=0\}.$$

If we restrict to $\mathcal{D}_0=\{\tilde{\varepsilon}=\mathcal{A}_r=0\}$, the separation function $\Delta$ is given by

$$\Delta(0,\mathcal{A})=\delta(A,B)$$

where $\delta(A,B)$ the function defined on (56). Thus, result follows directly from Lemma 7.11 and the Implicit Function Theorem. □

## 9. Blowing-up and blowing-down center manifolds

In this section, we will consider the *blowing-down* of the matching regions. These regions will define the domain of existence of local canard surfaces near the degenerate singularity.

Let us consider again the notation of Section 4. Thus, we let $(x, y, \varepsilon, \mathcal{A})$ be coordinates on an open subset $U = U_x \times U_y \times U_\varepsilon \times U_\mathcal{A}$, where $U_x \in (\mathbb{R}, 0)$ is an open connected set, $U_y = \mathbb{R}$, $U_\varepsilon \in (\mathbb{R}^+, 0)$ and $U_\mathcal{A} \in (\mathbb{R}^n, 0)$. Let

$$X = \varepsilon \frac{\partial}{\partial x} + F(x, y, \varepsilon, \mathcal{A}) \frac{\partial}{\partial y}$$

be a singular perturbation family on $U$, such that $\Gamma = \{y = \varepsilon = \mathcal{A} = 0\} \approx U_x$ is a curve of singularities. We suppose that $x = 0 \in \Gamma$ is an isolated degenerate singularity with multiplicity $\mu(X) = p$.

Further, we suppose that Transversality Hypothesis holds at $x = 0$, and that $\mathcal{A} = (a, b, \mathcal{A}_r)$ are the adapted parameters. Thus, we can consider the blowing-up map

$$\Phi \colon \overline{U} \to U$$

which is defined on Subsection 4.1.

Let us denote by $(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, A, B, \mathcal{A}_r) \in U_{\bar{\varepsilon}}$ the coordinates of the $\bar{\varepsilon}$-chart which is associated to the blowing-up. Recall that, in these coordinates, the map $\Phi$ is defined by

$$\widehat{\Phi}_{\bar{\varepsilon}} = \begin{cases} x = \tilde{\varepsilon}\, \tilde{x} \\ y = \tilde{\varepsilon}^{p+1}\, \tilde{y} \\ \varepsilon = \tilde{\varepsilon}^{p+1} \\ a_i = \tilde{\varepsilon}^{2p-i}\, A_i, & \text{for } 0 \leq i \leq 2p-1, \\ b_j = \tilde{\varepsilon}^{p-j}\, B_j, & \text{for } 0 \leq j \leq p-1. \end{cases} \tag{59}$$

Thus, restricted to the variables $(\varepsilon, a, b, \mathcal{A}_r)$, we have the polynomial map

$$\varphi \colon (\tilde{\varepsilon}, A_i, B_j, \mathcal{A}_r) \mapsto (\varepsilon, a_i, b_j, \mathcal{A}_r) = (\tilde{\varepsilon}^{p+1}, \tilde{\varepsilon}^{2p-i} A_i, \tilde{\varepsilon}^{p-j} B_j, \mathcal{A}_r), \tag{60}$$

where, $0 \leq i \leq 2p-1$ and $0 \leq j \leq p-1$.

Similarly, if we denote by $(\hat{x}, \hat{y}, \mathrm{e}, A, B, \mathcal{A}_r) = (\hat{x}, \hat{y}, \mathrm{e}, \widehat{\mathcal{A}}) \in U_{-x,\varepsilon}$ (respect. $U_{x,\varepsilon}$) the coordinates of the $(-x_\varepsilon)$-chart (respect. $x_\varepsilon$-chart), the blow-up map $\Phi$ is given in these coordinates by

$$\widehat{\Phi}_{\pm x_\varepsilon} = \begin{cases} x = \pm\hat{x} \\ y = \hat{x}^{p+1}\,\hat{y} \\ \varepsilon = \hat{x}^{p+1}\,\mathrm{e}^{p+1} \\ a_i = \hat{x}^{2p-i}\,\mathrm{e}^{2p-i} A_i, \\ b_j = \hat{x}^{p-j}\,\mathrm{e}^{p-j} B_j, \end{cases} \tag{61}$$

and, for each fixed $\hat{x} = \hat{x}_0 \geq 0$, the blowing-up restricted to the variables $(\varepsilon, a, b, \mathcal{A}_r)$ is the polynomial map

$$\phi_{\hat{x}_0} \colon (\mathrm{e}, A_i, B_j, \mathcal{A}_r) \mapsto \begin{cases} \varepsilon = \hat{x}_0^{p+1}\,\mathrm{e}^{p+1}, \\ a_i = \hat{x}_0^{2p-i}\,\mathrm{e}^{2p-i}\,A_i, \\ b_j = \hat{x}_0^{p-j}\,\mathrm{e}^{p-j}\,B_j, \\ \mathcal{A}_r = \mathcal{A}_r \end{cases}$$

(notice that $\phi_{\hat{x}_0}$ is a diffeomorphism outside the set $\{\mathrm{e} = 0\}$).

Recall from Section 3 that, given any non-degenerate point $x_0 \in \Gamma$ $(x_0 \neq 0)$ and any open subset $V \subset U_{\varepsilon,\mathcal{A}} \cap \{\varepsilon > 0\}$, we can consider an initial condition function

$$i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_0})$$

for $X$ at $x_0$. From Proposition 3.4 it follows that, given any point $x_1 \neq x_0$ on $\Gamma$, such that

- If $B_{x_0} > 0$, $x_1 < x_0$,
- If $B_{x_0} < 0$, $x_1 > x_0$, and
- $0 \notin \Gamma_{x_0,x_1}$.

There exists a unique dynamical center manifold

$$W(\Gamma_{x_0,x_1}) = \text{graph}\{y = w(x, \varepsilon, \mathcal{A})\}$$

defined over $\Gamma_{x_0,x_1} \times \mathcal{O}_{\Gamma_{x_0,x_1}}$ (where $\mathcal{O}_{\Gamma_{x_0,x_1}} \subset V$), such that $w(x_0, \varepsilon, \mathcal{A}) = i(\varepsilon, \mathcal{A})$.

Let us prove that, under appropriate hypothesis, this center manifold can be extended by blowing-up.

**Lemma 9.1.** *Suppose that* $\mathcal{B}_- := (-1)^p \mathcal{B}_0 < 0$ *and that* $x_0 < 0$. *Define* $\hat{x}_0 := -x_0$, *and suppose that there exist open subsets* $V_{\mathrm{e}} = (0, \mathrm{e}_0)$ *and* $V_{\widehat{\mathcal{A}}} = V_{A,B} \times V_{\mathcal{A}_r}$, *where* $\mathrm{e} > 0$, $V_{A,B} \subset \mathbb{R}^{3p}$ *has a compact closure and* $V_{\mathcal{A}_r} \subset (\mathbb{R}^{n-3p}, 0)$, *such that*

$$V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}} \subset \phi_{\hat{x}_0}^{-1}(V),$$

*and* $\hat{x}_0 \in V_x^-$ *(where* $V_x^-$ *is the set defined on Lemma 5.3). Then, the map*

$$I(\mathrm{e}, \widehat{\mathcal{A}}) := \left(\frac{1}{\hat{x}_0^p}\right) i \circ \phi_{\hat{x}_0}(\mathrm{e}, \widehat{\mathcal{A}})$$

*is an initial condition function for* $\overline{X}$ *at* $P_-$, *on the domain* $V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}$.

*Proof:* It suffices to verify that that

$$I \in C^\infty_{\text{flat}}(V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}, \{\mathrm{e} = 0\}, \widehat{W}_{-,\hat{x}_0})$$

where, we recall, $\widehat{W}_{-,\hat{x}_0}$ denotes the formal expansion of the center manifold of the blowed-up vector field $\overline{X}$ over $P_-$.

Notice that, by the uniqueness of these formal series expansions (see Lemmas 3.2 and 5.3), it is easy to see that $\widehat{W}_{-,\hat{x}_0}$ is precisely given by the *blowing-up* of the formal series $\widehat{W}_{x_0}$, i.e.

$$\widehat{W}_{-,\hat{x}_0}(\mathrm{e}, A, B) = \left(\frac{1}{\hat{x}_0^p}\right) \widehat{W}_{x_0} \circ \phi_{\hat{x}_0}(\mathrm{e}, A, B).$$

Thus, the result follows easily from Proposition 2.9. □

*Remark* 9.2. According to Proposition 5.6, this initial condition function $I(\mathrm{e}, \widehat{\mathcal{A}})$ uniquely defines a center manifold $W_-$ at $P_-$ (on some appropriate sub-domain). We shall say that such $W_-$ is an *extension* of the center manifold $W(\Gamma_{x_0,x_1})$.

Reciprocally, let $W_-$ be a center manifold at $P_-$, defined by the graph of a $C^\infty$ function

$$\hat{y} = w_-(\hat{x}, \mathrm{e}, \widehat{\mathcal{A}}),$$

with domain $N_{\hat{x}} \times N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}$. If we fix $\hat{x}_0 > 0 \in N_{\hat{x}}$, and define

$$I(\mathrm{e}, \widehat{\mathcal{A}}) = w_-(\hat{x}_0, \mathrm{e}, \widehat{\mathcal{A}}),$$

we have the converse to the above lemma:

**Lemma 9.3.** *On the above notation, the map*

$$i(\varepsilon, a, b) := (\hat{x}_0^p)\, I \circ \phi_{\hat{x}_0}^{-1}(\varepsilon, a, b)$$

*is an initial condition function for* $X$ *at* $x_0 = -\hat{x}_0$, *with domain of definition* $V = \phi_{\hat{x}_0}(N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}})$.

*Proof:* The argument is exactly the same which we have used on the proof of the previous lemma. □

*Remark* 9.4. Notice that, contrary to Lemma 9.1, the above lemma makes no hypothesis on the sign of $\mathcal{B}_-$.

