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GEODESIC FLOW ON $SO(4)$, KAC-MOODY LIE ALGEBRA AND SINGULARITIES IN THE COMPLEX t-PLANE

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Abstract

The article studies geometrically the Euler-Arnold equations associated to geodesic flow on $SO(4)$ for a left invariant diagonal metric. Such metric were first introduced by Manakov [**17**] and extensively studied by Mishchenko-Fomenko [**18**] and Dikii [**6**]. An essential contribution into the integrability of this problem was also made by Adler-van Moerbeke [**4**] and Haine [**8**]. In this problem there are four invariants of the motion defining in $\mathbb{C}^4 = \mathrm{Lie}(SO(4) \otimes \mathbb{C})$ an affine Abelian surface as complete intersection of four quadrics. The first section is devoted to a Lie algebra theoretical approach, based on the Kostant-Kirillov coadjoint action. This method allows us to linearizes the problem on a two-dimensional Prym variety $\mathrm{Prym}_\sigma(\mathsf{C})$ of a genus 3 Riemann surface C. In section 2, the method consists of requiring that the general solutions have the Painlevé property, i.e., have no movable singularities other than poles. It was first adopted by Kowalewski [**10**] and has developed and used more systematically [**3**], [**4**], [**8**], [**13**]. From the asymptotic analysis of the differential equations, we show that the linearization of the Euler-Arnold equations occurs on a Prym variety $\mathrm{Prym}_\sigma(\mathsf{\Gamma})$ of an another genus 3 Riemann surface $\mathsf{\Gamma}$. In the last section the Riemann surfaces are compared explicitly.

1. Lie algebra theoretical method

Consider the group $SO(4)$ and its Lie algebra $so(4)$ paired with itself, via the customary inner product

$$\langle X, Y \rangle = -\frac{1}{2} tr\,(X.Y)$$

where

$$X = (X_{ij})_{1\leq i,\, j\leq 4} = \sum_{i=1}^{6} x_i e_i = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4).$$

A left invariant metric on $SO(4)$ is defined by a non-singular symmetric linear map

$$\Lambda : so(4) \to so(4),\ X \mapsto \Lambda \cdot X$$

and by the following inner product: given two vectors gX and gY in the tangent space $SO(4)$ at the point $g \in SO(4)$

$$\langle gX, gY \rangle = \left\langle X, \Lambda^{-1} \cdot Y \right\rangle$$

regardless of g. Then the geodesic flow for this metric takes the following commutator form (Euler-Arnold equations):

$$\overset{\bullet}{X} = [X, \Lambda \cdot X]\,,\ \bullet \equiv \frac{d}{dt} \tag{1.1}$$

where

$$\begin{aligned} \Lambda \cdot X &= (\lambda_{ij} X_{ij})_{1\leq i,\, j\leq 4} \\ &= \sum_{i=1}^{6} \lambda_i x_i e_i = \begin{pmatrix} 0 & -\lambda_3 x_3 & \lambda_2 x_2 & -\lambda_4 x_4 \\ \lambda_3 x_3 & 0 & -\lambda_1 x_1 & -\lambda_5 x_5 \\ -\lambda_2 x_2 & \lambda_1 x_1 & 0 & -\lambda_6 x_6 \\ \lambda_4 x_4 & \lambda_5 x_5 & \lambda_6 x_6 & 0 \end{pmatrix} \in so(4). \end{aligned}$$

This flow is Hamiltonian with regard to the usual Kostant-Kirillov symplectic structure induced on the orbit

$$O = \left\{ Ad_g^*(X) = g^{-1} X g : g \in SO(4) \right\}$$

formed by the coadjoint action $Ad_g^*(X)$ of the group $SO(4)$ on the dual Lie algebra $so(4)^* \approx so(4)$. Let $z_1, z_2 \in so(4)$ and consider $\xi_1 = [X, z_1]$, $\xi_2 = [X, z_2]$ two tangent vectors to the orbit at the point $X \in so(4)$. Then the symplectic structure is defined by

$$\omega\,(X)\,(\xi_1, \xi_2) = \langle X, [z_1, z_2] \rangle\,.$$

This orbit is 4-dimensional and is defined by setting two trivial quadratic invariants H_1 and H_2 equal to generic constants c_1 and c_2:

$$\begin{aligned} H_1 &= \sqrt{\det X} = x_1 x_4 + x_2 x_5 + x_3 x_6 = c_1 \\ H_2 &= -\frac{1}{2} tr\left(X^2\right) = x_1^2 + x_2^2 + \cdots + x_6^2 = c_2. \end{aligned} \tag{1.2}$$

Fonctions H defined on the orbit lead to Hamiltonian vector fields

$$\overset{\bullet}{X} = [X, \nabla H] .$$

In particular

$$H = \frac{1}{2} \langle X, \lambda \cdot X \rangle = \frac{1}{2} \left(\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_6 x_6^2 \right) , \tag{1.3}$$

induces geodesic motion (1.1). The constants of the motion are given by the two quadratic invariants H_1, H_2 (1.2) and the Hamiltonian H (1.3). Since the system is Hamiltonian on a 4-dimensional symplectic manifold

$$\{H_1 = c_1\} \cap \{H_2 = c_2\}$$

to make it completely integrable, one needs one independent invariant. Under Manakov's conditions [**17**]:

$$\begin{cases} \lambda_1 = \dfrac{\beta_2 - \beta_3}{\alpha_2 - \alpha_3} \\ \lambda_2 = \dfrac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3} \\ \lambda_3 = \dfrac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2} \\ \lambda_4 = \dfrac{\beta_1 - \beta_4}{\alpha_1 - \alpha_4} \\ \lambda_5 = \dfrac{\beta_2 - \beta_{43}}{\alpha_2 - \alpha_4} \\ \lambda_6 = \dfrac{\beta_3 - \beta_4}{\alpha_3 - \alpha_4} \end{cases} , \alpha_i, \beta_i \text{ arbitrary}, \prod_{i<j} (\alpha_i - \beta_j) \neq 0 \tag{1.4}$$

