A NOTE ON INVERSE LIMITS
OF CONTINUOUS IMAGES OF ARCS

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Abstract

The main purpose of this paper is to prove some theorems concerning inverse systems and limits of continuous images of arcs. In particular, we shall prove that if \( X = \{X_a, p_{ab}, A\} \) is an inverse system of continuous images of arcs with monotone bonding mappings such that \( \text{cf}(\text{card}(A)) \neq \omega_1 \), then \( X = \lim X \) is a continuous image of an arc if and only if each proper subsystem \( \{X_a, p_{ab}, B\} \) of \( X \) with \( \text{cf}(\text{card}(B)) = \omega_1 \) has the limit which is a continuous image of an arc (Theorem 18).

1. Inverse limits of hereditarily locally connected continua

An arc (or ordered continuum) is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval \( I = [0, 1] \).

A space \( X \) is said to be an IOK (IOC) if there exists an ordered compact (connected) space \( K \) and a continuous surjection \( f : K \to X \). Frequently, we will say that a space \( X \) is a continuous image of an arc if \( X \) is an IOC.

The cardinality of a set \( A \) will be denoted by \( \text{card}(A) \). We assume that \( \text{card}(A) \) is the initial ordinal number. The cofinality of a cardinal number \( m \) will be denoted by \( \text{cf}(m) \).

Keywords. Inverse system and limit, continuous image of an arc.
A continuum $X$ is said to be hereditarily locally connected if each subcontinuum of $X$ is locally connected. A continuum $X$ is said to be finitely Suslinian \cite{17} if there do not exist open sets $U$ and $V$, and an infinite collection $\mathcal{K}$ of pairwise disjoint subcontinua of $X$ such that $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$ and $K \cap V \neq \emptyset$ and $K \cap U \neq \emptyset$ for each $K$ in $\mathcal{K}$. Each finitely Suslinian continuum is hereditarily locally connected. A continuum $X$ is said to be rim-finite (rim-countable) if it has a basis $\mathcal{B}$ such that $\text{card}(\text{Bd}(U)) < \aleph_0$ (card(Bd(U)) $\leq \aleph_0$) for each $U \in \mathcal{B}$. Each rim-finite continuum is finitely Suslinian. Each hereditarily locally connected continuum is a continuous image of an arc \cite[Theorem 3.4]{11}.

In the paper \cite[Problem 9.10]{8} the authors asked when the inverse limit of an inverse system $X = \{X_a, p_{ab}, A\}$ of hereditarily locally connected continua with monotone surjective bonding mappings $p_{ab}$ is a continuous image of an arc.

If $X = \{X_a, p_{ab}, A\}$ is an inverse system of hereditarily locally connected continua, then $X = \lim X$ need not be a hereditarily locally connected continuum since each locally connected metric continuum of dimension 1 (= curve) is the limit of an inverse sequence of rim-finite continua with surjective monotone bonding mappings \cite[Theorem 2.2]{13}.

In the present section we shall define a class of hereditarily locally connected continua such that each inverse limit of such spaces and monotone bonding mappings has a hereditarily locally connected limit.

In Appendix we review some definitions and known results needed in this section.

We say that an inverse system $Y = \{Y_a, q_{ab}, B\}$ is a subsystem of $X = \{X_a, p_{ab}, A\}$ if $B \subset A$, $Y_a = X_a$ and each $q_{ab}$ is $p_{ab}$.

We start with the following theorem.

**Theorem 1.** Let $X$ be the limit of an inverse system $X = \{X_a, p_{ab}, A\}$ of hereditarily locally connected continua $X_a$ such that the bonding mappings $p_{ab} : X_b \rightarrow X_a$ are monotone surjections. Then $X$ is a hereditarily locally continuum if and only if each countable inverse subsystem of $X$ has a hereditarily locally connected limit.

**Proof:** By Theorem 29 $X$ is homeomorphic to the limit of $X_\sigma = \{X_\Delta, p_{\Delta}, A_\sigma\}$, where $A_\sigma$ is the family of all nonempty countable directed subsets of $A$. If $X$ is hereditarily locally connected then each $X_\Delta$ is hereditarily locally connected as a monotone image of a hereditarily locally connected continuum $X$ (see Lemma 28). Conversely, if each $X_\Delta$ is hereditarily locally connected, then $X$ is hereditarily locally connected (Theorem 27). $\blacksquare$
The following theorem is a generalization of the well-known result of G. T. Whyburn [20, p. 81] which asserts that a metric continuum $X$ is hereditarily locally connected if and only if each cyclic element (see Appendix) $Z \subseteq X$ is hereditarily locally connected.

**Theorem 2.** A locally connected continuum $X$ is hereditarily locally connected if and only if each cyclic element of $X$ is hereditarily locally connected.