Of course, we have analogous results for $\mathcal{B}_+$. Namely,

**Lemma 9.5.** (1) *Suppose that $\mathcal{B}_+ > 0$ and that $x_0 > 0$. Then, the map*

$$I(\mathrm{e}, \widehat{\mathcal{A}}) := \left(\frac{1}{x_0^p}\right) i \circ \phi_{x_0}(\mathrm{e}, \widehat{\mathcal{A}})$$

*is an initial condition function for $\overline{X}$ at $P_+$. Its domain of definition can be chosen to be any open subset $V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}} \subset \phi_{x_0}^{-1}(V)$ with compact closure.*

(2) *Conversely, let $W_+ = \mathrm{graph}\{\hat{y} = w_+(\hat{x}, \mathrm{e}, \widehat{\mathcal{A}})\}$ be a center manifold at $P_+$, with domain $N_{\hat{x}} \times N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}$, and choose an arbitrary point $\hat{x}_0 > 0 \in N_{\hat{x}}$. Then, if we let $I(\mathrm{e}, \widehat{\mathcal{A}}) := w_+(\hat{x}_0, \mathrm{e}, \mathcal{A})$, the function*

$$i(\varepsilon, a, b) := (\hat{x}_0^p) I \circ \phi_{\hat{x}_0}^{-1}(\varepsilon, a, b)$$

*is an initial condition function for $X$ at $x_0 = \hat{x}_0$, with domain of definition $V = \phi_{\hat{x}_0}(N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}})$.*

Let us denote by $\Gamma_{<0}, \Gamma_{>0} \subset \Gamma$ the open intervals on $\Gamma$ defined by

$$\Gamma_{<0} := \{x \in U_x \mid x < 0\} \quad \text{and} \quad \Gamma_{>0} := \{x \in U_x \mid x > 0\}.$$

Then, the above lemmas imply the following result:

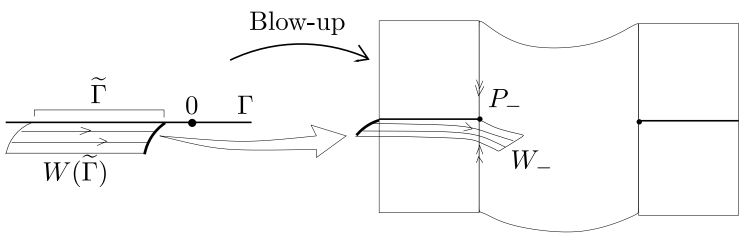



FIGURE 14. For $\mathcal{B}_- < 0$, $W(\widetilde{\Gamma})$ extends to a center manifold at $P_-$.

**Proposition 9.6.** *Suppose that $\mathcal{B}_- < 0$ (respect. $\mathcal{B}_+ > 0$), and let $W(\widetilde{\Gamma})$ be any center manifold with domain $\widetilde{\Gamma} \times \mathcal{O}$, where*

$$\widetilde{\Gamma} \subset \Gamma_{<0} \quad \textit{(respect. } \widetilde{\Gamma} \subset \Gamma_{>0}\textit{)}$$

*is a compact connected subinterval and $\mathcal{O} \subset \{(\varepsilon, \widehat{\mathcal{A}}) \mid \varepsilon > 0\}$. Then, for any sufficiently small positive constant $\hat{x}_0 = |x_0| \in U_x$, and any open subset with compact closure $V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}} = (0, \varepsilon_0) \times (V_{A,B} \times V_{\mathcal{A}_r})$ such that*

$$V_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}} \subset \phi_{\hat{x}_0}^{-1}(\mathcal{O}),$$

*there exists a smaller constant $0 < \mathrm{e}_1 < \mathrm{e}_0$ such that $W(\widetilde{\Gamma})$ uniquely extends as a center manifold $W_-$ (respect. $W_+$) at $P_-$ (respect. $P_+$) with domain of definition*

$$N_{\hat{x}} \times N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}},$$

*where $N_{\mathrm{e}} := (0, \mathrm{e}_1)$ and $N_{\hat{x}} := [0, \hat{x}_0]$.*

*Proof:* From Lemmas 9.1 and 9.5(1), we know that the function $i(\varepsilon, \mathcal{A})$ whose graph is given by

$$W(\widetilde{\Gamma}) \cap \{x = x_0\}$$

is (after the pull-back by $\phi_{\hat{x}_0}$) an initial condition function for $\overline{X}$ at $P_-$ (respect. $P_+$). Thus, the result follows from Propositions 5.6 and 5.8. $\square$

For shortness, we shall use the (self-explaining) notations

$$W(\Gamma_{<0}) \overset{\mathcal{B}_-<0}{\longrightarrow} W_- \quad \text{and} \quad W(\Gamma_{>0}) \overset{\mathcal{B}_+>0}{\longrightarrow} W_+ \tag{62}$$

to summarize the results of the above proposition.

Similarly, we obtain the following result:

**Proposition 9.7.** *Let $W_-$ (respect. $W_+$) be any center manifold at $P_-$ (respect. $P_+$), with domain $N_{\hat{x}} \times N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}$. Suppose that $\mathcal{B}_- > 0$ (respect. $\mathcal{B}_+ < 0$). Then, for any positive constant $\hat{x}_0 > 0 \in N_{\hat{x}}$, and any compact subinterval*

$$\widetilde{\Gamma} \subset \Gamma_{<0} \quad \textit{(respect. } \widetilde{\Gamma} \subset \Gamma_{>0}\textit{)},$$

*there exists a neighborhood $N \subset U_\varepsilon \times U_{\mathcal{A}}$ of $\{\varepsilon = \mathcal{A} = 0\}$ (depending on $\widetilde{\Gamma}$) such that $W_-$ (respect. $W_+$) uniquely extends as a center manifold $W(\widetilde{\Gamma})$, over the domain $\widetilde{\Gamma} \times \mathcal{O}$, where*

$$\mathcal{O} := \phi_{\hat{x}_0}(N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}) \cap N.$$

*Proof:* It suffices to consider Lemmas 9.3 and 9.5(2).

Notice that the hypothesis $\mathcal{B}_- > 0$ (respect. $\mathcal{B}_+ > 0$) implies that $B_x > 0$ (respect. $B_x < 0$) for any $x < 0 \in U_x$ (respect. $x > 0 \in U_x$). Thus, we can apply Proposition 3.4 to conclude. □

As in (62), we shall use the notation

$$W_- \stackrel{\mathcal{B}_- > 0}{\longrightarrow} W(\Gamma_{<0}) \quad \text{and} \quad W_+ \stackrel{\mathcal{B}_+ < 0}{\longrightarrow} W(\Gamma_{>0}) \tag{63}$$

to refer to results of the above proposition.

## 10. Existence of canard surfaces (with Transversality Hypothesis)

We are now ready to prove the main results of this paper. On the following enunciates, let $X$ be a singular perturbation family of transition type, defined on a domain $(x, y, \varepsilon, \mathcal{A}) \subset U_x \times U_y \times U_\varepsilon \times U_\mathcal{A}$. Further, suppose that

(a) $x = 0$ is the unique degenerate singularity on

$$\Gamma := \{y = \varepsilon = \mathcal{A} = 0\},$$

with multiplicity $\mu(X) = p$.

(b) $X$ satisfies the Transversality Hypothesis at $x = 0$.

Thus, up to a diffeomorphism on the space of parameters, we can suppose that

$$(\varepsilon, \mathcal{A}) = (\varepsilon, a, b, \mathcal{A}_r) \in U_{\varepsilon,\mathcal{A}}$$

are adapted parameters.

Below, we let $V \subset \{\varepsilon > 0\}$ be an open subset in $U_\varepsilon \times U_\mathcal{A}$ such that there exist positive constants $S, R > 0$ for which

$$\{0 < \varepsilon < S, \ \|\mathcal{A}\| < R\} \subset V. \tag{64}$$

Also, we let $\widetilde{\Gamma}$ be an arbitrary compact connected subset of $\Gamma$ which contains $x = 0$. Such $\widetilde{\Gamma}$ can be expressed as a closed subinterval

$$\widetilde{\Gamma} = [x_0, x_1]$$

where $x_0, x_1 \in U_x$ and $x_0 < 0 < x_1$.

### 10.1. Canard regions on $(\mathbf{s},\mathbf{s})$ and $(\mathbf{u},\mathbf{u})$ transitions.

Our first result is the following:

**Theorem 10.1.** *Suppose that the degenerate point $x = 0$ is either in the $(\mathbf{s},\mathbf{s})$ or $(\mathbf{u},\mathbf{u})$ case and that $X$ verifies the Transversality Hypothesis at this point. Further, let $i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_{\text{init}}})$ be an initial condition for $X$ at $x_{\text{init}}$, where $x_{\text{init}}$ is chosen as follows:*

- *In the $(\mathbf{s},\mathbf{s})$ case, $x_{\text{init}} := x_0$;*
- *In the $(\mathbf{u},\mathbf{u})$ case, $x_{\text{init}} := x_1$.*

*Then, there exists an open semi-analytic set of the form*

$$\mathcal{O}_{\widetilde{\Gamma}} = \begin{cases} 0 < \varepsilon < r \\ |\mathcal{A}_r| < s \\ |a_i|^{p+1} < k_i |\varepsilon|^{2p-i}, & \text{for } 0 \le i \le 2p-1 \\ |b_j|^{p+1} < l_j |\varepsilon|^{p-j}, & \text{for } 0 \le j \le 2p-1 \end{cases} \tag{65}$$

*(defined by strictly positive constants $r$, $s$, $k_i$ and $l_j$), and a unique $C^\infty$ function*

$$\begin{array}{rccc} w\colon & \widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}} & \longrightarrow & \mathbb{R} \\ & (x, (\varepsilon, \mathcal{A})) & \longmapsto & y = w(x, \varepsilon, \mathcal{A}) \end{array}$$

*such that*

(i) *$w(x_{\text{init}}, \varepsilon, \mathcal{A}) = i(\varepsilon, \mathcal{A})$, for each $(\varepsilon, \mathcal{A}) \in \mathcal{O}_{\widetilde{\Gamma}}$;*

(ii) *$W(\widetilde{\Gamma}) := \text{graph}\{y = w(x, \varepsilon, \mathcal{A})\}$ is an invariant set for $X$;*

(iii) *$w$ has a continuous extension to the closure of $\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}$, in such a way that $w(x, 0, 0) \equiv 0$. Moreover, such extension is $C^\infty$ at each point*

$$(x, \varepsilon, \mathcal{A}) \in \overline{\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}}$$

*such that $x \neq 0$ or $\varepsilon \neq 0$, and blow-up $C^\infty$ at $\{x = \varepsilon = 0\}$ (according to the definitions of Section 2).*

*Proof:* Let us suppose that $x = 0$ is in the $(\mathbf{s},\mathbf{s})$-case. The proof on the $(\mathbf{u},\mathbf{u})$-case is completely analogous by reversing the sense of the flow.

We construct the pair $(\mathcal{O}_{\widetilde{\Gamma}}, w)$ in three steps, which can be schematically represented as

$$\begin{aligned} &(1) \qquad W(\Gamma_{<0}) \overset{B_- <0}{\longrightarrow} W_-, \\ &(2) \qquad W_- \overset{B_+ <0}{\longrightarrow} W_+, \qquad \text{and} \\ &(3) \qquad W_+ \overset{B_+ <0}{\longrightarrow} W(\Gamma_{>0}). \end{aligned}$$

Let us describe each step in details.

*Step (1):* From Proposition 3.4 we know that, for each $x_0' \in U_x$ with

$$x_0 < x_0' < 0$$

there exists an open neighborhood $N_0 \subset U_{\varepsilon,\mathcal{A}}$ of $\{\varepsilon = \mathcal{A} = 0\}$ and a unique $C^\infty$ function $w_0(x, \varepsilon, \mathcal{A})$ defined on $\Gamma_0 \times \mathcal{O}_0$, where $\Gamma_0 := [x_0, x_0']$ and $\mathcal{O}_0 = V \cap N_0$, such that

(a) $w_0(x_0, \varepsilon, \mathcal{A}) = i(\varepsilon, \mathcal{A})$ for each $(\varepsilon, \mathcal{A}) \in \mathcal{O}_0$;

(b) $W(\Gamma_0) := \text{graph}\{y = w_0(x_0, \varepsilon, \mathcal{A})\}$ is an invariant set;

(c) $w_0 \in C^\infty(\Gamma_0 \times \mathcal{O}_0, \{\varepsilon = 0\}, \widehat{W})$.