the Lax flow (1.1) can be transformed into the following Lax-type equation (with an indeterminate h):

$$\begin{array}{c} \underbrace{\begin{array}{c} (X \overset{\bullet}{+} \alpha h) = [X + \alpha h, \Lambda X + \beta h] \\ \alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_4) \\ \beta = \operatorname{diag}(\beta_1, \ldots, \beta_4) \end{array}} \\ \Updownarrow \\ \overbrace{\begin{array}{c} \overset{\bullet}{X} = [X, \Lambda \cdot X] \Leftrightarrow (1.1) \\ [X, \beta] + [\alpha, \Lambda \cdot X] = 0 \Leftrightarrow (1.4) \\ [\alpha, \beta] = 0 \text{ trivially satisfied for diagonal matrices.} \end{array}} \end{array} \tag{1.5}$$

Consider the Kac-Moody extension ($n = 4$)

$$\mathcal{L} = \widetilde{gl(n)} = \left\{ \sum_{-\infty}^{N} A_i h^i : A_i \in gl(n),\ N \text{ arbitrary} \right\}$$

of $gl(n)$ with the bracket

$$[A(h), B(h)] = \left[\sum A_i h^i, \sum B_j h^j\right] = \sum_k \left(\sum_{i+j=k} [A_i, B_j] \right) h^k$$

and the nondegenerate, invariant inner product

$$\langle A(h), B(h) \rangle = \left\langle \sum A_i h^i, \sum B_j h^j \right\rangle = \sum_{i+j=-1} tr(A_i B_j).$$

This Lie algebra has a natural decomposition

$$\mathcal{L} = \mathcal{L}_{-\infty,-1} + \mathcal{L}_{0,\infty}, \quad \mathcal{L}_{ij} = \left\{ \sum_{i \geq 0} A_k h^k \right\}.$$

Observe that $\mathcal{L}_{-\infty,-1}^{\perp} = \mathcal{L}_{-\infty,-1}$ and $\mathcal{L}_{0,\infty}^{\perp} = \mathcal{L}_{0,\infty}$ where $\perp$ is taken with respect the form above. The infinite-dimensional Lie group underlying $\mathcal{L}_{-\infty,-1}$ acts coadjointly on the dual Kac-Moody Lie algebra $\mathcal{L}_{-\infty,-1}^{*} \approx \mathcal{L}_{0,\infty}^{\perp} = \mathcal{L}_{0,\infty}$, according to the rule of customary conjugation followed by registering the non-negative powers of h only. The orbits described in this way come equipped with a symplectic structure with Poisson bracket

$$\{H_1, H_2\}(\alpha) = \left\langle \alpha, \left[\nabla_{\mathcal{L}_{-\infty,-1}^{*}} H_1, \nabla_{\mathcal{L}_{-\infty,-1}^{*}} H_2\right] \right\rangle$$

where $\alpha \in \mathcal{L}_{-\infty,-1}^{*}$ and $\nabla_{\mathcal{L}_{-\infty,-1}^{*}} H \in \mathcal{L}_{-\infty,-1}$. The functions defined on this orbit are all in involution and the flow (1.5) evolves on the coadjoint orbit through the point $X + ah \in \mathcal{L}_{0,\infty}$, $X \in so(4)$. By the Adler-Kostant-Symes theorem [**1**], [**9**], [**24**], the coefficients of $z^i h^i$ appearing in the Riemann surface:

$$\mathsf{C} : \left\{ (z,h) \in \mathbb{C}^2 : \det(X + ah - zI) = 0 \right\} \tag{1.6}$$

associated to the equation (1.5), are invariant of the system in involution for the symplectic structure of this orbit. Also the flows generated by these invariants can be realized as straight lines on the Abelian variety defined by the periods of the Riemann surface C. Explicitly, equation (1.6) looks as follows

$$(1.7)\quad \mathsf{C} : H_1^2(X) + H_4(X)\, h^2 - H_3(X)\, zh + H_2(X)\, z^2 + \prod_{i=1}^{4} (\alpha_i h - z) = 0$$

with $H_1(X) = c_1$, $H_2(X) = c_2$ defined by (1.2), $H_3(X) = 2H = c_3$ by (1.3) and a 4^{th} quadratic invariant of the form

$$(1.8)\qquad H_4(X) = \mu_1 x_1^2 + \mu_2 x_2^2 + \cdots + \mu_6 x_6^2 = c_4$$

where

$$\mu_1 = \frac{\gamma_2 - \gamma_3}{\alpha_2 - \alpha_3}, \quad \mu_4 = \frac{\gamma_1 - \gamma_4}{\alpha_1 - \alpha_4}$$

$$\mu_2 = \frac{\gamma_1 - \gamma_3}{\alpha_1 - \alpha_3}, \quad \mu_5 = \frac{\gamma_2 - \gamma_4}{\alpha_2 - \alpha_4}$$

$$\mu_3 = \frac{\gamma_1 - \gamma_2}{\alpha_1 - \alpha_2}, \quad \mu_6 = \frac{\gamma_3 - \gamma_4}{\alpha_3 - \alpha_4}.$$

For generic choice of the c_i, C is a Riemann surface of genus 3 and it has a natural involution

$$\sigma : \mathsf{C} \to \mathsf{C},\ (z, h) \mapsto (-z, -h)$$

due to the skew-symmetry of the matrix X. Therefore the Jacobian variety $\mathrm{Jac}(\mathsf{C})$ of C (cf. [**7**] for definitions) splits up into an even and old part: the even part is an elliptic curve $\mathsf{C}_0 = \mathsf{C}/\sigma$ and the odd part is a 2-dimensional Abelian surface $\mathrm{Prym}_\sigma(\mathsf{C})$ called the Prym variety:

$$\mathrm{Jac}(\mathsf{C}) = \mathsf{C}_0 + \mathrm{Prym}_\sigma(\mathsf{C}).$$

The van Moerbeke-Mumford linearization method [**19**] provides then an algebraic map from the complex affine variety

$$\bigcap_{i=1}^{4} \{H_i(X) = c_i\} \subset \mathbb{C}^6$$

to the Jacobi variety $\mathrm{Jac}(\mathsf{C})$. By the antisymmetry of C, this map sends this variety to the Prym variety $\mathrm{Prym}_\sigma(\mathsf{C})$:

$$\bigcap_{i=1}^{4} \{H_i(X) = c_i\} \to \mathrm{Prym}_\sigma(\mathsf{C}),\ p \mapsto \sum_{k=1}^{3} s_k$$

and the complex flows generated by the constants of the motion are straight lines on $\mathrm{Prym}_\sigma(\mathsf{C})$. Finally, we have the