**Proof:** By Theorems 31 and 29 there exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ of metric locally connected spaces such that $p_{ab}$ are monotone and $X$ is homeomorphic to $\lim X$. Let us prove that each $X_a$ is hereditarily locally connected. It suffices to prove that each cyclic element $Z_a$ of $X_a$ is hereditarily locally connected. By Lemma 34 there exists a cyclic element $Z$ of $\lim X$ such $p_a(Z) \supseteq Z$. Since $\lim X$ is homeomorphic to $X$, $Z$ is hereditarily locally connected. This means that $p_a(Z)$ is hereditarily locally connected. It follows that $Z_a$ is hereditarily locally connected since $Z_a \subseteq p_a(Z)$. We infer that each $X_a$ is hereditarily locally connected since $X_a$ is a metric continuum. From Theorem 27 it follows that $X$ is hereditarily locally connected. \hfill \square

A surjective mapping $f : X \to Y$ is said to be **hereditarily monotone** [5, pp. 16–17] if for each subcontinuum $K$ of $X$ the restriction $f \mid K : K \to f(K)$ is monotone.

If $f : X \to Y$ and $g : Y \to Z$ are hereditarily monotone mappings, then $gf : X \to Z$ is hereditarily monotone [5, p. 29, (5.3)].

**Lemma 3.** If $Z$ is a cyclic element of $X$, then the canonical retraction $\rho : X \to Z$ is hereditarily monotone.

**Proof:** Let $\rho : X \to Z$ (see Appendix, p. 13) and let $K$ be any subcontinuum of $X$. Let us prove that the restriction $\rho_K : K \to \rho_K(K) \subseteq Z$ is monotone. Consider the following cases.

a) $K \subseteq Z$. Now $\rho_K$ is the identity and is monotone.

b) $K \subseteq X \setminus Z$. It is clear that $K$ is a subset of some component $J$ of $X \setminus Z$. Let $\{z_J\} = \text{Bd}(J)$. Now, $\rho_K(K) = \{z_J\}$. This means that $\rho_K^{-1}(z_J) = K$. Hence $\rho_K$ is monotone.

c) $K \cap Z \neq \emptyset$ and $K \cap (X \setminus Z) \neq \emptyset$. In this case

$$K = \left(K \cap Z\right) \cup \left\{K \cap J : J \text{ is a component of } X \setminus Z, K \cap J \neq \emptyset\right\}.$$
We infer that
\[ \rho_K(K) = \left( K \cap Z \right) \cup \{ \{ z_J \} : J \text{ is a component of } X \setminus Z, \{ z_J \} = \text{Bd}(J) \}. \]

If \( x \in \rho_K(K) \) such that \( x \neq z_J \) for all components \( J \) of \( X \setminus Z \), then \( \rho_K^{-1}(x) = \{ x \} \). Hence \( \rho_K^{-1}(x) \) is connected. If \( x = z_J \) for some component \( J \), then
\[ \rho_K^{-1}(x) = \{ z_J \} \cup \left\{ K \cap J_i : \{ z_J \} = \text{Bd}(J_i), i \in I \right\}. \]

It suffices to prove that each set \( \text{Cl}(J_i) \cap K \) is connected. Suppose that \( \text{Cl}(J_i) \cap K \) is not connected. There exists a component \( L \) of \( \text{Cl}(J_i) \cap K \) such that \( z_J \notin L \). By virtue of the normality of \( X \) it follows that there exists a pair \( U, V \) of disjoint open sets such that \( L \subseteq U \) and \( z_J \in V \). We may assume that \( \text{Cl}(U) \subseteq J \) since \( J \) is open. For each point \( x \) of \( K \setminus (U \cup V) \) there exists an open set \( U_x \) such that \( x \in U_x \) and \( U_x \cap U = \emptyset \). Let \( W \) be the union of \( V \) and all the sets \( U_x, x \in K \setminus (U \cup V) \). It is clear that \( U \cap W = \emptyset \) and \( K \subseteq U \cup W \). This is impossible since \( K \) is connected. Hence, \( \text{Cl}(J_i) \cap K \) is connected and \( \rho_K \) is monotone.

**Theorem 4.** Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system of continua and hereditarily monotone bonding mappings. Then the projections \( p_a, a \in A \), are hereditarily monotone. Moreover, if each \( X_a \) is hereditarily locally connected, then \( X = \lim X \) is hereditarily locally connected.

**Proof:** Let \( K \) be a subcontinuum of \( X \). For each \( a \in A \), \( K_a = p_a(K) \) is a subcontinuum of \( X_a \). We have the inverse system \( K = \{ K_a, p_{ab} | K_b, A \} \). From the definition of hereditarily monotone mapping it follows that each mapping \( p_{ab} | K_b \) is a monotone mapping. This means that the projections \( p_a | K \) are monotone [2, pp. 462–463]. From [2, Theorem 6.1.28] it follows that \( K \) is locally connected. Thus, \( X \) is hereditarily locally connected.