By hypothesis (64), if we let $\hat{x}_0' = |x_0'|$, the set

$$\phi_{\hat{x}_0'}^{-1}(\mathcal{O}_0)$$

contains a neighborhood of $\{\mathrm{e} = \widehat{\mathcal{A}} = 0\}$ on the coordinates of the $(-x_\varepsilon)$-chart. Therefore, since $B_- < 0$, it follows from Proposition 9.6 that (for $|x_0'|$ sufficiently small) the center manifold $W(\Gamma_0)$ uniquely extends as a center manifold

$$W_- = \text{graph}\{\hat{y} = \widehat{w}_-(\hat{x}, \mathrm{e}, \widehat{\mathcal{A}})\},$$

at $P_-$, defined on a domain of the form

$$N_{\hat{x}} \times N_{\mathrm{e}} \times V_{\widehat{\mathcal{A}}}, \quad \text{where } N_{\hat{x}} = [0, \hat{x}_0'],\ N_{\mathrm{e}} = (0, \mathrm{e}_0)$$

and $V_{\widehat{\mathcal{A}}}$ is an open neighborhood of $\{A = B = \mathcal{A}_r = 0\}$.

*Step (2):* Since $B_+ < 0$, it follows from Lemma 8.1 that, associated to $W_-$, there exists a unique matching center manifold at $P_+$,

$$W_+ = \text{graph}\{\hat{y} = \widehat{w}_+(\hat{x}, \hat{\varepsilon}, \widehat{\mathcal{A}})\}$$

defined on a domain of the form

$$N_{\hat{x}}' \times N_{\mathrm{e}}' \times V_{\widehat{\mathcal{A}}}', \quad \text{where } N_{\hat{x}}' = [0, \hat{x}_1'],\ N_{\mathrm{e}}' = (0, \mathrm{e}_1)$$

and $V_{\widehat{\mathcal{A}}}' \subset V_{\widehat{\mathcal{A}}}$.

Furthermore, if we let $(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, A, B, \mathcal{A}_r)$ denotes the coordinates of the $\bar{\varepsilon}$-chart, the matching region $\mathcal{O}(W_-, W_+)$ contains an open neighborhood of $\{\tilde{\varepsilon} = \widetilde{\mathcal{A}} = 0\}$ in $U_{\tilde{\varepsilon},\widetilde{\mathcal{A}}}$. Thus, one can choose strictly positive constants $r$, $s$, $k_i$ and $l_j$ such that the set

$$\Omega := \begin{cases} 0 < \tilde{\varepsilon} < r, \\ |\mathcal{A}_r| < s, \\ |A_i| < k_i, & \text{for } 0 \leq i \leq 2p-1 \\ |B_j| < l_j, & \text{for } 0 \leq j \leq p-1 \end{cases} \tag{66}$$

is strictly contained in $\mathcal{O}(W_-, W_+)$. Moreover, in the coordinates of the $\bar{\varepsilon}$-char, one can write

$$W_- \equiv W_+ = \operatorname{graph}\{\tilde{y} = \widetilde{w}(\tilde{x}, \tilde{\varepsilon}, \widetilde{\mathcal{A}})\}$$

for some function $\widetilde{w} \in C^\infty(\mathbb{R} \times \mathcal{O}(W_-, W_+))$.

*Step (3):* From Proposition 9.7, there exists a neighborhood $N_1 \subset U_{\varepsilon,\mathcal{A}}$ of $\{\varepsilon = \mathcal{A} = 0\}$ and a unique $C^\infty$ function $w_1(x, \varepsilon, \mathcal{A})$ defined on $\Gamma_1 \times \mathcal{O}_1$, where $\Gamma_1 := [x_1', x_1]$ and

$$\mathcal{O}_1 = \phi_{\hat{x}_1'}(N_{\mathrm{e}}' \times V_{\widehat{\mathcal{A}}}') \cap N_1$$

such that $W_+$ uniquely extends as a center manifold

$$W(\Gamma_1) := \operatorname{graph}\{y = w_1(x, \varepsilon, \mathcal{A})\},$$

over the interval $\Gamma_1$, for a function $w_1$ verifies the same properties (a), (b) and (c) which are verified by $w_0$.

Let us now consider the open set

$$\mathcal{O}_{\widetilde{\Gamma}} := \varphi(\Omega)$$

where $\varphi$ is the polynomial map on (60). Easy computations show that $\mathcal{O}_{\widetilde{\Gamma}}$ is the semi-analytic set given on the enunciate. Moreover, if the constants $r$, $s$ in (66) are chosen sufficiently small, it is clear that

$$\mathcal{O}_{\widetilde{\Gamma}} \subset \mathcal{O}_0 \cap \mathcal{O}_1.$$

On such domain, we can consider the *blowing-down* of the functions $\widehat{w}$, $\widetilde{w}$ and $\widehat{w}_+$. In terms of the coordinates $(x, y, \varepsilon, \mathcal{A})$, such blowing-downs are given, respectively, by

$$\begin{aligned} y &= w_-(x, \varepsilon, \mathcal{A}) := |x|^p \cdot \widehat{w}_-(|x|, \phi_{\hat{x}_0'}^{-1}(\varepsilon, \mathcal{A})) \\ y &= \widetilde{\mathbf{w}}(x, \varepsilon, \mathcal{A}) := \varepsilon^{p/(p+1)} \cdot \widetilde{w}(x\varepsilon^{1/(p+1)}, \varphi^{-1}(\varepsilon, \mathcal{A})) \\ y &= w_+(x, \varepsilon, \mathcal{A}) := x^p \cdot \widehat{w}_+(x, \phi_{\hat{x}_1'}^{-1}(\varepsilon, \mathcal{A})). \end{aligned}$$

Therefore, by construction, there exists a function $w \in C^\infty(\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}})$ which is defined as

$$\begin{aligned} w(x,\varepsilon,\mathcal{A}) &:= w_0(x,\varepsilon,\mathcal{A}), && \text{for } x \in [x_0, x_0'] \\ w(x,\varepsilon,\mathcal{A}) &:= w_-(x,\varepsilon,\mathcal{A}), && \text{for } x \in [x_0', 0) \\ w(0,\varepsilon,\mathcal{A}) &:= \widetilde{\mathbf{w}}(0,\varepsilon,\mathcal{A}), && \text{for } x = 0 \\ w(x,\varepsilon,\mathcal{A}) &:= w_+(x,\varepsilon,\mathcal{A}), && \text{for } x \in (0, x_1'] \\ w(x,\varepsilon,\mathcal{A}) &:= w_1(x,\varepsilon,\mathcal{A}), && \text{for } x \in [x_1', x_1]. \end{aligned}$$

The graph of such function clearly defines an invariant set for $X$. Moreover, since $\mathcal{O}_{\widetilde{\Gamma}}$ is semi-analytic (and, in particular, subanalytic), Lemma 2.8 easily implies the extension properties listed on item (iii) of the enunciate. This proves the theorem. □

*Remark* 10.2. In view of Remark 8.3, there exists a particular case where the above theorem can be strengthened as follows: *Suppose that*

$$\mathcal{Q}_0 = 0$$

*(where $\mathcal{Q}_0$ is the constant defined on (12)). Then, the region $\mathcal{O}_{\widetilde{\Gamma}}$ defined in (65) exists for an arbitrary large choice of positive constants $k_i$, $l_j$, provided that we choose $r, s > 0$ sufficiently small.*

### 10.2. Canard regions on $(\mathbf{u}, \mathbf{s})$ transition.

**Theorem 10.3.** *Suppose that the degenerate singularity at $x = 0$ is in the $(\mathbf{u}, \mathbf{s})$ case, and that $X$ verifies the Transversality Hypothesis at this point. Then, there exists an open semi-analytic set*

$$\mathcal{O}_{\widetilde{\Gamma}} = \begin{cases} 0 < \varepsilon < r \\ |\mathcal{A}_r| < s \\ |a_i|^{p+1} < k_i|\varepsilon|^{2p-i}, & \textit{for } 0 \le i \le 2p-1 \\ |b_j|^{p+1} < l_j|\varepsilon|^{p-j}, & \textit{for } 0 \le j \le 2p-1 \end{cases}$$

*(defined by strictly positive constants $r$, $s$, $k_i$ and $l_j$), and a unique $C^\infty$ function*

$$\begin{aligned} w\colon \quad \widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}} &\longrightarrow \mathbb{R} \\ (x, (\varepsilon, \mathcal{A})) &\longmapsto y = w(x,\varepsilon,\mathcal{A}) \end{aligned}$$

*such that*

(i) $w(0,\varepsilon,\mathcal{A}) \equiv 0$*;*

(ii) $W(\widetilde{\Gamma}) := \operatorname{graph}\{y = w(x,\varepsilon,\mathcal{A})\}$ *is an invariant set for $X$;*

(iii) *$w$ has a continuous extension to the closure of $\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}$, in such a way that $w(x,0,0) \equiv 0$. Moreover, such extension is $C^\infty$ at each point*

$$(x, \varepsilon, \mathcal{A}) \in \overline{\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}}$$

*such that $x \neq 0$ or $\varepsilon \neq 0$, and blow-up $C^\infty$ at $\{x = \varepsilon = 0\}$.*

*Proof:* The proof is very similar to the proof of Theorem 10.1. Here, instead of fixing the initial condition function at some non-degenerate point $x_{\text{init}} \neq 0$, we firstly construct local matching center manifolds $W_-$ at $P_-$ and $P_+$, respectively, and then extend these manifolds to the regular segment $\widetilde{\Gamma} \setminus \{0\}$.

Thus, the construction of the pair $(\mathcal{O}_{\widetilde{\Gamma}}, w)$ is now made on the following steps:

$$(1) \qquad W_- \overset{B_+<0}{\longrightarrow} W_+,$$

$$(2) \qquad W_- \overset{B_->0}{\longrightarrow} W(\Gamma_{<0}),$$

$$(3) \qquad W_+ \overset{B_+<0}{\longrightarrow} W(\Gamma_{>0}).$$

Let us briefly describe each step.

