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FLUCTUATIONS OF BROWNIAN MOTION WITH DRIFT

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Abstract

Consider 3 dimensional Brownian motion started on the unit sphere $\{|x| = 1\}$ with initial density ρ. Let ρ_t be the first hitting density on the sphere $\{|x| = t + 1\}$, $t > 0$. Then the linear operators T_t defined by $T_t\ \rho = \rho_t$ form a semigroup with an infinitesimal generator which is approximately the square root of the Laplacian. This paper studies the analogous situation for Brownian motion with a drift $\mathbf{b}$, where $\mathbf{b}$ is small in a suitable scale invariant norm.

Chapter 1. Introduction

In two previous papers [**CR**], [**CO**] we studied the Dirichlet problem for an elliptic equation on a domain in $\mathbb{R}^3$. Let B_R be the ball of radius R in $\mathbb{R}^3$ centered at the origin, $0 < R < \infty$. Consider the problem

$$(1.1)\qquad \begin{aligned} (-\Delta - \mathbf{b}(x) \cdot \nabla)u(x) &= f(x), \quad && x \in B_R, \\ u(x) &= 0, && x \in \partial B_R. \end{aligned}$$

A function $g : \mathbb{R}^3 \to \mathbb{C}$ is said to be in the Morrey space M_p^q, $1 \le p \le q < \infty$, if $|g|^p$ is locally integrable and there is a constant C such that

$$(1.2)\qquad \int_Q |g|^p\, dx \le C^p |Q|^{1-p/q},$$

for all cubes $Q \subset \mathbb{R}^3$. Here $|Q|$ denotes the volume of Q. The norm of g, $\|g\|_{q,p}$ is defined as the minimum C for which (1.2) holds. In [**CR**] we proved the following:

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Theorem 1.1. *Suppose* $1 < r < p \leq q$, $1 < p \leq 3$, *and* $|\mathbf{b}| \in M_p^3$, $f \in M_r^q$, *for some* q *with* $q > 3/2$. *Then there exists* $\epsilon > 0$ *depending only on* r, p, q *such that if* $\|\mathbf{b}\|_{3,p} < \epsilon$ *then the Dirichlet problem (1.1) is solvable. Further, there is a constant* C *depending only on* r, p, q *such that the solution* u *of (1.1) satisfies the inequality,*

$$\|u\|_\infty \leq CR^{2-3/q}\|f\|_{q,r}.$$

The condition $\|\mathbf{b}\|_{3,p}$ small for some p, $1 < p \leq 3$, includes the two important cases when $|\mathbf{b}(x)| = \epsilon/|x|$ and $|\mathbf{b}| \in L^3(\mathbb{R}^3)$, $\|\mathbf{b}\|_3 = \epsilon$, $\epsilon \ll 1$. Theorem 1.1 is a perturbative result. Writing the solution of (1.1) as a perturbation series in $\mathbf{b}$ one can show that the series converges if $\|\mathbf{b}\|_{3,p} \ll 1$. Observe also that Theorem 1.1 is a scale invariant theorem. The result for general R can be obtained by a scaling argument from the result for a particular value of R. Since all the results in this paper have the same scaling property we shall take $R = 1/4$ from here on.

It is well known [**SV**] that the solution of the Dirichlet problem (1.1) has a representation as an expectation value with respect to Brownian motion with drift $\mathbf{b}$. Let $X_\mathbf{b}(t)$, $t \geq 0$, denote the drift process started at time 0. If $X_\mathbf{b}(0) \in B_{1/4}$ let τ be the minimum time t such that $X_\mathbf{b}(t) \in \partial B_{1/4}$. Then the solution u of (1.1) is given by

$$u(x) = E_x\left[\int_0^\tau f(X_\mathbf{b}(t))\,dt\right], \quad x \in B_{1/4},$$

where E_x denotes that the expectation value is taken conditioned on $X_\mathbf{b}(0) = x$. Theorem 1.1 tells us therefore something about the behavior of the diffusion process $X_\mathbf{b}(t)$ when $\|\mathbf{b}\|_{3,p} \ll 1$. It gives us similar estimates on the expected time the diffusion spends in a subset of $B_{1/4}$ before exiting $\partial B_{1/4}$ to those one has for standard Brownian motion.