A surjective mapping \( f : X \to Y \) is said to be **cyclically hereditarily monotone** if for each cyclic element \( Z \) of \( X \) the restriction \( f | Z \) is hereditarily monotone.

**Theorem 5.** Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system of hereditarily locally connected continua \( X_a \) and cyclically hereditarily monotone bonding mappings \( p_{ab} \). Then \( X = \lim X \) is hereditarily locally connected.
Proof: By Theorem 2 it suffices to prove that each cyclic element $Z$ of $X$ is hereditarily locally connected. There exists an inverse system (Theorem 35) $\mathcal{Z} = \{Z_a, g_{ab}, A\}$ such that $Z_a$ is a cyclic element of $X_a$, $g_{ab} = \rho_a \circ (f_{ab} | Z_b)$ for all $a \leq b \in A$, and $Z$ is homeomorphic to $\lim Z$. This means that $g_{ab}$ is hereditarily monotone since $\rho_a$ is hereditarily monotone (Lemma 3). From Theorem 4 it follows that $Z$ is hereditarily locally connected. Thus, $X$ is hereditarily locally connected.

A space $X$ is in class $\mathcal{H}_m$ if $X$ is a hereditarily locally connected continuum and $X$ contains no non-degenerate metric subcontinuum. Each space in class $\mathcal{H}_m$ is rim-finite [19, Theorem 1].

**Theorem 6.** Let $X = \{X_n, p_{mn}, N\}$ be an inverse sequence with monotone surjective bonding mappings. If each $X_n$ is in class $\mathcal{H}_m$, then $X = \lim X$ is in class $\mathcal{H}_m$. Moreover, $X$ is rim-finite.

Proof: Each $X_n$ is a continuous image of an arc [11, Corollary 3.5] since each $X_n$ is hereditarily locally connected. Thus, $X$ is a continuous image of an arc (Theorem 14). Let us prove that $X$ contains no non-degenerate metric subcontinuum. Suppose that $Y$ is a non-degenerate metric subcontinuum of $X$. Then there exists a $n \in N$ such that $p_m(Y)$ is a non-degenerate metric subcontinuum of $X_m$ for each $m \geq n$. This is impossible since $X_m$ is in class $\mathcal{H}_m$. We infer that $X$ contains no non-degenerate metric subcontinua. By virtue of Theorem 21 it follows that $X$ is hereditarily locally connected. Moreover, from Theorem 1 of [19] it follows that $X$ is rim-finite.

**Theorem 7.** Let $X = \{X_a, p_{ab}, A\}$ be an inverse system with monotone surjective bonding mappings. If each $X_a$ is in class $\mathcal{H}_m$, then $X = \lim X$ is in class $\mathcal{H}_m$. Moreover, $X$ is rim-finite.

Proof: Apply Theorem 1 and Theorem 23.

A hereditarily locally connected continuum $X$ is in class $\mathcal{H}_{zm}$ if each cyclic element $Z$ of $X$ is in class $\mathcal{H}_m$.

**Theorem 8.** Let $X = \{X_a, p_{ab}, A\}$ be an inverse system with monotone surjective bonding mappings. If each $X_a$ is in class $\mathcal{H}_{zm}$, then $X = \lim X$ is in class $\mathcal{H}_{zm}$.

Proof: Let $Z$ be a cyclic element of $X$. By Theorem 35 there exists an inverse system $(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$ such that $Z_\gamma$ is homeomorphic to $\lim \text{inv}(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$, where $Z_\gamma$ is a cyclic element of $X_\gamma$ and $g_{\gamma\gamma'}$ is monotone. By Theorem 7 $Z$ is in class $\mathcal{H}_m$. From Theorem 2 it follows that $X$ is hereditarily locally connected. Hence, $X$ is in class $\mathcal{H}_{zm}$.
2. Special classes of continuous images of arcs

Theorem 9 [9]. Let $X$ be a locally connected continuum such that for each pair of distinct points $a$, $b$ in $X$, there exists a continuous onto map $f : X \rightarrow [c,d]$ such that $f(a) = c$ and $f(b) = d$ and $[c,d]$ is a non-metrizable arc. If $X$ is rim-metrizable or rim-scattered or monotonically normal, then $X$ is a continuous image of an arc.

A locally connected continuum is said to be a $\textit{NTT-space}$ if for each pair of distinct points $a$, $b$ in $X$, there exists a continuous onto map $f : X \rightarrow [c,d]$ such that $f(a) = c$ and $f(b) = d$ and $[c,d]$ is a non-metrizable arc.

Theorem 10. Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of $\textit{NTT}$-spaces with monotone surjective bonding mappings. Then $X = \lim X$ is a $\textit{NTT}$-space.