*Step (1):* First of all, consider the coordinates $(\tilde{x}, \tilde{y}, \tilde{\varepsilon}, \widetilde{\mathcal{A}})$ of the $\bar{\varepsilon}$-chart and an arbitrary $C^\infty$ function

$$\tilde{y} = \widetilde{I}(\tilde{\varepsilon}, \widetilde{\mathcal{A}}),$$

defined on some open neighborhood $U$ of $\{\tilde{\varepsilon} = \widetilde{\mathcal{A}} = 0\}$, such that $I(0,0) = 0$. Then, it follows from Lemma 8.2 that, possibly restricting $U$ to some smaller neighborhood, the following holds: There exist unique center manifolds $W_-$ at $P_-$ and $W_+$ at $P_+$, both defined on a domain of the form

$$N'_x \times N'_{\mathrm{e}} \times V'_{\widehat{\mathcal{A}}}, \quad \text{where } N'_x = [0, x'_0],\ N'_{\mathrm{e}} = (0, \varepsilon'_0]$$

and $V'_{\widehat{\mathcal{A}}}$ is an open neighborhood of $\{\widehat{\mathcal{A}} = 0\}$, such that

- $U \subset \mathcal{O}^- \cap \mathcal{O}^+$, and
- $W_- \cap \Sigma = W_+ \cap \Sigma = \operatorname{graph}\{y = I(\tilde{\varepsilon}, \widehat{\mathcal{A}})\}$ where $\Sigma = \{x = 0\}$.

In other words, the matching region $\mathcal{O}(W_+, W_-)$ contains $U$.

In particular, if we consider the identically zero function $I(\tilde{\varepsilon}, \widehat{\mathcal{A}}) \equiv 0$, there exist matching center manifolds $W_-$ and $W_+$ such that

$$W_- \cap \Sigma = W_+ \cap \Sigma = \{y = 0\}$$

for $(\tilde{\varepsilon}, \widetilde{\mathcal{A}})$ in some small neighborhood of $\{\tilde{\varepsilon} = \widehat{\mathcal{A}} = 0\}$. Thus, we can find sufficiently small strictly positive constants $r$, $s$, $k_i$ and $l_j$ such that a set $\Omega$, defined like in (66), is strictly contained in the matching region $\mathcal{O}(W_-, W_+)$.

*Steps (2) and (3):* Using Proposition 9.7, we can uniquely extend the center manifold $W_-$ (respect. $W_+$) as a center manifold over the interval $\Gamma_0 = [x_0, x_0']$ (respect. $\Gamma_1 = [x_1', x_1]$),

$$W_0 := \text{graph}\{y = w_0(x, \varepsilon, \mathcal{A})\}$$

(respect. $W_1 := \text{graph}\{y = w_1(x, \varepsilon, \mathcal{A})\}$), where $(\varepsilon, \mathcal{A})$ belong to the domain

$$\mathcal{O}_0 := N \cap \phi_{\hat{x}_0'}(N_\mathrm{e}' \times V_{\widehat{\mathcal{A}}}')$$

(respect. $\mathcal{O}_1 := N \cap \phi_{\hat{x}_1'}(N_\mathrm{e}' \times V_{\widehat{\mathcal{A}}}')$), for some sufficiently small neighborhood $N$ of $\{\varepsilon = \mathcal{A} = 0\}$.

Now, if we define $\mathcal{O}_{\widetilde{\Gamma}} := \varphi(\Omega)$, it is clear that it has the form given on the enunciate, and that

$$\mathcal{O}_{\widetilde{\Gamma}} \subset \mathcal{O}_0 \cap \mathcal{O}_1$$

provided that we choose the constants $r$, $s$ sufficiently small. Therefore, it suffices to construct the function $w \in C^\infty(\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}})$ as we have done on the proof of Theorem 10.1. It is easy to verify that it satisfies all the required conditions. $\square$

*Remark* 10.4. This theorem can be strengthened in the special case $\mathcal{Q}_0 = 0$ exactly as in Remark 10.2.

### 10.3. Canard regions on $(\mathbf{s}, \mathbf{u})$ transition.

**Theorem 10.5.** *Suppose that the degenerate singularity at $x = 0$ is in the $(\mathbf{s}, \mathbf{u})$ case, and that $X$ verifies the Transversality Hypothesis at this point. Let*

$$i_0 \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_0}) \quad \textit{and} \quad i_1 \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_1})$$

*be arbitrary initial condition for $X$ at $x_0$ and $x_1$, respectively. Then, there exists an open semi-analytic set*

$$\mathfrak{B} = \begin{cases} 0 < \varepsilon < r \\ |\mathcal{A}_r| < s \\ |a_i|^{p+1} < k_i\, |\varepsilon|^{2p-i}, & \textit{for } 0 \le i \le 2p-1 \\ |b_j|^{p+1} < l_j\, |\varepsilon|^{p-j}, & \textit{for } 0 \le i \le 2p-1 \end{cases}$$

*(defined by strictly positive constants $r$, $s$, $k_i$, $l_j$), a unique closed codimension one submanifold*

$$\mathcal{O}_{\widetilde{\Gamma}} \subset \mathfrak{B}$$

*which can be defined as the graph of a $C^\infty$ function $\alpha_s$ on $\mathfrak{B}_s := \mathfrak{B} \cap \{a_s = 0\}$,*

$$\mathcal{O}_{\widetilde{\Gamma}} := \operatorname{graph}\{a_s = \alpha_s(\varepsilon, a', b, \mathcal{A}_r)\}$$

*(where $a' = (a_0, \dots, a_{s-1}, a_{s+1}, \dots, a_{2p-1})$); and a $C^\infty$ function*

$$\begin{array}{rccl} w\colon & \widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}} & \longrightarrow & \mathbb{R} \\ & (x, (\varepsilon, \mathcal{A})) & \longmapsto & y = w(x, \varepsilon, \mathcal{A}) \end{array}$$

*such that*

(i) *$w(x_0, \varepsilon, \mathcal{A}) = i_0(\varepsilon, \mathcal{A})$, and $w(x_1, \varepsilon, \mathcal{A}) = i_1(\varepsilon, \mathcal{A})$, for each $(\varepsilon, \mathcal{A}) \in \mathcal{O}_{\widetilde{\Gamma}}$;*

(ii) *$\alpha_s$ has a blow-up $C^\infty$ extension at $\{\varepsilon = 0\}$;*

(iii) *$W(\widetilde{\Gamma}) := \operatorname{graph}\{y = w(x, \varepsilon, \mathcal{A})\}$ is an invariant set for $X$;*

(iv) *$w$ has a continuous extension to the closure of $\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}$, in such a way that $w(x, 0, 0) \equiv 0$. Moreover, such extension is $C^\infty$ at each point*

$$(x, \varepsilon, \mathcal{A}) \in \overline{\widetilde{\Gamma} \times \mathcal{O}_{\widetilde{\Gamma}}}$$

*such that $x \neq 0$ or $\varepsilon \neq 0$, and blow-up $C^\infty$ at $\{x = \varepsilon = 0\}$.*

*Proof:* The definition of $\mathfrak{B}$, $\alpha_s$ and $w$ is made in two steps

(1) $\quad W(\Gamma_{<0}) \overset{B_- < 0}{\longrightarrow} W_-, \quad$ and $W(\Gamma_{>0}) \overset{B_+ > 0}{\longrightarrow} W_+,$

(2) $\quad$ Compute $\mathcal{O}(W_-, W_+)$.

The step (1) has already been described on the proofs of Theorems 10.1 and 10.3.

*Step (2):* We have to compute the matching region $\mathcal{O}(W_-, W_+)$. By Lemma 8.4, the intersection $\mathcal{O}_- \cap \mathcal{O}_+$ of the maximal domain of extension of such manifolds contains an open neighborhood $U$ of $\{\tilde{\varepsilon} = \widetilde{\mathcal{A}} = 0\}$. Moreover, from Proposition 8.5 it follows that, for any even index $0 \leq s \leq 2p - 2$, there exists some smaller neighborhood $U_s \subset U$ of the origin and there exists a unique $C^\infty$ function

$$\mathbf{a}_s\colon U_s \cap \{A_s = 0\} \to \mathbb{R} \tag{67}$$

such that the matching region $\mathcal{O}(W_-, W_+)$, when restricted to $U_s$, is the graph of $\mathbf{a}_s$.

Thus, we can choose strictly positive constants $r$, $s$, $k_i$ and $l_j$ such that the set $\Omega$, defined like in (66), is strictly contained in $U_s$, and moreover,

$$\text{graph}\{A_s = \mathbf{a}_s|_{\Omega_s}\} \subset \Omega$$

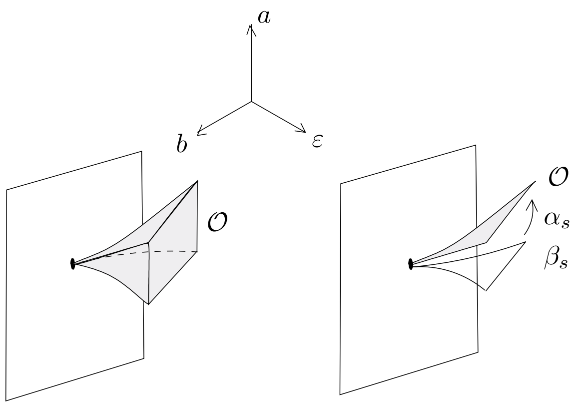
where we define $\Omega_s := \Omega \cap \{A_s = 0\}$.

Now, it suffices to define $\mathfrak{B} := \varphi(\Omega)$, $\mathcal{O}_{\widetilde{\Gamma}} := \varphi(\mathcal{O}(W_-, W_+) \cap \Omega)$, and let $\alpha_s \colon \mathfrak{B}_s \to \mathbb{R}$ to be the blowing-down of the function $\mathbf{a}_s$, which is given by

$$\alpha_s(\varepsilon, \widehat{a}, b, \mathcal{A}_r) := \varepsilon^{\frac{2p-s}{p+1}} \cdot \mathbf{a}_s \circ \phi^{-1}(\varepsilon, \widehat{a}, b, \mathcal{A}_r), \tag{68}$$

(where $\varphi$ is the map defined on (60)).

If we construct the function $w$ as in the end of proof of Theorem 10.1 it is clear that is satisfies all the enunciated properties. □




$(u, u)$, $(s, s)$ and $(u, s)$ Cases

$(s, u)$ Case

FIGURE 15. The regions defined on Theorems 10.1, 10.3 and 10.5.

*Remark* 10.6. Notice that, in the special case $\mathcal{Q}_0 = 0$ it follows from Subsection 7.1 that the distance function $\delta(A, B)$ is a linear function on the parameters $A$, defined for all $(A, B) \in \mathbb{R}^{3p}$ (see (50)). In particular, the restriction of the function $\mathbf{a}_s(\mathrm{e}, \widehat{A}, B, \mathcal{A}_r)$ in (67) to $\{\mathrm{e} = \mathcal{A}_r = 0\}$ is explicitly given by

$$\mathbf{a}_s(0, \widehat{A}, B, 0) = -\frac{1}{C_s} \sum_{i=0, i \neq s}^{2p-1} C_i\, A_i.$$

Thus, the above result can be strength exactly as in Remarks 10.2 and 10.4.

### 10.4. Smooth parameterization of the canard regions.

To work out the examples, it is also convenient to consider the natural parameterization of the canard region $\mathcal{O}_{\widetilde{\Gamma}}$ which are provided by the proofs of the previous theorems.