Theorem 1. *Let* $\mathrm{Prym}_\sigma(\mathsf{C})$ *be the Prym variety of the Riemann surface* C *(1.7). Under conditions (1.4), the Euler-Arnold equations (1.1) can be linearized on* $\mathrm{Prym}_\alpha(\mathsf{C})$.

2. Structure of the singularities in the complex t-plane and the integrability of the Euler-Arnold equations

First we recall several basics concepts. Consider a completely integrable Hamiltonian system

$$X_H : \overset{\bullet}{x} = JH'(x)\,,\; x \in \mathbb{R}^{2n+k},\; \bullet \equiv \frac{d}{dt},\; ' \equiv \frac{\partial}{\partial x} \tag{2.1}$$

where H is the Hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial x}, J\frac{\partial H_j}{\partial x} \right\rangle$$

satisfies the Jacobi identity. Let g^t be the corresponding phase flow. The system possesses $n+k$ independent polynomial invariants $H_1, \ldots, H_{n+k}$ (Casimir functions) of which k lead to zero vector fields $JH'_{n+i}(x) = 0$, $1 \le i \le k$, the n remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$). For most values of $c_i \in \mathbb{R}$, the invariant manifolds $\bigcap_{i=1}^{n+k} \left\{H_i = c_i,\, x \in \mathbb{R}^{2n+k}\right\}$ are compact, connected and by a theorem of Arnold-Liouville [**5**], are diffeomorphic to real tori $\mathbb{R}^n$/Lattice on which the flows $g_i^t(x)$ defined by the vector fields X_{H_i}, $1 \le i \le n$, are straight lines motions.

Let now $x \in \mathbb{C}^{2n+k}$, $t \in \mathbb{C}$ and $Z \subset \mathbb{C}^{2n+k}$ a non-empty Zariski open set. Note that the map

$$\Psi : (H_1, \ldots, H_{n+k}) : \mathbb{C}^{2n+k} \to \mathbb{C}^{n+k}$$

is submersive on Z, i.e., $dH_1(x), \ldots, dH_{n+k}(x)$ are linearly independent on Z. Let

$$\begin{aligned} I &= \Psi\left(\mathbb{C}^{2n+k} \backslash Z\right) \\ &= \left\{c = (c_i) \in \mathbb{C}^{n+k} : \exists\, x \in \Psi^{-1}(c) \text{ with } dH_1(x) \wedge \cdots \wedge dH_{n+k}(x) = 0\right\} \end{aligned}$$

be the set of critical values of Ψ and let $\overline{I}$ be the Zariski closure of I in $\mathbb{C}^{n+k}$. Recall [**4**], [**22**] that the system (2.1) is algebraically completely integrable if, for every $c \in \mathbb{C}^{n+k}\backslash\overline{I}$, the fibre $\mathbf{A} = \Psi^{-1}(c)$ is the affine part of an Abelian variety $\widetilde{\mathbf{A}} \cong \mathbb{C}^n$/Lattice, the flows $g_i^t(x)$, $x \in \mathbf{A}$, $t \in \mathbb{C}$, defined by the vector fields X_{H_i}, $1 \le i \le n$ are straight lines motions on $\mathbb{C}^n$/Lattice and the coordinates $x_i = x_i(t_1, \ldots, t_n)$ are meromorphic in $(t_1, \ldots, t_n)$.

Adler and van Moerbeke [**3**], [**4**] have developed and used the following necessary algebraic complete integrability criterion, inspired by the work of S. Kowalewski [**10**]: if the Hamiltonian system (2.1) is algebraically completely integrable, then each x_i blows up for some value of $t \in \mathbb{C}$ and whenever it blows up, the solution $x(t)$ behaves as a Laurent series

$$x_i = t^{-k_i}\left(x_i^{(0)} + x_i^{(1)}t + x_i^{(2)}t^2 + \cdots\right),\ k_i \in \mathbb{Z},\ \text{ some } k_i > 0$$

which admits dim(phase space) $-\ 1 = m - 1$ free parameters. To explain the criterion, if the Hamiltonian flow (2.1) is algebraically completely integrable, it means that the variables x_i are meromorphic on the torus $\mathbb{C}^n/\text{Lattice}$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathsf{S} \subset \mathbb{C}^n/\text{Lattice}$. By the algebraic complete integrability definition, the flow (2.1) is a straight line motion on $\mathbb{C}^n/\text{Lattice}$ and thus it must hit the divisor S in at least one place. Moreover through every point of S, there is a straight line motion and therefore a laurent expansion around that point of intersection. Hence the differential equation must admit Laurent expansions which depend on the $n-1$ parameters defining S and the $n+k$ constants c_i defining the torus $\mathbb{C}^n/\text{Lattice}$, the total count is therefore $m - 1 = \text{dim(phase space)} - 1$ parameters.