Proof: It is known that $X$ is locally connected continuum. Let $x$, $y$ be a pair of distinct points of $X$. There exists an $a \in A$ such that $p_b(x) \neq p_b(y)$ for each $b \geq a$. Since $X_b$ is a $\textit{NTT}$-space there exists a non-metrizable arc $[c,d]$ and a surjective mapping $f : X_b \rightarrow [c,d]$ such that $f(x) = c$ and $f(y) = d$. Considering the mapping $f_{ab} : X \rightarrow [c,d]$ we infer that $X$ is a $\textit{NTT}$-space.

Theorem 11. Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of $\textit{NTT}$-spaces and monotone surjective bonding mappings. If $X = \lim X$ is rim-metrizable or rim-scattered or monotonically normal, then $X$ is a continuous image of an arc.

Proof: By Theorem 10 $X$ is a $\textit{NTT}$-space. Apply Theorem 9.

Theorem 12. Let $X = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed inverse system of spaces $X_a$ such that for each pair $x_a$, $y_a$ of points of $X_a$ the subspace $X_a \setminus \{x_a, y_a\}$ is connected, $a \in A$. If each $X_a$ is a continuous image of an arc and each $p_{ab}$ is a monotone surjection, then $X = \lim X$ is a continuous image of an arc if and only if there exists an $a \in A$ such that $p_b : X \rightarrow X_b$ is a homeomorphism for each $b \geq a$ (if and only if $X$ is metrizable).

Proof: By Theorem 2 of [18] each $X_a$ is metrizable. We shall prove that for each pair $x$, $y$ of points of $X$ the subspace $Y = X \setminus \{x,y\}$ is connected. Suppose that $Y$ is not connected. Then there exists a pair $U$, $V$ of disjoint open subsets of $X$ such that $Y = U \cup V$. Moreover,
there exists an \( a \in A \) such that \( p_b(x) \neq p_b(y) \), \( b \geq a \). The sets \( U_a = \{ x_a : x_a \in X_a, p_a^{-1}(x_a) \subset U \} \) and \( V_a = \{ x_a : x_a \in X_a, p_a^{-1}(x_a) \subset V \} \) are disjoint and open. Now we have \( X \setminus \{ p_a(x), p_a(y) \} = U_a \cup V_a \). This is impossible since \( X \setminus \{ p_a(x), p_a(y) \} \) is connected. Hence, \( Y = X \setminus \{ x, y \} \) is connected. It follows that if \( X \) is a continuous image of an arc, then it is metrizable ([18, Theorem 2]). From Theorem 26 it follows that there exists a \( b \in A \) such that \( p_b : X \rightarrow X_b \) is a homeomorphism for every \( c \geq b \). Conversely, if such \( b \in A \) exists, then \( X \) is a continuous image of an arc.

**Theorem 13.** Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system of spaces \( X_a \) such that for each pair \( x_a, y_a \) of points of \( X_a \) the subspace \( X_a \setminus \{ x_a, y_a \} \) is connected, \( a \in A \). If each \( X_a \) is a continuous image of an arc and each \( p_{ab} \) is a monotone surjection, then \( X = \lim X \) is a continuous image of an arc if and only if there exists a countable subsystem \( Y \) of \( X \) such that \( \lim Y \) is homeomorphic to \( X \).

**Proof:** Consider the inverse system \( X = \{ X_a, p_{ab}, A \} \) from Theorem 29 and apply Theorem 12.

### 3. Inverse systems and subsystems

**Theorem 14** [8, Theorem 5.1]. Let \( X = \{ X_n, p_{mn}, \mathbb{N} \} \) be an inverse sequence with monotone surjective bonding mappings. If each \( X_n \) is the continuous image of an arc, then \( X = \lim X \) is the continuous image of an arc.

**Theorem 15** [4, Theorem 2.17]. Let \( X = \{ X_a, p_{ab}, A \} \) be a well-ordered inverse system such that \( \text{cf}(A) \neq \omega_1 \). If the mappings \( p_{ab} \) are monotone surjections and if the spaces \( X_a \) are the continuous images of arcs, then \( X = \lim X \) is the continuous image of an arc.

**Remark 16.** Theorem 15 is not true if \( \text{cf}(A) = \omega_1 \). This is shown by the following example of Nikkel [10]. Let \( L \) denote the long interval \([2, \infty)\). For each ordinal number \( \alpha, 0 < \alpha < \omega_1 \), let \( f_\alpha : [0, 1] \times L \rightarrow [0, 1] \times [0, \alpha] \) be defined by

\[
f(s, t) = \begin{cases} (s, t) & \text{if } t \leq_L \alpha \\ (s, \alpha) & \text{if } \alpha \leq_L t.
\end{cases}
\]