**The $(\mathbf{u},\mathbf{u})$, $(\mathbf{s},\mathbf{s})$ and $(\mathbf{u},\mathbf{s})$ cases.** If we let $\Omega$ be the set defined on (66) and $\varphi$ be defined as in (60), the polynomial map

$$\Omega \ni (\tilde{\varepsilon}, A, B, \mathcal{A}_r) \overset{\varphi}{\longmapsto} (\varepsilon, a, b, \mathcal{A}_r) \in \mathcal{O}_{\widetilde{\Gamma}}$$

is an analytic injective parameterization of $\mathcal{O}_{\widetilde{\Gamma}}$.

**The $(\mathbf{s},\mathbf{u})$ case.** If we let $\Omega_s := \Omega \cap \{A_s = 0\}$ and let $\mathbf{a}_s \colon \Omega_s \to \mathbb{R}$ be the function in (67), the map

$$\widehat{\varphi}(\tilde{\varepsilon}, \widehat{A}, B, \mathcal{A}_r) := \varphi(\tilde{\varepsilon}, A, B, \mathcal{A}_r)|_{A_s = \mathbf{a}(\tilde{\varepsilon}, \widehat{A}, B, \mathcal{A}_r)}$$

defines a $C^\infty$ injective parameterization of $\mathcal{O}_{\widetilde{\Gamma}}$,

$$\Omega_s \ni (\tilde{\varepsilon}, \widehat{A}, B, \mathcal{A}_r) \overset{\widehat{\varphi}}{\longmapsto} (\varepsilon, a, b, \mathcal{A}_r) \in \mathcal{O}_{\widetilde{\Gamma}}.$$

## 11. The induction map and improved existence results

We are now interested in studying a singular perturbation family

$$X = \varepsilon \frac{\partial}{\partial x} + F(x, y, \varepsilon, \mathcal{A}) \frac{\partial}{\partial y}$$

on the same conditions of the previous section, but which not necessarily verifies the Transversality Hypothesis at $x = 0$.

There exists a canonical way to define a *larger* family $\mathfrak{X}$, verifying the Transversality Hypothesis, which induces $X$. To explicitly compute $\mathfrak{X}$ it suffices to write the expansion

$$F(x, y, \varepsilon, \mathcal{A}) = \sum_{i=0}^{\infty} F_i(x, \varepsilon, \mathcal{A})\, y^i$$

where

$$F_0(x, \varepsilon, \mathcal{A}) = \sum_{i=0}^{2p-1} a_i(\varepsilon, \mathcal{A}) x^i + O(x^{2p}) \tag{69}$$

and

$$F_1(x,\varepsilon,\mathcal{A}) = \sum_{j=0}^{p-1} b_j(\varepsilon,\mathcal{A})\, x^j + O(x^p). \tag{70}$$

The new family $\mathfrak{X}$ is now defined as

$$\mathfrak{X} = \varepsilon\frac{\partial}{\partial x} + \mathfrak{F}(x,y,\varepsilon,a,b,\mathcal{A})\frac{\partial}{\partial y}$$

where the function $\mathfrak{F}$ is obtained from $F$ by considering

$$(a_0,\dots,a_{2p-1}) \in \mathbb{R}^{2p} \quad \text{and} \quad (b_0,\dots,b_{p-1}) \in \mathbb{R}^p$$

as new free parameters. The *induction map*, which makes $X$ a sub-family of $\mathfrak{X}$, is given by

$$\begin{array}{rcl} P\colon (\mathbb{R}^+,0)\times(\mathbb{R}^n,0) & \longrightarrow & (\mathbb{R}^+,0)\times(\mathbb{R}^{3p},0)\times(\mathbb{R}^n,0) \\ (\varepsilon,\mathcal{A}) & \longmapsto & (\varepsilon,a,b,\mathcal{A}_r) = (\varepsilon,a(\varepsilon,\mathcal{A}),b(\varepsilon,\mathcal{A}),\mathcal{A}) \end{array} \tag{71}$$

with $P(0,0) = (0,0,0,0)$.

*Remark* 11.1. Notice that the parameters $\mathcal{A}$ become the *inessential parameters* of the new family $\mathfrak{X}$. Below, we shall simply denote them by $\mathcal{A}_r$.

### 11.1. Restriction/Extension of initial condition functions.

Let $x_0 \in \Gamma$ be a non-degenerate point. Then, it follows from Lemma 3.2 that (possibly restricting $U_{\mathcal{A}_r}$ to some smaller neighborhood of the origin) there exists a unique *formal center manifold* for $X$ at $x_0$,

$$\widehat{W}(x_0,\varepsilon,\mathcal{A}_r) = \sum_{i=0}^{\infty} w_i(x_0,\mathcal{A}_r)\varepsilon^i,$$

where $w_i(x_0,\cdot) \in C^\omega(U_{\mathcal{A}_r})$ for all $i \geq 0$.

Similarly, by restricting $U_{a,b,\mathcal{A}_r}$ to some smaller neighborhood of the origin, there exists a unique *formal center manifold* for $\mathfrak{X}$ at $x_0$,

$$\widehat{\mathfrak{W}}(x_0,\varepsilon,\mathcal{A}_r) = \sum_{i=0}^{\infty} \mathbf{w}_i(x_0,\mathcal{A}_r)\varepsilon^i,$$

where $\mathbf{w}_i(x_0,\cdot) \in C^\omega(U_{a,b,\mathcal{A}_r})$ for all $i \geq 0$. The following result establishes a connection between such expansions:

**Lemma 11.2.** *If we let* $(\varepsilon, a, b, \mathcal{A}_r) = P(\varepsilon, \mathcal{A}_r)$ *be the induction map defined on (71), then*

$$\widehat{W}(x_0, \varepsilon, \mathcal{A}_r) = \widehat{\mathfrak{W}}(x_0, P(\varepsilon, \mathcal{A}_r)).$$

*In particular, if we re-expand the right hand side in powers of* $\varepsilon$*, we see that each* $w_i(x_0, \mathcal{A}_r)$ *can be written as*

$$w_i(x_0, \mathcal{A}_r) = \mathbf{w}_i(x_0, P(0, \mathcal{A}_r)) + p_i(\mathbf{w}_{i-1}, \ldots, \mathbf{w}_0)$$

*for uniquely defined analytic functions* $p_i$*.*

*Proof:* It is an immediate consequence of the uniqueness of the above expansions. □

It follows that an initial condition function $I$ for $\mathfrak{X}$ at $x_0$ always *restricts* to an initial condition $i$ to $X$ at $x_0$.

**Proposition 11.3.** *Let* $I(\varepsilon, a, b, \mathcal{A}_r) \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{\mathfrak{W}}_{x_0})$ *be an initial condition function for* $\mathfrak{X}$ *at* $x_0$*. Then, supposing that* $V' := P^{-1}(V)$ *is non-empty, the function*

$$i(\varepsilon, \mathcal{A}_r) := I \circ P(\varepsilon, \mathcal{A}_r)$$

*belongs to* $C^\infty_{\text{flat}}(V', \{\varepsilon = 0\}, \widehat{W}_{x_0})$*, and is an initial condition function for* $X$ *at* $x_0$*.*

*Proof:* It is an easy consequence of the previous lemma. □

In the same way, if we let $i$ be an initial condition function for $X$ at $x_0$, it can be extended to an initial condition function $I$ for $\mathfrak{X}$:

**Proposition 11.4.** *Let* $i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widehat{W}_{x_0})$ *be an initial condition function for* $X$ *at* $x_0$*. Then, there exists a neighborhood* $V_{a,b}$ *of* $\{a = b = 0\}$ *in* $\mathbb{R}^{3p}$ *and an initial condition function*

$$I \in C^\infty_{\text{flat}}(V \times V_{a,b}, \{\varepsilon = 0\}, \widehat{\mathfrak{W}}_{x_0})$$

*such that* $i$ *is the restriction of* $I$ *(i.e.* $i = I \circ P$*).*

*Proof:* Using Whitney's Extension Theorem, we can choose a function

$$\widetilde{I} \in C^\infty_{\text{flat}}(U_{a,b,\mathcal{A}_r}, \{\varepsilon = 0\}, \widehat{\mathfrak{W}}_{x_0}).$$

Now, if we restrict $\widetilde{I}$ to the domain $U_{a,b,\mathcal{A}_r} \cap \{(\varepsilon, \mathcal{A}_r) \in V\}$, and define

$$I(\varepsilon, a, b, \mathcal{A}_r) := \widetilde{I}(\varepsilon, a, b, \mathcal{A}_r) - \widetilde{I}(P(\varepsilon, \mathcal{A}_r)) + i(\varepsilon, \mathcal{A}_r),$$

it is easy to see that $I$ satisfies the requirements of the enunciate. □

### 11.2. Dropping the Transversality Hypothesis.

Let us state the following condition, which links the multiplicity $\mu(X) = p$ at the degenerate point $0 \in \Gamma$, and the asymptotic behavior of the induction map $P(\varepsilon, \mathcal{A}_r)$:

> **Asymptotic Hypothesis:** The functions $a_i(\varepsilon, \mathcal{A}_r)$ and $b_j(\varepsilon, \mathcal{A}_r)$ which appear on (69) and (70) are such that
> $$a_i(\varepsilon, 0) = o(\varepsilon^{\frac{2p-i}{p+1}}), \quad \text{for } 0 \leq i \leq 2p-1$$
> $$b_j(\varepsilon, 0) = o(\varepsilon^{\frac{p-j}{p+1}}), \quad \text{for } 0 \leq j \leq p-1,$$

where, as usual, $g(x) = o(f(x))$ means that $\lim_{x \to 0} g/f = 0$.

Using this hypothesis, we can generalize the theorems of the previous section as follows:

**Theorem 11.5.** *On the enunciates of Theorems 10.1 and 10.3, replace the words* Transversality Hypothesis *by* Asymptotic Hypothesis*. Then, there exists an open semi-analytic set*

$$O_{\widetilde{\Gamma}} \subset \{\varepsilon > 0\},$$

*which contains $\{\varepsilon = \mathcal{A}_r = 0\}$ in its closure, and there exists a $C^\infty$ function $\mathbf{w}(x, \varepsilon, \mathcal{A}_r)$ defined on $\widetilde{\Gamma} \times O_{\widetilde{\Gamma}}$ which verifies exactly the same conditions (i), (ii) and (iii) on these enunciates.*

*Proof:* Let us suppose that $x = 0$ is in the $(\mathbf{s}, \mathbf{s})$ case. The proof on the other cases is very similar.

Suppose given an initial condition function $i \in C^\infty_{\text{flat}}(V, \{\varepsilon = 0\}, \widetilde{W}_{x_0})$ for $X$ at $x_0$. Using Proposition 11.4, we can extend it to an initial condition

$$I \in C^\infty_{\text{flat}}(V \times U_{a,b}, \{\varepsilon = 0\}, \widetilde{\mathfrak{W}}_{x_0})$$

for the family $\mathfrak{X}$ at $x_0$. Applying Theorem 10.1, we find an open semi-analytic set

$$\mathcal{O}_{\widetilde{\Gamma}} \subset U_{\varepsilon,a,b,\mathcal{A}_r}$$

which has the form (65) (for some collection of strictly positive constants $r$, $s$, $k_i$ and $l_j$) and a unique $C^\infty$ function $w(x,\varepsilon,a,b,\mathcal{A}_r)$ in $\widetilde{\Gamma}\times\mathcal{O}_{\widetilde{\Gamma}}$ which verifies conditions (i)–(iii).