In [**CO**] we proved a nonperturbative version of Theorem 1.1, allowing $\mathbf{b}$ to be only locally in a Morrey space M_p^3 with small norm. A key ingredient in the proof of this theorem was the fact that the fluctuations of $X_\mathbf{b}(t)$ did not increase as t increases, provided $\|\mathbf{b}\|_{3,p} \ll 1$. To be specific, suppose $0 < \rho \leq 1/2$, and the process $X_\mathbf{b}(t)$ starts on the sphere $\partial B_{(1-\rho)/4}$ with initial density function f, and $f_{\rho,\mathbf{b}}$ is the first hitting density on the sphere $\partial B_{1/4}$. It is evident, by conservation of probability, that the average value of f is the same as the average value of $f_{\rho,\mathbf{b}}$, $Avf = Avf_{\rho,\mathbf{b}}$. In [**CO**] we proved the following:

Theorem 1.2. *Suppose* $0 < \rho \leq 1/2$ *and* $1 < p \leq 3$. *Consider* f *and* $f_{\rho,\mathbf{b}}$ *to be functions on the unit sphere* S, *with* $f \in L^2(S)$. *Then for* $\delta > 0$ *there exists* ϵ, *depending only on* p, ρ, δ *such that, if* $\|\mathbf{b}\|_{3,p} < \epsilon$ *and* $\|f - Avf\|_2 \leq \delta|Avf|$, *one has* $f_{\rho,\mathbf{b}} \in L^2(S)$ *and* $\|f_{\rho,\mathbf{b}} - Avf_{\rho,\mathbf{b}}\|_2 \leq \delta|Avf_{\rho,\mathbf{b}}|$.

It is easy to see that Theorem 1.2 holds uniformly in ρ as $\rho \to 0$ for the case of Brownian motion, $\mathbf{b} \equiv 0$. Since $\mathbf{b}$ with $\|\mathbf{b}\|_{3,p} \ll 1$ is perturbative to Brownian motion it is natural to expect a similar uniformity when $\|\mathbf{b}\|_{3,p}$ is small. In this paper we prove a uniform version of Theorem 1.2. We cannot however use the L^2 norm to measure the oscillation of f and $f_{\rho,\mathbf{b}}$. We must use a finer norm which weights high Fourier modes more than low Fourier modes. This subtlety is closely related to the extra complication in the proof of Theorem 1.2 over Theorem 1.1. To prove Theorem 1.1 one shows that a certain integral operator is bounded on Morrey spaces. To prove Theorem 1.2 one needs to know that this same integral operator is bounded on a weighted Morrey space, where the weight of a point $x \in B_{1/4}$ decreases as x gets close to $\partial B_{1/4}$.

To define our new norm on functions f with domain S, let Δ_S be the Laplace operator on the unit sphere. For $k = 1, 2, \dots$, let E_k be the L^2 projection operator onto the space spanned by the eigenfunctions of $-\Delta_S$ with eigenvalues λ^2 satisfying $2^{k-1} < \lambda \leq 2^k$. Let E_0 be the projection onto the constant function. For $f : S \to \mathbb{C}$ and $\nu > 0$ we define $\|f\|_{2,\nu}$ by

$$\|f\|_{2,\nu} = \sup_{k \geq 0} 2^{\nu k} \|E_k f\|_2.$$

We then have the following:

Theorem 1.3. *Suppose* $0 < \rho \leq 1/2$, $1 < p \leq 3$, *and* $\nu > 0$ *is sufficiently small, depending only on* p. *Then for* $\delta > 0$ *there exists* $\epsilon > 0$ *depending only on* p, δ *such that if* $\|\mathbf{b}\|_{3,p} < \epsilon$ *and* $\|f - Avf\|_{2,\nu} \leq \delta|Avf|$, *one has* $\|f_{\rho,\mathbf{b}} - Avf_{\rho,\mathbf{b}}\|_{2,\nu} \leq \delta|Avf_{\rho,\mathbf{b}}|$.

We consider the relationship between the proof of Theorem 1.1 and the proof of Theorem 1.2. Let G_D be the Dirichlet Green's function for $-\Delta$ on $B_{1/4}$, whence G_D is given explicitly by the formula,

$$G_D(x, y) = \frac{1}{4\pi|x - y|} - \frac{1}{16\pi|y|}\,\frac{1}{|x - \bar{y}|},$$

where $\bar{y}$ is the reflection of y in $\partial B_{1/4}$. Let T be the integral operator on functions with domain $B_{1/4}$ given by

$$T\,f(x) = \int_{B_{1/4}} \mathbf{b}(x) \cdot \nabla_x G_D(x, y)\, f(y)\, dy, \quad x \in B_{1/4}. \tag{1.3}$$

It was shown in [**CR**] that Theorem 1.1 is a consequence of the fact that T is a bounded operator on the Morrey space M_r^q with norm, $\|T\|$, satisfying $\|T\| \le C\|\mathbf{b}\|_{3,p}$ for some constant C depending only on p, q, r.