The system (1.1) can be written in the form (2.1), with $m = 6$, H is given by (1.3),

$$J = \begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 \\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2 \\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1 \\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix} \in so(6)$$

and is explicitely given by

$$\begin{aligned}
\dot{x}_1 &= (\lambda_3 - \lambda_2)\, x_2x_3 + (\lambda_6 - \lambda_5)\, x_5x_6 \\
\dot{x}_2 &= (\lambda_1 - \lambda_3)\, x_1x_3 + (\lambda_4 - \lambda_4)\, x_4x_6 \\
\dot{x}_3 &= (\lambda_2 - \lambda_1)\, x_1x_2 + (\lambda_5 - \lambda_4)\, x_4x_5 \\
\dot{x}_4 &= (\lambda_3 - \lambda_5)\, x_3x_5 + (\lambda_6 - \lambda_2)\, x_2x_6 \\
\dot{x}_5 &= (\lambda_4 - \lambda_3)\, x_3x_4 + (\lambda_1 - \lambda_6)\, x_1x_6 \\
\dot{x}_6 &= (\lambda_2 - \lambda_4)\, x_2x_4 + (\lambda_5 - \lambda_1)\, x_1x_5.
\end{aligned}$$

Here we have $n = 2$, $k = 2$, $Z = \left\{x \in \mathbb{C}^6 : \Psi(x) \in \mathbb{C}^4 \backslash \overline{I}\right\}$ is a nonempty Zariski open set in $\mathbb{C}^6$ and

$$\mathbf{A} = \Psi^{-1}(c) = \bigcap_{i=1}^{4} \left\{H_i(x) = c_i,\, x \in \mathbb{C}^6\right\} \tag{2.2}$$

where $H_i(x)$ are given in (1.2), (1.3), (1.8).

The invariant variety $\mathbf{A}$ (2.2) is the fibre of a morphism from $\mathbb{C}^6$ to $\mathbb{C}^4$, thus $\mathbf{A}$ is a smooth affine surface for generic $c = (c_1, \ldots, c_4) \in \mathbb{C}^4$ and the main problem will be to complete $\mathbf{A}$ into an Abelian surface. So, the question I address is how does one find the compactification of $\mathbf{A}$ into an Abelian surface? This compactification is not trivial and the simplest one obtained as a closure:

$$\overline{\mathbf{A}} = \bigcap_{i=1}^{4} \left\{H_i(x) = c_i x_0^2\right\} \subset \mathbb{P}^6$$

i.e.,

$$\begin{aligned} x_1x_4 + x_2x_5 + x_3x_6 &= c_1x_0^2 \\ x_1^2 + x_2^2 + \cdots + x_6^2 &= c_2x_0^2 \\ \lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_6x_6^2 &= c_3x_0^2 \\ \mu_1x_1^2 + \mu_2x_2^2 + \cdots + \mu_6x_6^2 &= c_4x_0^2 \end{aligned}$$

where $[x_0, x_1, \ldots, x_6]$ are homogeneous coordinates on $\mathbb{P}^6$, does not lead to this result. (In the following we will not distinguish between x_1 as a homogeneous coordinates $[x_0, x_1]$ and as an affine coordinate x_1/x_0.) Indeed, an Abelian surface is not simply-connected and therefore cannot be projective complete intersection. In other words, if $\mathbf{A}$ is to be the affine part of an Abelian surface, $\overline{\mathbf{A}}$ must have a singularity somewhere along the locus at infinity

$$I = \overline{\mathbf{A}} \bigcap \{x_0 = 0\}\,.$$

A direct calculation shows that I is an ordinary double curve of $\overline{\mathbf{A}}$ except at 16 ordinary pinch points of $\overline{\mathbf{A}}$; the variety $\overline{\mathbf{A}}$ has a local analytic equation $x^2 = yz^2$. The reduced curve I_r is a smooth elliptic curve. Now, it's only after blowing up $\overline{\mathbf{A}}$ along the curve I_r that one gets the desired Abelian surface.

Theorem 2. *The divisor of poles of the functions $x_1, x_2, \ldots, x_6$ is a Riemann surface* S *of genus* 9*. For generic constants, the surface* $\mathbf{A}$ *(2.2) is the affine part of an Abelian surface* $\widetilde{\mathbf{A}}$ *obtained by gluing to* $\mathbf{A}$ *the divisor* S.

Proof: Consider points at infinity which are limit points of trajectories of the flow. There is a Laurent decomposition of such asymptotic solutions,

$$X(t) = t^{-1}\left(X^{(0)} + X^{(1)}t + X^{(2)}t^2 + \cdots\right) \tag{2.3}$$

which depend on dim(phase space) $- 1 = 5$ free parameters. Putting (2.1) into (1.1), solving inductively for the $X^{(k)}$, one finds at the 0^{th} step a non-linear equation,

$$X^{(0)} + \left[X^{(0)}, \Lambda \cdot X^{(0)}\right] = 0 \tag{2.4}$$

and at the k^{th} step, a linear system of equations

$$(\mathrm{L} - k\,\mathrm{I})\,X^{(k)} = \begin{cases} 0 & \text{for } k = 1 \\ \text{quadratic polynomial in } X^{(1)}, \ldots, X^{(k-1)} & \text{for } k > 1 \end{cases}$$

where L denotes the linear map

$$\mathrm{L}(Y) = \left[Y, \Lambda \cdot X^{(0)}\right] + \left[X^{(0)}, \Lambda \cdot Y\right] + Y = \text{Jacobian map of (2.4)}.$$

One parameter appear at the 0^{th} step, i.e., in the resolution of (2.4) and the 4 remaining ones at the k^{th} step, $k = 1, \ldots, 4$. Taking into account only solutions trajectories lying on the surface $\mathbf{A}$, we obtain one-parameter families which are parameterized by a Riemann surface. To be precise we search for the set S of Laurent solutions (2.3) restricted to the affine invariant surface $\mathbf{A}$, i.e.,