Now, from the expression (65), the Asymptotic Hypothesis necessarily implies that

$$P^{-1}(\mathcal{O}_{\widetilde{\Gamma}})$$

is a non-empty semi-analytic set which contains the origin in its adherence (independently of the constants $r$, $s$, $k_i$ and $l_j$). To prove this, it suffices to consider the restriction of the induction map to the curve

$$c_\varepsilon := \{\mathcal{A}_r = 0,\ \varepsilon > 0\}.$$

From the Asymptotic Hypothesis, it is clear that its image $P(c_\varepsilon)$ should be contained in $\mathcal{O}_{\widetilde{\Gamma}}$ for all sufficiently small $\varepsilon > 0$.

Thus, it suffices to define

$$O_{\widetilde{\Gamma}} := P^{-1}(\mathcal{O}_{\widetilde{\Gamma}}) \quad \text{and} \quad \mathbf{w}(x,\varepsilon,\mathcal{A}_r) := w(x, P(\varepsilon,\mathcal{A}_r)).$$

This proves the theorem. □

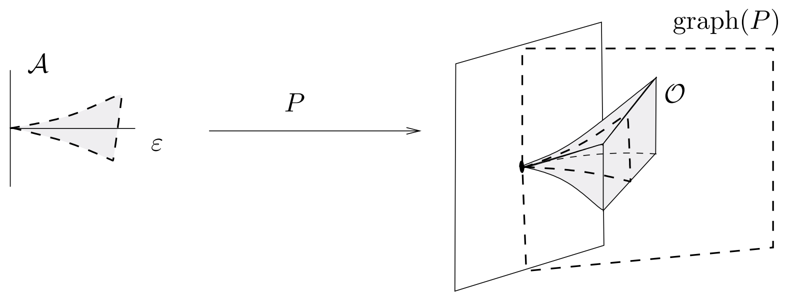



Figure 16. The canard region for the family $X$.

In order to prove an analogous generalization of Theorem 10.5, we need to make an stronger hypothesis concerning the induction map:

**Improved Transversality Hypothesis:** Let us write $\mathcal{A}_r = (\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$ and suppose that, up to a diffeomorphism

$$(\varepsilon, \mathcal{A}_r) = (\varepsilon, \gamma(\varepsilon, \mathcal{A}_r))$$

on the space of parameters (with $\gamma(0,0) = 0$), the following conditions hold:

(a) There exists an even index $0 \leq s \leq 2p - 2$ such that

$$a_s(\varepsilon, \mathcal{A}_r) = \mathcal{A}_0.$$

(b) If we consider the restriction of each $a_i(\varepsilon, \mathcal{A}_r)$ and $b_j(\varepsilon, \mathcal{A}_r)$ to the set

$$(\varepsilon, \mathcal{A}_0, 0) = \{\mathcal{A}_1 = \cdots = \mathcal{A}_{n-1} = 0\},$$

and make the substitution $\varepsilon = t^{p+1}$, $\mathcal{A}_0 = t^{2p-s}$, the following asymptotic conditions hold:

$$\begin{aligned} a_i(t^{p+1}, t^{2p-s}, 0) &= o(t^{2p-i}), \quad \text{for } 0 \leq i \neq s \leq 2p-1 \\ b_j(t^{p+1}, t^{2p-s}, 0) &= o(t^{p-j}), \quad \text{for } 0 \leq j \leq p-1. \end{aligned}$$

Notice, in particular, that the Improved Transversality Hypothesis always implies the Asymptotic Hypothesis.

**Lemma 11.6.** *Let us suppose that the induction map $P : U_{\varepsilon,\mathcal{A}_r} \to U_{\varepsilon,a,b,\mathcal{A}_r}$ is such that the Improved Transversality Hypothesis holds. Then there exists a polynomial map*

$$\psi : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^+ \times \mathbb{R}^n$$

*which has the form*

$$(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) \longmapsto \begin{cases} \varepsilon = \tilde{\varepsilon}^{p+1} \\ \mathcal{A}_0 = \tilde{\varepsilon}^{2p-s} \widetilde{\mathcal{A}}_0 \\ \mathcal{A}_i = \tilde{\varepsilon}^{\gamma_i} \widetilde{\mathcal{A}}_i, \qquad \text{for } 1 \leq i \leq n-1 \end{cases} \tag{72}$$

*for some collection of natural numbers $\gamma_1, \ldots, \gamma_{n-1} \in \mathbb{N} \setminus \{0\}$; such that, if we let $\varphi$ be the polynomial map defined on (60), there exists a unique analytic map*

$$\widetilde{P} : \mathcal{U}_{\varepsilon,\mathcal{A}_r} \to \mathcal{U}_{\varepsilon,A,B,\mathcal{A}_r}$$

*where* $\mathcal{U}_{\varepsilon,\mathcal{A}_r} := \psi^{-1}(U_{\varepsilon,\mathcal{A}_r})$ *and* $\mathcal{U}_{\varepsilon,A,B,\mathcal{A}_r} = \varphi^{-1}(U_{\varepsilon,a,b,\mathcal{A}_r})$, *such that the diagram*

$$\begin{array}{ccc} \mathcal{U}_{\varepsilon,A,B,\mathcal{A}_r} & \xrightarrow{\widetilde{P}} & \mathcal{U}_{\varepsilon,\mathcal{A}_r} \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\varphi} \\ U_{\varepsilon,a,b,\mathcal{A}_r} & \xrightarrow{P} & U_{\varepsilon,\mathcal{A}_r} \end{array}$$

*is commutative.*

*Proof:* Using the definition of $\varphi$ and $\psi$, and writing

$$\widetilde{P}(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) = (\tilde{\varepsilon}, A_i(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), B_j(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), \mathcal{A}_r(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r))$$

one sees that the relation $\varphi \circ \widetilde{P} = P \circ \psi$ is equivalent to the equations

$$\begin{aligned} \tilde{\varepsilon}^{2p-s}\, A_s(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) &= \tilde{\varepsilon}^{2p-s}\widetilde{\mathcal{A}}_0 \\ \tilde{\varepsilon}^{2p-i}\, A_i(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) &= a_i \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), \\ \tilde{\varepsilon}^{p-j}\, B_j(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) &= b_j \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), \\ \mathcal{A}_r(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) &= \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r). \end{aligned}$$

From the first equation, one immediately obtains

$$A_s(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) = \widetilde{\mathcal{A}}_0.$$

For the remaining equations, we can set

$$A_i(\tilde{\varepsilon}, \mathcal{A}_r) := \frac{a_i \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r)}{\tilde{\varepsilon}^{2p-i}} \quad \text{and} \quad B_j(\tilde{\varepsilon}, \mathcal{A}_r) := \frac{b_j \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r)}{\tilde{\varepsilon}^{p-j}}$$

provided that one verifies the following statement:

**Claim.** *There exists a choice of weights* $\gamma_1, \dots, \gamma_{n-1}$ *in (72) such that the functions* $A_i$ *and* $B_j$ *defined above are analytic.*

Indeed, it follows from the Improved Transversality Hypothesis that, if we write the expansion,

$$a_i(\varepsilon, \mathcal{A}_r) = f(\varepsilon, \mathcal{A}_0) + O(\mathcal{A}_1, \dots, \mathcal{A}_{n-1})$$

(where $O(\mathcal{A}_1, \dots, \mathcal{A}_{n-1})$ denote terms which have some $\mathcal{A}_i$, $1 \le i \le n-1$, as a factor), then

$$a_i \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) = o(\tilde{\varepsilon}^{2p-i}) + O(\tilde{\varepsilon}^{\gamma})$$

where $\gamma := \min\{\gamma_1, \dots, \gamma_{n-1}\}$. Similarly, we obtain that

$$b_j \circ \psi(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) = o(\tilde{\varepsilon}^{p-j}) + O(\tilde{\varepsilon}^{\gamma}).$$

Thus, if $\gamma$ is sufficiently large, $A_i$ and $B_j$ are analytic functions. This proves the claim and the lemma. $\square$

*Remark* 11.7. From the above proof, it follows that we can further assume that $\widetilde{P}$ has the form

$$\widetilde{P}(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) : \begin{cases} \tilde{\varepsilon} = \tilde{\varepsilon} \\ A_s = \widetilde{\mathcal{A}}_0 \\ A_i = \tilde{\varepsilon}\, f_i(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), & \text{for } 0 \leq i \neq s \leq 2p-1 \\ B_j = \tilde{\varepsilon}\, g_j(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r), & \text{for } 0 \leq j \leq p-1 \\ \mathcal{A}_r = \tilde{\varepsilon} H(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) \end{cases}$$

for some analytic functions $f_i$, $g_j$ and $H$.

As a consequence of this hypothesis, we can prove the following generalization of Theorem 10.5:

**Theorem 11.8.** *On the enunciate of Theorem 10.5, replace the words* Transversality Hypothesis *by* Improved Transversality Hypothesis. *Then, there exists an open semi-analytic set $B \subset \{\varepsilon > 0\}$, a submanifold*

$$O_{\widetilde{\Gamma}} \subset B$$

*which contains $\{\varepsilon = \mathcal{A}_r = 0\}$ in its closure, and a $C^\infty$ function $\mathbf{w}(x, \varepsilon, \mathcal{A}_r)$ defined on $\widetilde{\Gamma} \times O_{\widetilde{\Gamma}}$ which verifies exactly the same conditions (i)–(iv) on the enunciate of that theorem.*

*Proof:* Let $i_0$ and $i_1$ be arbitrary initial condition functions for $X$ at $x_0$ and $x_1$. Then, considering their extensions (given by Proposition 11.4)

$$I_0 \in C^\infty_{\text{flat}}(V \times U_{a,b}, \{\varepsilon = 0\}, \widetilde{\mathfrak{W}}_{x_0}) \quad \text{and}$$
$$I_1 \in C^\infty_{\text{flat}}(V \times U_{a,b}, \{\varepsilon = 0\}, \widetilde{\mathfrak{W}}_{x_1})$$

as initial condition functions for $\mathfrak{X}$ at $x_0$ and $x_1$, we can use Theorem 10.5 to define the sets $\mathfrak{B}$, the manifold

$$\mathcal{O}_{\widetilde{\Gamma}} := \text{graph}\{a_s = \alpha_s(\varepsilon, a', b, \mathcal{A}_r)\}$$

and the function $w(x, y, \varepsilon, a, b, \mathcal{A}_r)$.