For g a function with domain $\partial B_{1/4}$ let $v(y) = Pg(y)$, $y \in B_{1/4}$, be the solution of the Dirichlet problem,

$$\begin{aligned} \Delta v(y) &= 0, & y &\in B_{1/4}, \\ v(y) &= g(y), & y &\in \partial B_{1/4}. \end{aligned}$$

The function v is given explicitly by the Poisson formula,

$$Pg(y) = \frac{1}{\pi} \int_{\partial B_{1/4}} \frac{1/16 - |y|^2}{|y-x|^3} g(x)\, dx, \quad y \in B_{1/4}. \tag{1.4}$$

We can formally define an integral operator Q on the functions g by

$$Qg(x) = \int_{B_{1/4}} dy\, G_D(x,y)(I-T)^{-1}\mathbf{b} \cdot \nabla Pg(y), \quad x \in B_{1/4}, \tag{1.5}$$

where T is given by (1.3). This operator induces an operator on functions with domain S as follows: Let $f : S \to \mathbb{C}$ and denote also by f the function with domain $\partial B_{1/4}$ naturally induced by f. Consider now the function $Q_\rho f$ with domain $\partial B_{(1-\rho)/4}$ defined by

$$Q_\rho f(x) = Qf(x), \quad x \in \partial B_{(1-\rho)/4}. \tag{1.6}$$

We can think of $Q_\rho f$ as a function with domain S, whence Q_ρ is an operator on functions with domain S. In [**CO**] we proved that Q_ρ is a bounded operator on $L^2(S)$, $0 < \rho \le 1/2$, with norm $\|Q_\rho\|$ satisfying $\|Q_\rho\| \le C\|\mathbf{b}\|_{3,p}$ for some constant C depending only on ρ, p, provided $\|\mathbf{b}\|_{3,p} < \epsilon$ for some ϵ depending only on p. Theorem 1.2 is a consequence of this fact.

Observe that the proof of Theorem 1.2 must be more difficult than the proof of Theorem 1.1 since in the definition of Q_ρ one assumes that the inverse $(I-T)^{-1}$ exists. To prove Theorem 1.3 we need to know not only that Q_ρ is bounded on $L^2(S)$ for $0 < \rho \le 1/2$ but also to have a bound which is uniform as $\rho \to 0$. Let $\langle , \rangle$ denote the scalar product on $L^2(S)$. In section 2 we prove the following:

Theorem 1.4. *Let $0 < \rho \le 1/2$, $1 < p \le 3$. Then there exists C, $\epsilon > 0$ depending only on p, such that*

$$|\langle f, Q_\rho g\rangle| \le C\|\mathbf{b}\|_{3,p}\|f\|_2\|g\|_2,$$

provided $\|\mathbf{b}\|_{3,p} \le \epsilon$, f, $g \in L^2(S)$.

In order to prove Theorem 1.3 we need to know more detailed properties of Q_ρ than those given in Theorem 1.4. In particular we must know that if Q_ρ acts on a slowly varying function g then the slowly varying component of $Q_\rho g$ has norm bounded linearly in ρ as $\rho \to 0$. We also need to know that if g is highly oscillatory then the slowly varying component of $Q_\rho g$ has small norm. These properties of Q_ρ are summarised in the following:

Theorem 1.5. *Let $0 < \rho \le 1/2$, $1 < p \le 3$. Then there exists C, ϵ, $\eta > 0$ depending only on p such that $\|\mathbf{b}\|_{3,p} < \epsilon$ implies that*

$$|\langle E_{k'} f, Q_\rho E_k g\rangle| \le C\|\mathbf{b}\|_{3,p}\|E_{k'} f\|_2\|E_k g\|_2 \min[\rho 2^k, 1] \min[2^{\eta(k'-k)}, 1]$$
$$0 \le k,\ k' < \infty.$$

We prove Theorem 1.5 in section 3. Theorem 1.3 is a simple consequence of Theorems 1.4 and 1.5. It is proved in section 4.