(2.5)

$$\begin{aligned}
\mathsf{S} &= \text{closure of the continuous components of} \\
&\qquad \{\text{Laurent solutions } X(t) \text{ such that } H_i(X(t)) = c_i,\ 1 \le i \le 4\} \\
&= \bigcap_{i=1}^{4} \left\{t^0 - \text{coefficient of } H_i(X(t)) = c_i\right\} \\
&= \text{a Riemann surface whose affine equation is} \\
&\qquad \left\{ \begin{aligned} w^2 + c_1\left(x_5^{(0)}x_6^{(0)}\right)^2 + c_2\left(x_4^{(0)}x_6^{(0)}\right)^2 + c_3\left(x_4^{(0)}x_5^{(0)}\right)^2 + c_4 x_4^{(0)}x_5^{(0)}x_6^{(0)} \\ \equiv w^2 + F\left(x_4^{(0)}, x_5^{(0)}, x_6^{(0)}\right) = 0 \end{aligned} \right.
\end{aligned}$$

where w is an arbitrary parameter and where $x_4^{(0)}$, $x_5^{(0)}$, $x_6^{(0)}$ parametrizes the elliptic curve

$$\mathsf{E}:\begin{cases}\left(x_4^{(0)}\right)^2+\left(x_5^{(0)}\right)^2+\left(x_6^{(0)}\right)^2=0\\ \left(\beta x_5^{(0)}+\alpha x_6^{(0)}\right)\left(\beta x_5^{(0)}-\alpha x_6^{(0)}\right)=1\end{cases}\tag{2.6}$$

with (α,β) such that: $\alpha^2+\beta^2+1=0$.

The Riemann surface S is a two-sheeted ramified covering of the elliptic curve E and it is easy to check that the elliptic curve E is exactly the reduced curve I_r. The branch points are defined by the 16 zeroes of $F\left(x_4^{(0)},x_5^{(0)},x_6^{(0)}\right)$ on E. The Riemann surface S is unramified at infinity and by Riemann-Hurwitz's formula,

$$2g(\mathsf{S})-2=N\,(2g(\mathsf{E})-2)+R$$

where N is the number of sheets and R the ramification index, the genus $g(\mathsf{S})$ of S is 9. To show that $\mathbf{A}$ is the affine part of an Abelian surface $\widetilde{\mathbf{A}}$ with $\widetilde{\mathbf{A}}\backslash\mathbf{A}=\mathsf{S}$, we shall compute the invariants of $\widetilde{\mathbf{A}}$ and use Enriques classification of algebraic surfaces [**7**, p. 590]. We denote as usual by $K_{\widetilde{\mathbf{A}}}$ the canonical bundle, $\chi(\mathcal{O}_{\widetilde{\mathbf{A}}})$ the Euler characteristic and $q(\widetilde{\mathbf{A}})$ the irregularity of $\widetilde{\mathbf{A}}$. Now if $\phi:\widetilde{\mathbf{A}}\to\overline{\mathbf{A}}\subset\mathbb{P}^6$ is the normalization of $\overline{\mathbf{A}}$, then the pullback map on sections

$$\phi^*:H^0\left(\overline{\mathbf{A}},\mathcal{O}_{\overline{\mathbf{A}}}\right)\to H^0\left(\widetilde{\mathbf{A}},\mathcal{O}_{\widetilde{\mathbf{A}}}\right)$$

is an isomorphism and

$$K_{\widetilde{\mathbf{A}}}=\widetilde{K_{\overline{\mathbf{A}}}}-\mathsf{S}$$

where $\widetilde{K_{\overline{\mathbf{A}}}}=\phi^*\left(K_{\overline{\mathbf{A}}}\right)$ and so for H a hyperplane in $\mathbb{P}^6$,

$$K_{\widetilde{\mathbf{A}}}=\phi^*\left(\overline{\mathbf{A}}\cdot K_{\mathbb{P}^6}+\left(\sum_{i=1}^4\deg H_i\right)\cdot\mathsf{H}\right)-\mathsf{S}=0.$$

Also,

$$\begin{aligned}\chi\left(\mathcal{O}_{\widetilde{\mathbf{A}}}\right)&=\chi\left(\phi_*\mathcal{O}_{\widetilde{\mathbf{A}}}/\mathcal{O}_{\overline{\mathbf{A}}}\right)+\chi\left(\mathcal{O}_{\overline{\mathbf{A}}}\right)\\&=\chi\left(\phi_*\mathcal{O}_{\mathsf{S}}/\mathcal{O}_{\mathsf{E}}\right)+\chi\left(\mathcal{O}_{\overline{\mathbf{A}}}\right).\end{aligned}$$

Recall that the Riemann surface S (2.5) of genus 9, is a double cover ramified over 16 points of the elliptic curve E (2.6). We shall use the Koszul complex to compute $\chi\left(\mathcal{O}_{\overline{\mathbf{A}}}\right)$. In the local ring at each point of $\mathbb{P}^6$ the localizations of the 4 homogeneous polynomials H_i give a regular sequence, and the Koszul complex gives a canonical resolution

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-8) \to \mathcal{O}_{\mathbb{P}^6}(-6)^4 \to \mathcal{O}_{\mathbb{P}^6}(-4)^6 \to \mathcal{O}_{\mathbb{P}^6}(-2)^4 \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{\overline{V}_c} \to 0.$$

Thus $\chi\left(\mathcal{O}_{\overline{\mathbf{A}}}\right) = 8$, hence $\chi\left(\mathcal{O}_{\widetilde{\mathbf{A}}}\right) = 0$ and $q\left(\widetilde{\mathbf{A}}\right) = 2$. By Enriques-Kodaira's classifcation theorem [**7**, p. 590], it follows that $\widetilde{\mathbf{A}}$ is an Abelian surface and this concludes the proof of Theorem 2. ■

Theorem 3. *The flow (1.1) evolves on an Abelian surface* $\widetilde{\mathbf{A}} \cong \mathbb{C}^2/$*Lattice of polarization*

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \quad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