Each \( X_\alpha = [0, 1] \times [0, \alpha] \) is homeomorphic to \([0, 1] \times [0, 1]\) and it is a continuous image of an arc. Moreover, \( w(X_\alpha) = \aleph_0 \). Let \( f_{\alpha\beta} = f_\alpha | [0, 1] \times [0, \beta] \), \( \beta < \alpha \). We obtain an inverse system \( \{ X_\alpha, f_{\alpha\beta}, \alpha < \omega_1 \} \) whose limit is \([0, 1] \times L\) which is not a continuous image of an arc.
**Theorem 17.** Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces such that $\text{card}(A) > \aleph_0$. There exists a transfinite sequence $\{A_\alpha : \alpha < \text{card}(A)\}$ of directed subsets $A_\alpha$ of $A$ such that:

1. $\text{card}(A_\alpha) < \text{card}(A)$, $\alpha < \text{card}(A)$,
2. $\alpha < \beta < \text{card}(A)$ implies $A_\alpha \subseteq A_\beta$,
3. $A = \bigcup\{A_\alpha : \alpha < \text{card}(A)\}$,
4. each collection $\{X_a, p_{ab}, A_\alpha\}$ is an inverse system with limit $X_\alpha$,
5. if $\alpha < \beta < \text{card}(A)$ then there exists a mapping $q_{\alpha \beta} : X_\beta \to X_\alpha$,
6. $\lim X$ is homeomorphic to $\lim \{X_\alpha, q_{\alpha \beta}, \alpha < \beta < \text{card}(A)\}$,
7. if the mappings $p_{ab}$ are monotone, then the mappings $q_{\alpha \beta}$ are monotone.

**Proof:** The proof consists of several steps. Step 1 is from [7, pp. 238–239, Hilfssatz]. For the sake of the completeness we give the proof of Step 1.

**Step 1:** Let $\nu$ be any finite subset of $A$. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup \{\delta(\nu) : \nu \subset B \text{ and } \nu \text{ is finite}\}$. Put $F_{n+1} = F_1(F_n(B))$, and

$$F_\infty(B) = \bigcup\{F_n(B) : n \in \mathbb{N}\}.$$  

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \ldots \subseteq F_n(B) \subseteq \ldots$$

The set $F_\infty(B)$ is directed since each finite subset $\nu$ of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$. If $B$ is finite, then $\text{card}(F_\infty(B)) = \aleph_0$. If $\text{card}(B) \geq \aleph_0$, then we have $\text{card}(\{\delta(\nu) : \nu \in B\}) \leq \text{card}(B)\aleph_0$. We infer that $\text{card}(F_1(B)) \leq \text{card}(B)\aleph_0$. Similarly, $\text{card}(F_n(B)) \leq \text{card}(B)\aleph_0$. This means that $\text{card}(F_\infty(B)) \leq \text{card}(B)\aleph_0$. Thus

$$\text{card}(F_\infty(B)) \leq \text{card}(B)\aleph_0.$$  

Suppose that $\text{card}(A) > \aleph_0$. Put $\Omega = \text{card}(A)$. Hence, $A = \{a_\alpha : \alpha < \Omega\}$. Put $B_\alpha = \{a_\mu : \mu < \alpha < \Omega\}$. We have a transfinite sequence $\{B_\alpha : \alpha < \Omega\}$ such that

a) $\text{card}(B_\alpha) < \text{card}(A)$,
b) $\alpha < \beta < \Omega$ implies $B_\alpha \subseteq B_\beta$,
c) $A = \bigcup\{B_\alpha : \alpha < \Omega\}$.

Put $A_\alpha = F_\infty(B_\alpha)$.
Step 2: Assertions 1-3 follow from Step 1.

Step 3: Assertion 4 follows from the fact that each $A_\alpha$ is directed subset of $A$.

Step 4: Let us prove 5. From assertion 2 it follows that there exists a continuous mapping $q_{\alpha\beta} : X_\beta \rightarrow X_\alpha$ since each point $x \in X_\beta$ induces a collection $\{x_a : a \in A_\alpha\}$ which satisfies $p_{ab}(x_b) = x_a$, i.e., $\{x_a : a \in A_\alpha\}$ is a point of $X_\alpha$.

Step 5: It is obvious that there exists a mapping $H : \lim X \rightarrow \lim \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$ since each $x = (x_a : a \in A) \in \lim X$ induces a collection $\{x_a : a \in A_\alpha\}$ on each $A_\alpha$. Thus we have the mappings $H_\alpha : \lim X \rightarrow X_\alpha$, for each $\alpha < \text{card}(A)$. The mappings $H_\alpha$ induce a continuous mapping $H : \lim X \rightarrow \lim \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$. It remains to prove that $H$ is 1-1 and onto. Let us prove that $H$ is 1-1. Let $x, y \in \lim X$ and $x \neq y$. There exists an $a \in A$ such that $x_a \neq y_a$. From Step 1 it follows that there is an $A_\alpha$ such that $a \in A_\alpha$. Now, $x_a \neq y_a$ implies $H_\alpha(x) \neq H_\alpha(y)$ (see Step 5). This means that $H(x) \neq H(y)$. Hence $H$ is 1-1. In order to complete the proof it suffices to prove that $H$ is onto. Let $y = (y_\alpha : \alpha < \text{card}(A))$ be any point of $\lim \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\}$. Then $y_\alpha \in X_\alpha$. Thus, $y_\alpha$ is a thread in $X_\alpha$, i.e., $y = (x_a, a \in A_\alpha)$. We infer that for each $a \in A$ there exists a point $x_a \in X_\alpha$. It is readily to seen that $p_{ab}(x_b) = x_a$. Thus, $(x_a : a \in A)$ is a thread in $\lim X$ such that $H(x) = y$.