Chapter 2. Configuration Space Localization

To prove Theorem 1.4 we shall modify the proof in [**CO**] so that the estimates are uniform in ρ as $\rho \to 0$.

Define an operator A on functions g with domain $S_{1/4} = \{|x| = 1/4\}$ by

$$Ag(y) = |\mathbf{b}(y)|^{1/2} Pg(y), \quad |y| < 1/4. \tag{2.1}$$

Proposition 2.1. *A is a bounded linear operator from $L^2(S_{1/4})$ to $L^2(B_{1/4})$. There is a constant C depending only on $p > 1$ such that $\|A\| \le C\|\mathbf{b}\|_{3,p}^{1/2}$.*

We shall prove Proposition 2.1 by following the general lines of the proof of Theorem 1.2 of [**CR**]. If a function u is defined on the sphere $S_{1/4}$ then Au is defined on the ball $B_{1/4}$. Let Q_0 be a cube with side of length 1 and for $n = 1, 2, \dots$ let Q_n be the dyadic subcubes of Q_0 with side of length 2^{-n}. We define u_{Q_n} as follows:

$$u_{Q_n} = 0 \qquad \text{if } Q_n \cap S_{1/4} \text{ is empty,}$$

$$u_{Q_n} = |Q_n|^{-2/3} \int_{Q_n \cap S_{1/4}} |u(x)|\, dx, \quad \text{otherwise.}$$

For $\xi \in \mathbb{R}^3$ let $Q_0(\xi)$ be the unit cube centered at ξ with corresponding dyadic subcubes $Q_n(\xi)$. We then have the following:

Lemma 2.1. *There exists a universal constant C such that*

$$\int_{B_{1/4}} |Au(y)|^2\, dy \le C \int_{B_{1/4}} d\xi \sum_{n=0}^{\infty} \sum_{Q_n(\xi) \subset Q_0(\xi)} u^2_{Q_n(\xi)} \int_{Q_n(\xi)} |\mathbf{b}(y)|\, dy. \tag{2.2}$$

Proof: We have from the definition (2.1) that

$$Au(y) = |\mathbf{b}(y)|^{1/2} \frac{1}{16\pi} \int_{S_{1/4}} \frac{1 - 16|y|^2}{|y-x|^3}\, u(x)\, dx.$$

We estimate $Au(y)$ in the annulus

$$R_m = \left\{ y : \frac{1}{4}(1 - 2^{-m}) \le |y| < \frac{1}{4}(1 - 2^{-(m+1)}) \right\}, \quad m = 0, 1, 2, \dots .$$

Thus

$$|Au(y)| \le C\, |\mathbf{b}(y)|^{1/2}\, 2^{-m} \sum_{n=0}^{m} 2^{3n} \int_{S_{1/4} \cap \{|x-y| < 2^{-n-1}\}} |u(x)|\, dx,$$

for some universal constant C. Choosing $\alpha > 0$ and applying the Schwarz inequality to the RHS of the previous expression we have

$$|Au(y)|^2 \le C_\alpha\, |\mathbf{b}(y)|\, 2^{-2m(1-\alpha)} \sum_{n=0}^{m} 2^{2(3-\alpha)n} \left[\int_{S_{1/4} \cap \{|x-y| < 2^{-n-1}\}} |u(x)|\, dx \right]^2,$$

for some constant C_α depending only on α. Thus

$$\int_{B_{1/4}} |Au(y)|^2\, dy = \sum_{m=0}^{\infty} \int_{R_m} |Au(y)|^2\, dy$$

$$\leq C_\alpha \sum_{0\leq n\leq m<\infty} 2^{-2m(1-\alpha)}\, 2^{2(3-\alpha)n} \int_{R_m} dy|\mathbf{b}(y)|$$

$$\left[\int_{S_{1/4}\cap\{|x-y|<2^{-n-1}\}} |u(x)|\, dx\right]^2 .$$

Let $U_n = \cup_{m=n}^{\infty} R_m$, $n \geq 0$. Then it is clear from the previous expression that if $\alpha < 1$ there is a constant C_α depending only on α such that

$$\int_{B_{1/4}} |Au(y)|^2\, dy \leq C_\alpha \sum_{n=0}^{\infty} a_n,$$

where

$$a_n = \int_{U_n} dy|\mathbf{b}(y)| \left[2^{2n} \int_{S_{1/4}\cap\{|x-y|<2^{-n-1}\}} |u(x)|\, dx\right]^2 .$$

It is clear now that there exists a universal constant C such that

$$a_n \leq C \int_{B_{1/4}} d\xi \sum_{Q_n(\xi)\subset Q_0(\xi)} u_{Q_n}^2(\xi) \int_{Q_n(\xi)} |\mathbf{b}(y)|\, dy, \quad n \geq 0.$$

The result follows then from the last two inequalities. ■

Next we shall show that for fixed ξ the RHS of (2.2) is bounded by the L^2 norm of u.