Proof: Let

$$L \equiv \left\{f : f \text{ meromorphic on } \widetilde{\mathbf{A}},\ (f) + \mathsf{S} \geq 0\right\}$$

be the vector space of meromorphic functions on $\widetilde{\mathbf{A}}$ with at worst a simple pole along S and let

$$\chi(\mathsf{S}) = \dim H^0\left(\widetilde{\mathbf{A}}, \mathcal{O}(\mathsf{S})\right) - \dim H^1\left(\widetilde{\mathbf{A}}, \mathcal{O}(\mathsf{S})\right)$$

be the Euler characteristic of S. The adjunction formula and the Riemann-Roch theorem for divisors on Abelian surfaces imply that

$$g(\mathsf{S}) = \frac{K_{\widetilde{\mathbf{A}}} \cdot \mathsf{S} + \mathsf{S} \cdot \mathsf{S}}{2} + 1$$

$$\chi(\mathsf{S}) = p_a\left(\widetilde{\mathbf{A}}\right) + 1 + \frac{1}{2}\left(\mathsf{S} \cdot (\mathsf{S} - K_{\widetilde{\mathbf{A}}})\right)$$

where $g(\mathsf{S})$ is the geometric genus of S and $p_a\left(\widetilde{\mathbf{A}}\right)$ is the arithmetic genus of $\widetilde{\mathbf{A}}$. Since $\widetilde{\mathbf{A}}$ is an Abelian surface $\left(K_{\widetilde{\mathbf{A}}} = 0, p_a\left(\widetilde{\mathbf{A}}\right) = -1\right)$,

$$g(\mathsf{S}) - 1 = \frac{\mathsf{S} \cdot \mathsf{S}}{2} = \chi(\mathsf{S}).$$

Using Kodaira-Serre duality [**7**, p. 153], Kodaira-Nakano vanishing theorem [**7**, p. 154] and a theorem on theta-functions [**7**, p. 317], it is easy to see that

$$\begin{aligned} g(\mathsf{S})-1 &= \dim L(\mathsf{S}) \left(\equiv h^0(L)\right) \\ &= d_1 d_2 \end{aligned} \tag{2.7}$$

where $d_1, d_2 \in \mathbb{N}$, are the elementary divisors of the polarization $c_1(L)$ of $\widetilde{\mathbf{A}}$. Note that the natural reflection about the origin of $\mathbb{C}^2$, is given by

$$\sigma \equiv -\,\mathrm{id} : (x_0, x_1, \dots, x_6) \mapsto (-x_0, x_1, \dots, x_6)$$

and has 16 fixed points on $\widetilde{\mathbf{A}}$, given by the 16 branch points on S covering the 16 roots of the polynomial $F(x_4^0, x_5^0, x_6^0)$ (2.5). Since L is symmetric ($\sigma^* L \simeq L$), σ can be lifted to L as an involution $\widetilde{\sigma}$ in two ways differing in sign and for each section (theta-function) $s \in H^0(L)$, we therefore have $\widetilde{\sigma} s = \pm s$. Recall that a section $s \in H^0(L)$ is called even (resp. odd) if $\widetilde{\sigma} s = +s$ (resp. $\widetilde{\sigma} s = -s$). Under $\widetilde{\sigma}$ the vector space $H^0(L)$ splits into an even and odd subspace

$$H^0(L) = H^0(L)^{\text{even}} \oplus H^0(L)^{\text{odd}}$$

with $H^0(L)^{\text{even}}$ containing all the even sections and $H^0(L)^{\text{odd}}$ all odd ones. Using the inverse formula [**21**, p. 331], we see after a small computation that

(2.8)

$$h^0(L)^{\text{even}} \equiv \dim H^0(L)^{\text{even}} = \begin{cases} \dfrac{d_1 d_2}{2} + 2\left(1 + \left[\dfrac{d_2}{2}\right] - \dfrac{d_2}{2}\right) & \text{for even } d_1 \\ \dfrac{d_1 d_2}{2} + \left(1 + \left[\dfrac{d_2}{2}\right] - \dfrac{d_2}{2}\right) & \text{for odd } d_1 \end{cases}$$

$$h^0(L)^{\text{odd}} \equiv \dim H^0(L)^{\text{odd}} = \begin{cases} \dfrac{d_1 d_2}{2} - 2\left(1 + \left[\dfrac{d_2}{2}\right] - \dfrac{d_2}{2}\right) & \text{for even } d_1 \\ \dfrac{d_1 d_2}{2} - \left(1 + \left[\dfrac{d_2}{2}\right] - \dfrac{d_2}{2}\right) & \text{for odd } d_1. \end{cases}$$

Notice that $c_1(L) = \phi^*(\mathsf{H})$ and $(c_1(L)^2) = 16$ (since the degree of $\overline{\mathbf{A}}$ is 16). By the classification theory of ample line bundles on Abelian varieties, $\widetilde{\mathbf{A}} \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix

$$\Omega = \begin{pmatrix} d_1 & 0 & a & c \\ 0 & d_2 & c & b \end{pmatrix}, \quad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$$

according to (2.7), with

$$d_1 d_2 = h^0(L) = g(\mathsf{S}) - 1 = 8, \quad d_1 \mid d_2,\ d_i \in \mathbb{N}^*.$$

Hence we have two possibilities:

$$\text{(i)} \quad d_1 = 1, \quad d_2 = 8$$

$$\text{(ii)} \quad d_1 = 2, \quad d_2 = 4.$$

Fom formula (2.8), the corresponding line bundle L has in case (i), 5 even sections, 3 odd ones and in case (ii), 6 even sections, 2 odd ones. Now $x_1, \dots, x_6$ are 6 even sections, showing that case (ii) is the only alternative and the period matrix has the form

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \quad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0.$$