Step 6: Let us prove 7. If the mappings $p_{ab}$ are monotone, then from Lemma 28 it follows that the mappings $q_{\alpha\beta}$ are monotone.

Now we shall prove the main theorem of this section which is a generalization of Theorem 15.

**Theorem 18.** Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of continuous images of arcs with monotone bonding mappings. If $\text{cf}((\text{card}(A)) \neq \omega_1$, then $X = \lim X$ is a continuous image of an arc if and only if each proper subsystem $\{X_a, p_{ab}, B\}$ of $X$ with $\text{cf}(\text{card}(B)) = \omega_1$ has the limit which is a continuous image of an arc.

**Proof:** The “only if part”. If $X$ is a continuous image of an arc, then for each subsystem $\{X_a, p_{ab}, B\}$ there exists a natural projections $f_a : X \rightarrow \lim \{X_a, p_{ab}, B\}$. Hence, $\lim \{X_a, p_{ab}, B\}$ is a continuous image of an arc.
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**The “if” part.** By Theorem 17 there exists a well-ordered inverse system \( \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\} \) such that \( X \) is homeomorphic to \( \lim \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\} \). If \( \text{cf}(\text{card}(A)) \leq \omega_0 \), then we have an inverse subsequence of \( \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\} \) which is a cofinal subsystem of \( \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(A)\} \). By Theorem 14 \( X \) is a continuous image of an arc. Now, suppose that \( \text{cf}(\text{card}(A)) > \omega_1 \). By Theorem 17 it suffices to prove that each subsystem of \( \{X_\alpha, p_{\alpha\beta}, B\} \) of \( X = \{X_\alpha, p_{\alpha\beta}, A\} \) has the limit which is a continuous image of an arc.

We shall use the transfinite induction on \( \text{card}(B) \). If \( \text{card}(B) \leq \omega_0 \), then we use Theorem 14. If \( \text{card}(B) = \omega_1 \), then \( \text{lim} \{X_\alpha, p_{\alpha\beta}, B\} \) is a continuous image of an arc by assumption of Theorem. Let now \( \{X_\alpha, p_{\alpha\beta}, B\} \) be a subsystem of \( \{X_\alpha, p_{\alpha\beta}, A\} \) such that \( \text{card}(B) > \omega_1 \).

Suppose that Theorem is true for each subsystem of the cardinality \( < \text{card}(B) \). By Theorem 17 there exists a well-ordered inverse system \( \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(B)\} \) such that \( \text{lim} \{X_\alpha, p_{\alpha\beta}, B\} \) is homeomorphic to \( \text{lim} \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(B)\} \). Since each \( X_\alpha \) is the limit of a subsystem of the cardinality \( < \text{card}(B) \), we have the inverse system \( \{X_\alpha, q_{\alpha\beta}, \alpha < \beta < \text{card}(B)\} \) which satisfies the conditions of Theorem 15. Thus, \( \text{lim} \{X_\alpha, p_{\alpha\beta}, B\} \) is a continuous image of an arc. By the transfinite induction the proof is complete.

**Corollary 19.** Let \( X \) be a locally connected continuum. The following conditions are equivalent:

a) \( X \) is a continuous image of an arc.

b) If \( f : X \to Y \) is a continuous mapping and \( \text{cf}(\text{w}(Y)) = \omega_1 \), then \( Y \) is a continuous image of an arc.

**Proof:** a) \( \Rightarrow \) b). Obvious.

b) \( \Rightarrow \) a). By Theorem 31 there exists an inverse system \( X = \{X_\alpha, p_{\alpha\beta}, A\} \) such that \( X_\alpha \) are metric locally connected continua, \( p_{\alpha\beta} \) are monotone mappings and \( X \) is homeomorphic to \( \text{lim} X \). If \( Y = \{X_\alpha, p_{\alpha\beta}, B\} \) is any subsystem of \( \{X_\alpha, p_{\alpha\beta}, A\} \) with \( \text{cf}(\text{card}(B)) = \omega_1 \), then there exists a natural projection \( P : X \to \text{lim} Y \). By b) it follows that \( \text{lim} Y \) is a continuous image of an arc since \( \text{w}(\text{lim} Y) = \aleph_1 \). Applying Theorem 18 we complete the proof.