Lemma 2.2. *Let Q_0 be a cube in $\mathbb{R}^3$ with side of length 1 and dyadic subcubes Q_n with side of length 2^{-n}, $n = 1, 2, \ldots$. Then there is a constant C depending only on $p > 1$ such that*

$$\sum_{n=0}^{\infty} \sum_{Q_n\subset Q_0} u_{Q_n}^2 \int_{Q_n} |\mathbf{b}(y)|\, dy \leq C\, \|\mathbf{b}\|_{3,p}\, \|u\|_2^2. \tag{2.3}$$

Evidently Proposition 2.1 follows from Lemma 2.1 and Lemma 2.2. Observe that for fixed n one has

$$\sum_{Q_n\subset Q_0} u_{Q_n}^2 \int_{Q_n} |\mathbf{b}(y)|\, dy \leq \sum_{Q_n\subset Q_0} u_{Q_n}^2 \|\mathbf{b}\|_{3,p}\, 2^{-2n}$$

$$\leq C\, \|\mathbf{b}\|_{3,p} \sum_{Q_n\subset Q_0} \int_{Q_n\cap S_{1/4}} |u(x)|^2\, dx$$

$$\leq C\, \|\mathbf{b}\|_{3,p}\, \|u\|_2^2,$$

for some universal constant C. In order to do the summation with respect to n in (2.3) we need to resort to a Calderon-Zygmund decomposition. First we have

Lemma 2.3. *Let Q' be a cube in $\mathbb{R}^3$ with side of length $2^{-n_{Q'}}$, where $n_{Q'}$ is a nonnegative integer. Suppose for some $\varepsilon > 0$ one has $|Q|^\varepsilon\, u_Q \le |Q'|^\varepsilon\, u_{Q'}$ for all dyadic subcubes Q of Q'. Then if ε is sufficiently small there exists a constant C depending only on $p > 1$ such that*

$$\sum_{Q \subset Q'} u_Q^2 \int_Q |\mathbf{b}(y)|\, dy \le C\, \|\mathbf{b}\|_{3,p}\, |Q'|^{2/3}\, u_{Q'}^2.$$

Proof: We have for fixed $n \ge n_{Q'}$,

$$\begin{aligned}
&\sum_{Q_n \subset Q'} u_{Q_n}^2 \int_{Q_n} |\mathbf{b}(y)|\, dy \\
&\quad \le 2^{6(n-n_{Q'})\varepsilon} u_{Q'}^2 \sum_{\{Q_n : Q_n \subset Q',\, Q_n \cap S_{1/4} \ne \emptyset\}} \int_{Q_n} |\mathbf{b}(y)|\, dy \\
&\quad \le 2^{6(n-n_{Q'})\varepsilon}\, u_{Q'}^2 \int_{Q' \cap \{y : 1 > |y| > 1 - 2^{-n}\sqrt{3}\}} |\mathbf{b}(y)|\, dy \\
&\quad \le 2^{6(n-n_{Q'})\varepsilon} u_{Q'}^2 \operatorname{meas}\Big(Q' \cap \{y : 1 > |y| > 1 - 2^{-n}\sqrt{3}\}\Big)^{1-1/p} \\
&\qquad\qquad \left[\int_{Q'} |\mathbf{b}(y)|^p\, dy\right]^{1/p} \\
&\quad \le C\, 2^{6(n-n_{Q'})\varepsilon}\, u_{Q'}^2 \Big(2^{-2n_{Q'}}\, 2^{-n}\Big)^{1-1/p} \|\mathbf{b}\|_{3,p}\, |Q'|^{1/p-1/3} \\
&\quad = C\, \|\mathbf{b}\|_{3,p}\, |Q'|^{2/3}\, u_{Q'}^2\, 2^{(n-n_{Q'})(6\varepsilon+1/p-1)},
\end{aligned}$$

where C is a universal constant.