This completes the proof of Theorem 3. ■

Theorem 4. *The Abelian surface* $\widetilde{\mathbf{A}}$ *which completes the affine surface* $\mathbf{A}$ *(2.2) is the Prym variety* $\operatorname{Pry} m_\sigma(\Gamma)$ *of the genus* 3 *Riemann surface* Γ*:*

$$\Gamma : \begin{cases} w^2 = -c_1\left(x_5^0 x_6^0\right)^2 - c_2\left(x_6^0\right)^2 z - c_3\left(x_5^0\right)^2 z + c_4 y \\ y^2 = z\left(\alpha^2 z - 1\right)\left(\beta^2 z + 1\right) \end{cases}$$

for the involution $\sigma : \Gamma \to \Gamma$, $(w, y, z) \mapsto (-w, y, z)$ *interchanging the two sheets of the double covering* $\Gamma \to \Gamma_0$, $(w, y, z) \mapsto (y, z)$ *where* Γ_0 *is the elliptic curve defined by*

$$\Gamma_0 : y^2 = z\left(\alpha^2 z - 1\right)\left(\beta^2 z + 1\right).$$

Proof: After substitution $z \equiv \left(x_4^0\right)^2$, the Riemann surface S can also be seen as a four-sheeted unramified covering of another Riemann surface Γ, determined by the equation

$$\Gamma : G(w, z) \equiv \left[w^2 + c_1\left(x_5^0 x_6^0\right)^2 + c_2\left(x_6^0\right)^2 z + c_3\left(x_5^0\right)^2 z\right]^2 - c_4^2\left(x_5^0 x_6^0\right)^2 z = 0.$$

It is straighforward to verify that the equations (2.6) are equivalent to $\left(x_5^0\right)^2 = \beta^2 z + 1$ and $\left(x_6^0\right)^2 = \alpha^2 z - 1$. To compute the genus of Γ, we observe that the Riemann surface Γ is invariant under an involution

$$\sigma : \Gamma \to \Gamma,\ (w, z) \curvearrowright (-w, z). \tag{2.9}$$

Let us consider a map

$$\rho : \Gamma \to \Gamma_0 \equiv \Gamma/\sigma,\ (w, y, z) \curvearrowright (y, z)$$

of the Riemann surface Γ onto an elliptic curve $\Gamma_0 \equiv \Gamma/\sigma$, that is given by the equation

$$\Gamma_0 : y^2 = z\left(\alpha^2 z - 1\right)\left(\beta^2 z + 1\right). \tag{2.10}$$

The genus of the Riemann surface

$$\Gamma : \begin{cases} w^2 = -c_1\left(x_5^0 x_6^0\right)^2 - c_2\left(x_6^0\right)^2 z - c_3\left(x_5^0\right)^2 z + \eta \\ y^2 = z\left(\alpha^2 z - 1\right)\left(\beta^2 z + 1\right) \end{cases} \tag{2.11}$$

is calculated easily by means of the map ρ. The latter is two-sheeted ramified covering of the elliptic curve Γ_0 and it has 4 branch points. Using the Riemann-Hurwitz formula, we obtain $g(\Gamma) = 3$. I now will proceed to show that the Abelian surface $\widetilde{\mathbf{A}}$ can be identified as Prym variety $\mathrm{Prym}_\sigma(\Gamma)$. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a basis of cycles in the Riemann surface Γ, with the intersection indices $a_i o a_j = b_i o b_j = 0$, $a_i o b_j = \delta_{ij}$, such that $\sigma(a_1) = a_3$, $\sigma(b_1) = b_3$, $\sigma(a_2) = -a_2$, $\sigma(b_2) = -b_2$ for the involution σ (2.9). By the Poincaré residu map [**7**, p. 221], the 3 holomorphic 1-forms ω_0, ω_1, ω_2 in Γ are the differentials

$$P(w,z)\frac{dz}{\left(\partial G/\partial w\right)(w,z)}\ |_{G(w,z)=0} = P(w,z)\frac{dz}{4wy}$$

for P a polynomial of degree $\leq \deg G - 3 = 1$. Therefore

$$\omega_0 = \frac{dz}{y}, \quad \omega_1 = \frac{zdz}{wy}, \quad \omega_2 = \frac{dz}{wy} \tag{3.5}$$

form a basis of holomorphic differentials on Γ and obviously $\sigma^*(\omega_0) = \omega_0$, $\sigma^*(\omega_k) = -\omega_k$ $(k = 1, 2)$ for the involution σ (2.9). It is well known that the period matrix Ω of $\mathrm{Prym}_\sigma(\Gamma)$ can be written as follows

$$\Omega = \begin{pmatrix} 2\displaystyle\int_{a_1}\omega_1 & \displaystyle\int_{a_2}\omega_1 & 2\displaystyle\int_{b_1}\omega_1 & \displaystyle\int_{b_2}\omega_1 \\ 2\displaystyle\int_{a_1}\omega_2 & \displaystyle\int_{a_2}\omega_2 & 2\displaystyle\int_{b_1}\omega_2 & \displaystyle\int_{b_2}\omega_2 \end{pmatrix}.$$

Let (dt_1, dt_2) be a basis of holomorphic 1-forms on $\widetilde{\mathbf{A}}$ such that $dt_j\ |_{\mathsf{S}} = \omega_j$, $(j = 1, 2)$,

$$L_{\Omega'} = \left\{\sum_{k=1}^{2} m_k \int_{a'_k}\begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} + n_k \int_{b'_k}\begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} : m_k,\, n_k \in \mathbb{Z}\right\}$$

the lattice associated to the period matrix

$$\Omega' = \begin{pmatrix} \int_{a_1'} dt_1 & \int_{a_2'} dt_1 & \int_{b_1'} dt_1 & \int_{b_2'} dt_1 \\ \int_{a_1'} dt_2 & \int_{a_2'} dt_2 & \int_{b_1'} dt_2 & \int_{b_2'} dt_2 \end{pmatrix}$$

where (a_1', a_2', b_1', b_2') is a basis of $H_1(\widetilde{\mathbf{A}}, \mathbb{Z})$ and let

$$\widetilde{\mathbf{A}} \to \mathbb{C}^2/L_{\Omega'} : P \curvearrowright \int_{P_0}^{P} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix}$$

be the uniformizing map. By the Lefschetz theorem on hyperplane section [**7**, p. 156], the map $H_1(\mathsf{S}, \mathbb{Z}) \to H_1(\widetilde{\mathbf{A}}, \mathbb{Z})$ induced by the inclusion $\mathsf{S} \hookrightarrow \widetilde{\mathbf{A}}$ is surjective and consequently we can find 4 cycles a_1', a_2', b_1', b_2' on the Riemann surface S such that

$$\Omega' = \begin{pmatrix} \int_{a_1'} \omega_1 & \int_{a_2'} \omega_1 & \int_{b_1'} \omega_1 & \int_{b_2'} \omega_1 \\ \int_{a_1'} \omega_2 & \int_{a_2'} \omega_2 & \int_{b_1'} \omega_2 & \int_{b_2'} \omega_2 \end{pmatrix}$$

and

$$L_{\Omega'} = \left\{ \sum_{k=1}^{2} m_k \int_{a_k'} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + n_k \int_{b_k'} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} : m_k,\, n_k \in \mathbb{Z} \right\}.$$