**Corollary 20.** Let \( X \) be a locally connected continuum such that \( \text{w}(X) > \aleph_{\omega_1} \). The following conditions are equivalent:

a) \( X \) is a continuous image of an arc.

b) If \( f : X \to Y \) is a continuous mapping and \( \text{w}(Y) = \aleph_1 \), then \( Y \) is a continuous image of an arc.
4. Appendix

Theorem 21 [15, Theorem 4]. If $X$ is a connected IOK and $X$ contains no nondegenerate metric subcontinuum, then $X$ is hereditarily locally connected.

Theorem 22 [17, Theorem]. If $X$ is a continuum containing no non-degenerate metric subcontinuum, then $X$ is finitely Suslinian if and only if $X$ is a continuous image of an arc.

We say that $X = \{X_a, p_{ab}, A\}$ is $\sigma$-directed if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of $A$ there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Theorem 23 [8, Theorem 9.8]. If $X$ is the limit of a $\sigma$-directed inverse system of finitely Suslinian continua, then $X$ is finitely Suslinian.

Theorem 24 [8, Theorem 9.9]. Let $X = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed inverse system of rim-finite continua with surjective bonding mappings. Then $X = \lim X$ is a rim-finite continuum.

Theorem 25. Let $X = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed inverse system of compact spaces with surjective bonding mappings and limit $X$. Let $Y$ be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.

Proof: Let $B$ be a countable basis of $Y$ and let $V$ be a collection of all finite subfamilies of $B$ which cover $X$. Clearly, card($V$) = $\aleph_0$. Hence, $V = \{V_n : n \in \mathbb{N}\}$. For each $V_n$ $f^{-1}(V_n) = \{f^{-1}(U) : U \in V_n\}$ is a covering of $X$. There exists an $a(n) \in A$ such that for each $b \geq a(n)$ there is a cover $V_{nb}$ of $X_b$ with $P_b^{-1}(V_{nb}) \prec f^{-1}(V_n)$. From the $\sigma$-directedness of $A$ it follows that there is an $a \in A$ such that $a \geq a(n)$, $n \in \mathbb{N}$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ is degenerate. Suppose that there exists a pair $u, v$ of distinct points of $Y$ such that $u, v \in f(p_b^{-1}(x_b))$. Then there exists a pair $x, y$ of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let $U, V$ be a pair of disjoint open sets of $Y$ such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y \setminus \{u, v\}\}$. There exists a covering $V_n \in V$ such that $V_n \prec \{U, V, X \setminus \{u, v\}\}$. We infer that there is a covering $V_{nb}$ of $X_b$ such that $p_b^{-1}(V_{nb}) \prec f^{-1}(V_n)$. It follows that $p_b(x) \neq p_b(y)$ since $x$ and $y$ lie in the disjoint members of the covering $f^{-1}(V_n)$. This is impossible since $x, y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define $g_b : X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is
clear that \( g_b p_b = f \). Let us prove that \( g_b \) is continuous. Let \( U \) be open in \( Y \). Then \( g_b^{-1}(U) \) is open since \( p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U) \) is open and \( p_b \) is quotient (as a closed mapping).

**Theorem 26.** Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-directed inverse system of compact spaces with limit \( X \). A closed subspace \( Y \) of \( X \) is metrizable if and only if there exists an \( a \in A \) such that \( p_a \mid Y : Y \to p_a(Y) \) is a homeomorphism for each \( b \geq a \).

**Proof:** Consider the inverse system \( Y = \{p_a(Y), p_{ab} \mid p_b(Y), A\} \) with limit \( Y \) and the identity mapping \( i : Y \to Y \). Apply Theorem 25.

**Theorem 27** [3, Corollary 3]. Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-directed inverse system of hereditarily locally connected continua \( X_a \). Then \( X = \lim X \) is hereditarily locally connected.

The following lemma follows from Theorem 10 and Corollary on p. 69 of [14]. See also [12, Lemma 3.5].

**Lemma 28.** Let \( X = \{X_a, p_{ab}, A\} \) be an inverse system of compact spaces with monotone bonding surjections, \( X = \lim X \), \( Y \) be a compact space and \( m_a : Y \to X_a \), \( a \in A \), be monotone surjections such that \( m_a = p_{ab} m_b \) for any \( a, b \in A \), \( a \leq b \). Moreover, let \( m : Y \to X \) denote the map induced by \( m_a \), \( a \in A \). Then \( m \) is also a monotone surjection. Moreover, each projection \( p_a : X \to X_a \), \( a \in A \), is a monotone surjection.