If we now choose ε to satisfy $\varepsilon < (1-1/p)/6$, then we have

$$\begin{aligned}
\sum_{Q \subset Q'} u_Q^2 \int_Q |\mathbf{b}(y)|\, dy &\le C\, \|\mathbf{b}\|_{3,p}\, |Q'|^{2/3}\, u_{Q'}^2 \sum_{n=n_{Q'}}^{\infty} 2^{(n-n_{Q'})(6\varepsilon+1/p-1)} \\
&\le C_p\, \|\mathbf{b}\|_{3,p}\, |Q'|^{2/3}\, u_{Q'}^2,
\end{aligned}$$

where the constant C_p depends only on $p > 1$. ■

Let Q_0 be a unit cube in $\mathbb{R}^3$. We make a Calderon-Zygmund decomposition of Q_0 based on the criterion in Lemma 2.3. In particular we define a sequence of families $\mathcal{F}_j$ of dyadic subcubes of Q_0, $j = 0, 1, 2, \dots$ as follows: $\mathcal{F}_0 = \{Q_0\}$. Let $G_1 \subset Q_0$ be defined as

$$G_1 = \{x \in Q_0 : |Q|^\varepsilon u_Q \le |Q_0|^\varepsilon u_{Q_0} \quad \text{for all dyadic subcubes } Q \text{ of } Q_0 \text{ with } x \in Q\}.$$

Then there is a unique finite family $\mathcal{F}_1$ of disjoint dyadic subcubes of Q_0 such that

$$\bigcup_{Q \in \mathcal{F}_1} Q = Q_0 \backslash G_1.$$

Proceeding by induction as in section 2 of [**CR**], we can construct sets G_j and families $\mathcal{F}_j$, $j \ge 1$, with the properties

(a) $\bigcup_{j=1}^\infty G_j = Q_0$.

(b) $\bigcup_{Q \in \mathcal{F}_k} Q = Q_0 \backslash \bigcup_{j=1}^k G_j$.

(c) For any $Q \in \mathcal{F}_k$ let $\bar{Q} \in \mathcal{F}_{k-1}$ be the unique dyadic subcube containing Q. Then

$$|Q|^\varepsilon u_Q > |\bar{Q}|^\varepsilon u_{\bar{Q}}.$$

It is clear now from Lemma 2.3 that there is a constant C depending only on $p > 1$ such that

$$\sum_{n=0}^\infty \sum_{Q_n \subset Q_0} u_{Q_n}^2 \int_{Q_n} |\mathbf{b}(y)|\, dy \le C \, \|\mathbf{b}\|_{3,p} \sum_{j=0}^\infty \sum_{Q \in \mathcal{F}_j} |Q|^{2/3} \, u_Q^2.$$

The proof of Lemma 2.2 will be complete if we can show

Lemma 2.4. *There exists a constant C depending only on $p > 1$ such that*

$$\sum_{j=0}^\infty \sum_{Q \in \mathcal{F}_j} |Q|^{2/3} \, u_Q^2 \le C \|u\|_2^2. \tag{2.4}$$

Proof: We shall assume without lost of generality that $\|u\|_\infty < \infty$. Hence there exists an integer $t \ge 1$ such that $\mathcal{F}_t$ is empty, whence

$$Q_0 = \bigcup_{j=1}^{t} G_j.$$

Let us consider a particular $Q \in \mathcal{F}_j$, $0 \le j \le t-1$. It is evident that

$$Q \subset \bigcup_{m=j+1}^{t} G_m.$$

We wish to estimate the ratio $|Q \cap G_m \cap S_{1/4}|/|Q|^{2/3}$ for $m \ge j+1$. We have now

$$\begin{aligned}
|Q|^{2/3}\, u_Q &= \int_{Q\cap S_{1/4}} |u(x)|\,dx \ge \sum_{i=m}^{t} \int_{Q\cap G_i \cap S_{1/4}} |u|\,dx \\
&= \sum_{\bar{Q}\in\mathcal{F}_{m-1},\bar{Q}\subset Q} |\bar{Q}|^{2/3}\, u_{\bar{Q}} \ge \sum_{\bar{Q}\in\mathcal{F}_{m-1},\bar{Q}\subset Q} |\bar{Q}|^{2/3} \left(\frac{|Q|}{|\bar{Q}|}\right)^{\varepsilon} u_Q \\
&\ge 2^{3(m-j-1)\varepsilon}\, u_Q \sum_{\bar{Q}\in\mathcal{F}_{m-1},\bar{Q}\subset Q} |\bar{Q}|^{2/3} \\
&\ge c\, 2^{3(m-j-1)\varepsilon}\, u_Q \left| Q \cap \bigcup_{i=m}^{t} G_i \cap S_{1/4} \right|,
\end{aligned}$$