Recalling that $F(x_4^0, x_5^0, x_6^0)$ (2.5) has 4 zeroes on Γ_0 (2.10) and 16 zeroes on E (2.6), it follows that the 4 cycles a_1', a_2', b_1', b_2' on S which we look for are $2a_1$, a_2, $2b_1$, b_2 and they form a basis of $H_1(\widetilde{\mathbf{A}}, \mathbb{Z})$ such that

$$\Omega' = \begin{pmatrix} 2\int_{a_1} \omega_1 & \int_{a_2} \omega_1 & 2\int_{b_1} \omega_1 & \int_{b_2} \omega_1 \\ 2\int_{a_1} \omega_2 & \int_{a_2} \omega_2 & 2\int_{b_1} \omega_2 & \int_{b_2} \omega_2 \end{pmatrix} = \Omega$$

is a Riemann matrix. Thus, $\widetilde{\mathbf{A}}$ and $\mathrm{Prym}_\sigma(\Gamma)$ are two Abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_\Omega$. By Chow's theorem, $\widetilde{\mathbf{A}}$ and $\mathrm{Prym}_\sigma(\Gamma)$ are then algebraically isomorphic. This finishes the proof of Theorem 4. ■

3. Main observation

We know from section 1, that the linearization of the Euler-Arnold equations (1.1) takes place on the Prym variety $\mathrm{Prym}_\sigma(\mathsf{C})$ of the genus 3 Riemann surface C (1.7); the latter is a double ramified cover of an elliptic curve C_0. Also, from the asymptotic analysis (section 2) of the equations (1.1), the intersection $\mathbf{A}$ (2.2) of the four invariants (1.2), (1.3), (1.8) completes into an Abelian surface $\widetilde{\mathbf{A}}$ upon adding a Riemann surface S (2.5) of genus 9, which is a 4-fold unramified cover of a Riemann surface Γ (2.11) of genus 3; the latter is a double ramified cover of an elliptic curve Γ_0. The Abelian surface $\widetilde{\mathbf{A}}$ can also be identified as the Prym variety $\mathrm{Prym}_\sigma(\Gamma)$ and the problem linearizes on $\mathrm{Prym}_\sigma(\Gamma)$. From the fondamental exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\widetilde{\mathbf{A}}} \overset{\exp}{\to} \mathcal{O}^*_{\widetilde{\mathbf{A}}} \to 0$$

we get the map

$$\cdots \to H^1\left(\widetilde{\mathbf{A}}, \mathcal{O}^*_{\widetilde{\mathbf{A}}}\right) \to H^2\left(\widetilde{\mathbf{A}}, \mathbb{Z}\right) \to \cdots$$

i.e., the first Chern class of a line bundle on $\widetilde{\mathbf{A}}$. Therefore the group $\mathrm{Pic}^{\mathrm{o}}\left(\widetilde{\mathbf{A}}\right)$ of holomorphic line bundles on $\widetilde{\mathbf{A}}$ with Chern class zero (any line bundle with Chern class zero can be realized by constant multipliers) is given by

$$\mathrm{Pic}^{\mathrm{o}}\left(\widetilde{\mathbf{A}}\right) = H^1\left(\widetilde{\mathbf{A}}, \mathcal{O}_{\widetilde{\mathbf{A}}}\right) / H^1\left(\widetilde{\mathbf{A}}, \mathbb{Z}\right)$$

and is naturally isomorphic to the dual Abelian surface $\widetilde{\mathbf{A}}^{\circledast}$ of $\widetilde{\mathbf{A}}$ ($^{\circledast}$ means the dual Abelian surface). The relationship between $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}^{\circledast}$ is symmetric like the relationship between two vectors spaces set up a bilinear pairing. It is interesting to observe that the Abelian surfaces $\widetilde{\mathbf{A}} = \mathrm{Prym}_\sigma(\Gamma)$ obtained from the asymptotic analysis of the differential equations and $\mathrm{Prym}_\sigma(\mathsf{C})$ obtained from the orbits in the Kac-Moody Lie algebra are not identical but only isogeneous, i.e., one can be obtained from the other by doubling some periods and leaving other unchanged. The precise relation between these two Abelian surfaces is

$$\widetilde{\mathbf{A}} = (\mathrm{Prym}_\sigma(\mathsf{C}))^{\circledast}$$

i.e., they are dual of each other. In fact, the functions $x_1, \ldots, x_6$ are themselves meromorphic on $\widetilde{\mathbf{A}}$, while only their squares are on $\mathrm{Prym}_\sigma(\mathsf{C})$.

The final point we want to make is that the relationship between the Riemann surfaces Γ and C is quite intricate. As usual we let Θ the theta divisor on $\mathrm{Jac}(\Gamma)$, we have

$$\mathrm{Prym}_\sigma(C)\backslash\Pi = \Theta \cap \mathrm{Prym}_\sigma(C) = \Gamma$$

with Π a Zariski open set of $\mathrm{Prym}_\sigma(C)$. Also

$$\Theta \cap \widetilde{\mathbf{A}} = C$$

where Θ is a translate of the theta divisor of $\mathrm{Jac}(C)$ invariant under the involution σ.

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