Let \( X = \{X_a, p_{ab}, A\} \) be an inverse system. For each infinite subset \( \Delta_0 \) of \( (A, \leq) \) we define sets \( \Delta_n \), \( n = 0, 1, \ldots \), by the inductive rule \( \Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\} \), where \( m(x, y) \) is a member of \( A \) such that \( x, y \leq m(x, y) \). Let \( \Delta = \bigcup \{\Delta_n : n \in \mathbb{N}\} \). It is clear that \( \text{card}(\Delta) = \text{card}(\Delta_0) \). Moreover, \( \Delta \) is directed by \( \leq \) [8, Lemma 9.2]. For each directed set \( (A, \leq) \) we define

\[
A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \leq\}.
\]

Then \( A_\sigma \) is \( \sigma \)-directed by inclusion [8, Lemma 9.3]. If \( \Delta \in A_\sigma \), let \( X^\Delta = \{X_b, p_{ab}, \Delta\} \) and \( X_\Delta = \lim X^\Delta \). If \( \Delta, \Gamma \in A_\sigma \) and \( \Delta \subseteq \Gamma \), let \( p_{\Delta \Gamma} : X_\Gamma \to X_\Delta \) denote the map induced by the projections \( p_\delta^\Gamma : X_\Gamma \to X_\delta \), \( \delta \in \Delta \), of the inverse system \( X^\Gamma \). Now, we have the following theorem.

**Theorem 29** [8, Theorem 9.4]. If \( X = \{X_a, p_{ab}, A\} \) is an inverse system, then \( X_\sigma = \{X_\Delta, p_{\Delta \Gamma}, A_\sigma\} \) is a \( \sigma \)-directed inverse system and \( \lim X \) and \( \lim X_\sigma \) are canonically homeomorphic.
Theorem 30. Let $X$ be a compact space. There exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ of compact metric spaces $X_a$ and surjective bonding mappings $p_{ab}$ such that $X$ is homeomorphic to $\lim X$.

Proof: See [6, pp. 152, 164]. ■

Theorem 31. If $X$ is a locally connected compact space, then there exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric locally connected compact space, each $p_{ab}$ is a monotone surjection and $X$ is homeomorphic to $\lim X$. Conversely, the inverse limit of such system is always a locally connected compact space.

Proof: See [6, p. 163, Theorem 2]. ■

Theorem 32 [11, Corollary 2.9]. If $X$ is a hereditarily locally connected continuum, then there exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metrizable hereditarily locally connected continuum, each $p_{ab}$ is a monotone surjection and $X$ is homeomorphic to $\lim X$.

Let $X$ be a non-degenerate locally connected continuum. A subset $Y$ of $X$ is said to be a cyclic element of $X$ if $Y$ is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of locally connected continuum is again a locally connected continuum. We let

$L_X = \{Y \subset X : Y$ is a non-degenerate cyclic element of $X\}$.

Lemma 33 [8, Lemma 2.2]. If $C$ is a connected subset of $X$ and $Y \in L_X$, then $C \cap Y$ is connected (possibly void).

Lemma 34 [8, Lemma 2.3]. If $f : X \to X'$ is a monotone surjection, then for each $Y' \in L_{X'}$ there exists $Y \in L_X$ such that $Y' \subseteq f(Y)$. In particular, $L_X$ is non-empty if $L_{X'}$ is non-empty.

Let $Z$ be a cyclic element of $X$. For each component $J$ of $X \setminus Z$, let $\text{Bd}(J) = \{z_J\}$. We define a mapping [8, p. 5] $\rho : X \to Z$ such that $\rho(x) = x$ if $x \in Z$ and $\rho(x) = z_J$ if $x \in J$.

The mapping $\rho$ is a monotone continuous retraction. It is called the canonical retraction of $X$ onto $Z$. 
Theorem 35 [8, Theorem 2.7]. Let $Y$ be a cyclic locally connected continuum and $S=(Y_\gamma,f_{\gamma\gamma'},\Gamma)$ an inverse system such that $Y=\liminv S$ and each bonding mapping $f_{\gamma\gamma'}$ is a monotone surjection. For each $\gamma \in \Gamma$, let $Z_\gamma$ be either a cyclic element of $Y_\gamma$ or a one-point subset of $Y_\gamma$. Let $\rho_\gamma : Y_\gamma \to Z_\gamma$ denote the canonical retraction if $Z_\gamma$ is non-degenerate, and otherwise let $\rho_\gamma$ be the constant map. Suppose that some $Z_{\gamma_0}$ is non-degenerate, and that $Z_\gamma \subseteq f_{\gamma\gamma'}(Z_{\gamma'})$ for all $\gamma \leq \gamma' \in \Gamma$. Let $g_{\gamma\gamma'} = \rho_\gamma \circ (f_{\gamma\gamma'}|Z_{\gamma'})$ for all $\gamma \leq \gamma' \in \Gamma$. Then each $g_{\gamma\gamma'} : Z_{\gamma'} \to Z_\gamma$ is a monotone surjection and $Y = \liminv(Z_\gamma,g_{\gamma\gamma'},\Gamma)$.

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