for some universal constant $c > 0$. We conclude therefore that

$$|Q \cap G_m \cap S_{1/4}|/|Q|^{2/3} \le c^{-1} 2^{-3(m-j-1)\varepsilon}.$$

Next we write

$$\begin{aligned}
|Q|^{2/3}\, u_Q^2 &= |Q|^{-2/3} \left[\int_{Q\cap S_{1/4}} |u|\,dx \right]^2 \\
&= |Q|^{-2/3} \left[\sum_{m=j+1}^{t} \int_{Q\cap G_m\cap S_{1/4}} |u|\,dx \right]^2 \\
&\le |Q|^{-2/3} \left[\sum_{m=j+1}^{t} a_m^2 \right] \sum_{m=j+1}^{t} a_m^{-2} \left[\int_{Q\cap G_m\cap S_{1/4}} |u|\,dx \right]^2,
\end{aligned}$$

for any positive sequence a_m. We choose a_m to be given by

$$a_m = \left[(3/2)^{3(m-j-1)\varepsilon}\,|Q\cap G_m\cap S_{1/4}|/|Q|^{2/3}\right]^{1/2}.$$

Then, in view of the last two inequalities we have

$$|Q|^{2/3}\,u_Q^2 \le C\sum_{m=j+1}^{t}\left(\frac{2}{3}\right)^{3(m-j-1)\varepsilon}|Q\cap G_m\cap S_{1/4}|^{-1}\left[\int_{Q\cap G_m\cap S_{1/4}}|u|\,dx\right]^2,$$

for some constant C depending only on $\varepsilon > 0$. We conclude then that

$$\sum_{Q\in\mathcal{F}_j}|Q|^{2/3}\,u_Q^2 \le C\sum_{m=j+1}^{t}\left(\frac{2}{3}\right)^{3(m-j-1)\varepsilon}\int_{G_m\cap S_{1/4}}|u|^2\,dx.$$

Now if we sum this last inequality with respect to j and use the fact that the sets G_j are disjoint we obtain the inequality (2.4). ■

We return now to the operator Q_ρ of (1.6). The proof in section 4 of [**CO**] that Q_ρ is a bounded operator on L^2 had three ingredients. The first stage was to show that the function $\mathbf{b}\cdot\nabla Pg$ with domain $B_{1/4}$ is in an appropriate Morrey space, This Morrey space, $M_r^{3/2}(B_{1/4})$, is defined to be the set of functions $h : B_{1/4} \to \mathbb{R}$ such that

$$\int_{Q\cap B_{1/4}}(1/4-|y|)^r|h(y)|^r\,dy \le C^r|Q|^{1-2r/3},$$

on all cubes Q. The norm of h, $\|h\|_{3/2,r}$, is the minimum C satisfying the previous inequality. It was shown in [**CO**] that $\mathbf{b}\cdot\nabla Pg$ is in $M_r^{3/2}(B_{1/4})$ provided r satisfies $r/(2-r) < p$. We see now how this follows from Proposition 2.1. By the Harnack inequality there is a constant C such that

$$\begin{aligned}(1/4-|y|)|\mathbf{b}(y)\cdot\nabla Pg(y)| &\le C\,|\mathbf{b}(y)|\,|Pg(y)|\\ &= C\,|\mathbf{b}(y)|^{1/2}\,|Ag(y)|,\quad |y|<1/4.\end{aligned}$$

Hence for any cube Q one has

$$\begin{aligned}&\int_{Q\cap B_{1/4}}(1/4-|y|)^r|\mathbf{b}(y)\cdot\nabla Pg(y)|^r\,dy\\ &\qquad\le C\left[\int_{B_{1/4}}|Ag(y)|^2\,dy\right]^{r/2}\left[\int_Q|\mathbf{b}(y)|^{r/(2-r)}\,dy\right]^{1-r/2}\\ &\qquad\le C\,\|A\|^r\,\|g\|_2^r\,\|\mathbf{b}\|_{3,p}^{r/2}\,|Q|^{1-2r/3}\\ &\qquad\le C_1\|g\|_2^r\,\|\mathbf{b}\|_{3,p}^r\,|Q|^{1-2r/3}.\end{aligned}$$

If follows that if $r/(2-r) < p$ the function $\mathbf{b} \cdot \nabla Pg$ is in $M_r^{3/2}(B_{1/4})$ and

$$\|\mathbf{b} \cdot \nabla Pg\|_{3/2,r} \le C \, \|\mathbf{b}\|_{3,p} \, \|g\|_2$$

for some constant C depending only on $p > 1$.