Now, we shall use Lemma 11.6 in order to define $B$ and $O_{\widetilde{\Gamma}}$. Keeping the notation of that lemma, let $\mathbf{a}_s \colon U_s \to \mathbb{R}$ be the function in (67), which defines the matching region $\mathcal{O}(W_-, W_+)$, i.e.

$$\mathcal{O} = \text{graph}\{A_s = \mathbf{a}_s(\tilde{\varepsilon}, \widehat{A}, B, \mathcal{A}_r)\}.$$

Then, the set

$$S := \{(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) \mid \widetilde{\mathcal{A}}_0 = \mathbf{a}_s \circ \widetilde{P}\}$$

defines the *pull-back* of the matching region under the map $\widetilde{P}$. From Remark 11.7, we can prove the following statement:

**Claim.** *There exists an open neighborhood $V$ of $\{\tilde{\varepsilon} = \widetilde{\mathcal{A}}_r = 0\}$ in $\mathbb{R}^+ \times \mathbb{R}^n$ and a unique $C^\infty$ function*

$$\widetilde{\mathcal{A}}_0 = \mathfrak{a}_0(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_1, \dots, \widetilde{\mathcal{A}}_{n-1})$$

*defined on $V \cap \{\widetilde{\mathcal{A}}_0 = 0\}$, with $\mathfrak{a}_0(0,0) = 0$, such that*

$$S := \operatorname{graph}\{\widetilde{\mathcal{A}}_0 = \mathfrak{a}_0(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_1, \dots, \mathcal{A}_{n-1})\} \quad \textit{on } V.$$

Indeed, since $\mathbf{a}_s(0,0) = 0$ and $\widetilde{P}(0,0) = 0$ (see Remark 11.7), we know that the point $(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_r) = (0,0)$ is contained in $S$. Moreover,

$$\frac{\partial}{\partial \mathcal{A}_0}(\widetilde{\mathcal{A}}_0 - \mathbf{a}_s \circ \widetilde{P})(0,0) = 1.$$

Thus, it suffices to use the Implicit Function Theorem to define $\mathfrak{a}_0$. This proves the claim.

Since $V$ is an open neighborhood of the origin, there exist some strictly positive integers $r$, $k_i$ $(0 \le i \le n-1)$ such that the open $(n+1)$-cube,

$$\widetilde{B} := \{0 < |\varepsilon| < r,\ |\widetilde{\mathcal{A}}_i| < k_i\ (0 \le i \le n-1)\} \tag{73}$$

is such that $V \cap \{\widetilde{\mathcal{A}}_0 = 0\}$ contains $\widetilde{B}_0 := \widetilde{B} \cap \{\widetilde{\mathcal{A}}_0 = 0\}$ and

$$\operatorname{graph}\{\mathfrak{a}_0|_{\widetilde{B}_0}\}$$

is strictly contained in $\widetilde{B}$. Thus, to conclude, it suffices to define $B := \psi(\widetilde{B})$,

$$O_{\widetilde{\Gamma}} := \operatorname{graph}\{\mathcal{A}_0 = \varepsilon^{2p-s}\mathfrak{a}_0 \circ \psi^{-1}(\varepsilon, \mathcal{A}_1, \dots, \mathcal{A}_{n-1})\}, \tag{74}$$

and $\mathbf{w}(x, \varepsilon, \mathcal{A}_r) := w(x, P(\varepsilon, \mathcal{A}_r))$. This proves the theorem. □

*Remark* 11.9. Similarly to Subsection 10.4, the above proof provides a natural $C^\infty$ parameterization for $O_{\widetilde{\Gamma}}$: If we let $\widetilde{B}$ be the cube defined on (73), and define the map

$$\widehat{\psi}(\tilde{\varepsilon}, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}) := \psi(\tilde{\varepsilon}, \mathcal{A}_0, \dots, \mathcal{A}_{n-1})|_{\mathcal{A}_0 = \mathfrak{a}_0(\tilde{\varepsilon}, \widetilde{\mathcal{A}}_1, \dots, \mathcal{A}_{n-1})}$$

with domain $\widetilde{B}_0 := \widetilde{B} \cap \{\mathcal{A}_0 = 0\}$, then

$$\widetilde{B}_0 \ni (\tilde{\varepsilon}, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}) \xmapsto{\widehat{\psi}} (\varepsilon, \mathcal{A}_r) \in O_{\widetilde{\Gamma}}$$

defines a $C^\infty$ injective parameterization of $O_{\widetilde{\Gamma}}$.

## 12. Some examples worked out

### 12.1. An example local study.

Let $X_{\varepsilon,\alpha}$ be given by

$$X_{\varepsilon,\alpha} = \varepsilon \frac{\partial}{\partial x} + \left[\alpha x^s + x^p y(1+O(x)) + O(y^2)\right] \frac{\partial}{\partial y},$$

where $\alpha \in (\mathbb{R}, 0)$ is a one-dimensional control parameter, $p \geq 1$ is odd and $0 \leq s \leq 2p-1$ is an even index. Then, the local canard problem has a positive answer.

Indeed, there exists a transversal family $X_{\varepsilon,a,b,\alpha}$ which induces $X_{\varepsilon,a}$, with the induction map given by

$$P(\varepsilon,\alpha) : \begin{cases} a_i = 0, & 0 \leq i \leq 2p-1, \quad i \neq s, \\ b_j = 0, & 0 \leq j \leq p-1, \quad \text{and} \\ a_s = \alpha. \end{cases}$$

Notice that $P$ clearly satisfies the Improved Transversality Hypothesis.

Let us consider a sufficiently small interval $U_x \subset \mathbb{R}$ such that $x = 0$ is the unique degenerate singular point of $X_{\varepsilon,\alpha}$ on

$$\Gamma = \{y = \varepsilon = \alpha = 0,\, x \in U_x\},$$

and let us fix arbitrary initial condition functions $i_0(\varepsilon,\alpha)$ and $i_1(\varepsilon,\alpha)$ at points $x_0, x_1 \in U_x$, with $x_0 < 0 < x_1$.

Now, it suffices to apply Theorem 11.8. It follows that there exists an open interval $I = (0, \varepsilon_0)$, a unique $C^\infty$ function

$$\alpha = \alpha(\varepsilon)$$

defined on $I$, and an unique $C^\infty$ function $w(x,\varepsilon)$ defined on $[x_0, x_1] \times I$ such that the surface

$$W = \mathrm{graph}\{y = w(x,\varepsilon)\}$$

is an invariant surface for the vector field $X_{\varepsilon,\alpha(\varepsilon)}$, with the boundary conditions

$$w(x_k, \varepsilon, \alpha(\varepsilon)) = i_k(\varepsilon, \alpha(\varepsilon)),$$

for $k = 0, 1$.

Using Remark 11.9, we can also consider the $C^\infty$ parameterization of the canard region, which is given by

$$(\mathbb{R}^+, 0) \ni \rho \mapsto (\varepsilon, \alpha) = (\rho^{p+1}, \mathbf{a}(\rho)) \in (\mathbb{R}^+, 0) \times (\mathbb{R}, 0),$$

for some uniquely defined $C^\infty$ function $\mathbf{a}\colon (\mathbb{R}^+,0) \to (\mathbb{R},0)$. Thus, the surface

$$W = \operatorname{graph}\{y = w(x, \rho^{p+1})\}$$

is a canard surface for the restricted vector field $X^\gamma_\rho = X_{\varepsilon(\rho),\alpha(\rho)}$, which is given by

$$X^\gamma_\rho := \rho^{p+1}\frac{\partial}{\partial x} + \left[\mathbf{a}(\rho)x^s + x^p y(1+O(x)) + O(y^2)\right]\frac{\partial}{\partial y}.$$

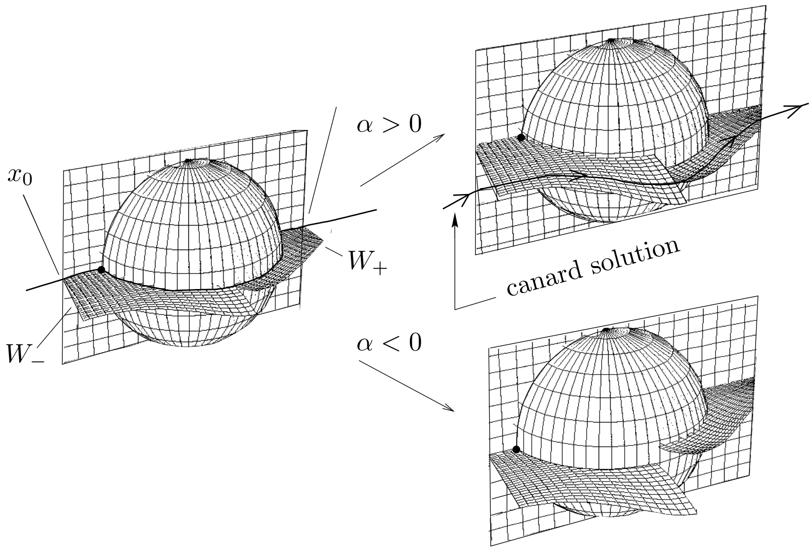



FIGURE 17. The control parameter $\alpha$ is a function of $\varepsilon$.

**12.2. An example of global study.**

Let us consider a global singular perturbation family on the cylinder $E = \mathbb{S}^1 \times \mathbb{R}$, given by

$$X_{\varepsilon,\alpha} = \varepsilon\frac{\partial}{\partial\theta} + \big(\alpha\cos(\theta) + r\,\sin(\theta) + r^2 H(r,\theta,\varepsilon,\alpha)\big)\frac{\partial}{\partial r}$$

where $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R}$, $(\varepsilon,\alpha) \in U_\varepsilon \times U_\alpha$ (with $U_\varepsilon \in (\mathbb{R}^+,0)$ and $U_\alpha \in (\mathbb{R},0)$) and $H$ is an arbitrary analytic function.

For the parameter values $\{\varepsilon = \alpha = 0\}$, the set

$$\Gamma = \{r = 0\} \approx \mathbb{S}^1$$

is a curve of singularities of $X_{\varepsilon,\alpha}$. Let us prove the following statement: There exists a $C^\infty$ curve $\gamma$ on the parameter space, having the form

$$\begin{aligned} \gamma\colon [0,\delta) &\longrightarrow \mathbb{R}^+ \times \mathbb{R} \\ \rho &\longmapsto (\varepsilon,\alpha) = (\rho^2, \alpha(\rho)) \quad \text{where } \alpha(0) = 0, \end{aligned}$$

such that, for each $\rho \neq 0$, the restricted family $X_{\rho^2,\alpha(\rho)}$ has a limit cycle $l_\rho$, and

$$l_\rho \longrightarrow \Gamma \quad \text{as } \rho \to 0$$

(for the Hausdorff metric on compact sets).

*Remark* 12.1. In the Bifurcation Theory terminology, this implies the *cyclicity* of $\Gamma$ (in the family $X_{\varepsilon,\alpha}$) is strictly positive.

To prove such statement, it suffices to prove that the (global) canard problem has a positive answer for such family.

Notice that all singularities on $\Gamma \setminus \{\theta = 0, \pi\}$ are non-degenerate, and both degenerate points $\theta = 0$ and $\theta = \pi$ have multiplicity $p = 1$. The first point $\theta = 0$ is in the $(\mathbf{s}, \mathbf{u})$ case (i.e. presents an stable-unstable transition), while $\theta = \pi$ is in the $(\mathbf{u}, \mathbf{s})$ case (i.e. an unstable-stable transition).

Let us firstly study the point $\theta = \pi$. Locally at this point, the family $X$ can be embedded into the local transversal family

$$\mathfrak{X}_\pi = \varepsilon \frac{\partial}{\partial x} + \big(a_0 + a_1 x + O(x^2) + y(b_0 - x + O(x^2)) + y^2 H\big) \frac{\partial}{\partial y}$$

where $x \in U_x \in (\mathbb{R}, 0)$, $y \in \mathbb{R}$ and $(a_0, a_1, b_0) \in (\mathbb{R}^3, 0)$. From Theorem 10.3, for each compact connected segment $\widetilde{\Gamma} \subset U_x$, such family has a canard region of the form

$$\mathcal{O}_{\widetilde{\Gamma}} = \begin{cases} 0 < \varepsilon < r \\ |\alpha| < s \\ |a_0|^2 < k_0\, |\varepsilon|^2 \\ |a_1|^2 < k_1\, |\varepsilon| \\ |b_0|^2 < l_0\, |\varepsilon| \end{cases}$$

for some strictly positive constants $r$, $s$, $k_0$, $k_1$, $l_0$.

To compute the induction map $P_\pi(\varepsilon,\alpha) = (\varepsilon, a, b, \alpha)$ (which realizes $X$ a sub-family of $\mathfrak{X}_\pi$) it suffices to compute the Taylor expansion of $\sin(\theta)$ and $\cos(\theta)$ at $\theta = \pi$. This gives

$$P_\pi \colon (\varepsilon,\alpha) \longmapsto (\varepsilon, a_0, a_1, b_0, \alpha) = (\varepsilon, -\alpha, 0, 0, \alpha).$$

This induction map clearly satisfies the Asymptotic Hypothesis (see Subsection 11.2), which implies that the pull-back of $\mathcal{O}_{\widetilde{\Gamma}}$ under $P_\pi$ is a nonempty semi-analytic set. Explicitly, it is given by

$$O_{\widetilde{\Gamma},\pi} := P_\pi^{-1}(\mathcal{O}_{\widetilde{\Gamma}}) = \begin{cases} 0 < \varepsilon < R \\ |\alpha|^2 < k_0\,|\varepsilon|^2 \end{cases}$$

(where $R = \min\{r, s\}$).

Thus, there exists a $C^\infty$ function $\mathbf{w}(\theta,\varepsilon,\alpha)$ defined on the region $\widetilde{\Gamma} \times O_{\widetilde{\Gamma},\pi}$ such that

$$W = \operatorname{graph}\{r = \mathbf{w}(\theta,\varepsilon,\alpha)\}$$

defines an invariant surface for $X_{\varepsilon,\alpha}$ over $\widetilde{\Gamma}$. Let us denote by $\theta_0, \theta_1 \in \mathbb{S}^1$ the two endpoints of $\widetilde{\Gamma}$, where

$$\theta_0 \in (-\pi, 0) \quad \text{and } \theta_1 \in (0,\pi).$$

If we define $I_k(\varepsilon,\alpha) := \mathbf{w}(\theta_k,\varepsilon,\alpha)$ ($k = 0,1$), it follows from Whitney Extension Theorem that such functions can be extended to all $(\varepsilon,\alpha) \in U_\varepsilon \times U_\alpha$. That is, there exist $C^\infty$ functions $i_0(\varepsilon,\alpha)$, $i_1(\varepsilon,\alpha)$ in $C^\infty(U_\varepsilon \times U_\alpha)$ such that

$$i_0|_{O_{\widetilde{\Gamma},\pi}} = I_0 \quad \text{and} \quad i_1|_{O_{\widetilde{\Gamma},\pi}} = I_1.$$

Moreover, using a similar argument to the one used in the proof of Lemma 11.4, we can prove that $i_0$ and $i_1$ can be chosen as *initial condition functions* for $X$, at $x_0$ and $x_1$ respectively (because the sets $U = O_{\widetilde{\Gamma},\pi}$ and $V = \{\varepsilon = 0\}$ are *regularly situated*).

Let us now pass to the point $\theta = 0$. At this point, the family $X$ can be locally embedded into the transversal family

$$\mathfrak{X}_0 = \varepsilon\frac{\partial}{\partial x} + \left(a_0 + a_1 x + O(x^2) + y(b_0 + x + O(x^2)) + y^2 H\right)\frac{\partial}{\partial y}.$$

The induction map is given by

$$P_0 \colon (\varepsilon,\alpha) \longmapsto (\varepsilon, a_0, a_1, b_0, \alpha_0) = (\varepsilon, \alpha, 0, 0, \alpha),$$

and it clearly satisfies the Improved Transversality Hypothesis.

Therefore, we can apply Theorem 11.8 using the above functions $i_0$ and $i_1$ above as initial condition functions. It follows that there exists an open interval $B_0 = (0, \varepsilon_0)$ and a unique $C^\infty$ function $a(\varepsilon)$ defined on $B_0$ such that the canard region at $\theta = 0$ is given by

$$O_0 = \text{graph}\{\alpha = a(\varepsilon)\}.$$

To conclude, we have to prove (possibly taking restrictions to some smaller neighborhood of $\{\varepsilon = 0\}$), that the condition

$$O_0 \subset O_{\widetilde{\Gamma},\pi} \tag{75}$$

is verified. Indeed, such condition proves the existence of a global invariant surface $W = \{r = \mathbf{w}(\theta, \varepsilon, \alpha)\}$, with domain $\mathbb{S}^1 \times O_0$.

To prove (75), it is easier to consider the parameterization of $O_0$ which is given by the blowing-up $\psi$. From Remark 11.9, we can write $O_0$ as the image of a $C^\infty$ curve

$$(0, \delta) \ni \rho \overset{\gamma}{\longmapsto} (\rho^2, \alpha(\rho)) \in U_x \times U_\alpha$$

where $\alpha(\rho)$ has the form

$$\alpha(\rho) = \rho^2 \mathfrak{a}_0(\rho)$$

(see (74)), for some $C^\infty$ function $\mathfrak{a}_0(\rho)$ with $\mathfrak{a}_0(0) = 0$. Thus, it follows that $\alpha(\rho) = O(\rho^3)$ and the curve $\gamma(\rho)$ is clearly contained in the canard region $O_\pi$ for all $\rho > 0$ sufficiently small. This proves (75).

## 13. Appendix on normal forms

### 13.1. Normal form near semi-hyperbolic singularities.

First of all, we state the following special case of a result from [**B**] (the enunciate below is taken from [**Du-R**]).

**Theorem 13.1.** *Let $X(x, y, z, \mathcal{A})$ be a $C^\infty$ vector field on $\mathbb{R}^3 \times \mathbb{R}^n$ (where $\mathcal{A} = (a_0, \ldots, a_{n-1})$), having the following properties:*

- (i) *$X$ has the functions $g_i = a_i$ (for $0 \le i \le n-1$) and $F(x, z) = x^p z^q$ (where $p, q \in \mathbb{N} \setminus \{0\}$ are relatively prime) as first integrals;*
- (ii) *$DX_0$ has exactly one non-zero eigenvalue, and the related eigenspace is given by $\{x = z = \mathcal{A} = 0\}$.*

*Let $W$ be a $C^k$ center manifold of $X$ at $0$, for some $k \ge 1$. Then, there exists a local $C^k$ change of coordinates $\psi$ of the form*

$$\psi \colon (x, y, z, \mathcal{A}) \longmapsto (\phi_1(x, y, z, \mathcal{A}), \phi_2(x, y, z, \mathcal{A}), \phi_3(x, y, z, \mathcal{A}), \mathcal{A})$$

*with*

$$F(\phi_1(x, y, z, \mathcal{A}), \phi_2(x, y, z, \mathcal{A})) = F(x, z) \quad \textit{and} \quad \phi(W) = \{y = 0\},$$

*and a strictly positive $C^k$ function $f(x,y,z,\mathcal{A})$ such that*

$$[f\cdot\phi_*X](x,y,z,\mathcal{A}) = \pm y\frac{\partial}{\partial y} + Y(x,z,\mathcal{A}) \tag{76}$$

*with $Y$ being a vector field of class $C^k$, such that $Y(g_i)=Y(F)=0$.*

If the above equality holds, we shall say that the germ of $X$ at the origin in $C^k$*-equivalent* to the expression on the right hand side of (76).

The following results are immediate consequences of this theorem. Below, given a real number $\lambda \neq 0$, we shall denote by

$$\operatorname{sgn}(\lambda) := \frac{\lambda}{|\lambda|} \in \{-1,1\}$$

its corresponding sign.

**Corollary 13.2.** *Let $X_{\varepsilon,a}$ be a germ of singular perturbation of transition type, and let $x\in\Gamma$ be a non-degenerate point (i.e. a semi-hyperbolic singularity of $X_{0,0}$). Then, the germ of $X_{\varepsilon,a}$ at $p$ is $C^k$-equivalent to*

$$Y = \varepsilon\frac{\partial}{\partial x} + s\cdot y\frac{\partial}{\partial y}$$

*where $s=\operatorname{sgn}(B_x)$ ($B_x$ being the non-zero eigenvalue of $DX_{0,0}$ at $x$).*

*Proof:* We refer the reader to [**Du-R**, Proposition 4]. □

Let us now suppose that $x\in\Gamma$ is a degenerate point of $X_{\varepsilon,a}$ and that the Transversality Hypothesis holds at $x$. Then we can also find normal forms for the points which are contained in the semi-hyperbolic set $P_-$ and $P_+$ (see (31)).

**Corollary 13.3.** *Let $p_-\in P_-$ be an arbitrary normal hyperbolic singularity. Then, the germ of $\overline{X}$ at $p_-$ is $C^k$-equivalent to*

$$Y_- = (s_-)\cdot y\frac{\partial}{\partial y} + \mathrm{e}^{p+1}f(x,y,\mathrm{e},\mathcal{A})\left(-x\frac{\partial}{\partial x}+\mathrm{e}\frac{\partial}{\partial \mathrm{e}}\right)$$

*for $s_- := \operatorname{sgn}(\mathcal{B}_-)$ and some strictly positive $C^k$ function $f$. Similarly, the germ of $\overline{X}$ at an arbitrary point $p_+\in P_+$ is $C^k$-equivalent to*

$$Y_+ = (s_+)\cdot y\frac{\partial}{\partial y} + \mathrm{e}^{p+1}f(x,y,\mathrm{e},\mathcal{A})\left(x\frac{\partial}{\partial x}-\mathrm{e}\frac{\partial}{\partial \mathrm{e}}\right)$$

*for some strictly positive $C^k$ function $f$ and $s_+ := \operatorname{sgn}(\mathcal{B}_+)$.*

*Proof:* The proof is obtained by easy modifications on the proof of [**Du-R**, Proposition 5]. □

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