The second stage was to prove that, for T the integral operator given by (1.3), the function $(I-T)^{-1}\mathbf{b} \cdot \nabla Pg$ is also in $M_r^{3/2}(B_{1/4})$. Thus the perturbation due to the integral operator T is small in the sense that it preserves the space in which $\mathbf{b} \cdot \nabla Pg$ lies. We shall do something analagous here. For $y \in \mathbb{R}^3$ let us define a vector $\mathbf{n}(y)$ by

$$\begin{aligned} \mathbf{n}(y) &= 0 && \text{if } \mathbf{b}(y) = 0. \\ &= \mathbf{b}(y)/|\mathbf{b}(y)| && \text{otherwise.} \end{aligned}$$

Evidently one has $|\mathbf{n}(y)| \le 1$ for all $y \in \mathbb{R}^3$. Let $T_{\rm sym}$ be the integral operator on functions with domain $B_{1/4}$ which has kernel

$$|\mathbf{b}(x)|^{1/2}\mathbf{n}(x) \cdot \nabla_x G_D(x,y)|\mathbf{b}(y)|^{1/2}, \quad x,\, y \in B_{1/4}.$$

Then we may formally write

$$(I-T)^{-1}\mathbf{b}\cdot\nabla Pg(y) = |\mathbf{b}(y)|^{1/2}(I-T_{\rm sym})^{-1}|\mathbf{b}|^{1/2}\mathbf{n}\cdot\nabla Pg(y), \quad y \in B_{1/4}.$$

Let $L^2_{\rm weight}(B_{1/4})$ be the weighted L^2 space of functions $h : B_{1/4} \to \mathbb{C}$ such that

$$\|h\|^2_{2,\rm weight} = \int_{B_{1/4}} (1/4 - |y|)^2 \, |h(y)|^2 \, dy < \infty.$$

From Proposition 2.1 and Harnack the function $|\mathbf{b}|^{1/2}\mathbf{n} \cdot \nabla Pg$ is in this space.

Proposition 2.2. *The operator $T_{\rm sym}$ is bounded on the space $L^2_{\rm weight}(B_{1/4})$ and there exists a constant $C > 0$ depending only on $p > 1$ such that*

$$\|T_{\rm sym}\| \le C \, \|\mathbf{b}\|_{3,p}.$$

Remark 2.1. It is interesting to compute the information that the results of Olsen [**O**] give about the operator $T_{\rm sym}$. If we apply Theorem 2 of [**O**] then

$$|T_{\rm sym}h(x)| \le T_g \, f(x), \quad x \in B_{1/4},$$

where $f(y) = |\mathbf{b}(y)|^{1/2}|h(y)|$ and $g(x) = |\mathbf{b}(x)|^{1/2}/2\pi$. Then if we assume h is in $L^2(B_{1/4})$ we have in the notation of Theorem 2 of [**O**] $p = 3/2$, $r = 3/2$, $v = 6$, and u, q can be taken slightly larger than 2, 1 respectively. Hence $T_g f \in M_s^t$ where

$$\frac{1}{t} = \frac{1}{v} + \frac{1}{r} + \frac{1}{p} - 1 = \frac{1}{6} + \frac{2}{3} + \frac{2}{3} - 1 = \frac{1}{2},$$
$$\frac{1}{s} = \frac{1}{v} + \frac{1}{q} + \frac{1}{p} - 1 = \frac{1}{6} + \frac{1}{q} + \frac{2}{3} - 1 = \frac{1}{q} - \frac{1}{6}.$$

Now if $s = 2$ then $T_{\text{sym}}h$ is also in $L^2(B_{1/4})$. Evidently $s = 2$ implies $q = 3/2$ which implies $q/(2-q) = 3$ whence we need to have $\mathbf{b} \in L^3(\mathbb{R}^3)$. We can do better than this by combining various theorems of [**O**].

Proposition 2.3. *The operator T_{sym} is bounded on the space $L^2(B_{1/4})$ and there exists a constant $C > 0$ depending only on $p > 1$ such that*
$$\|T_{\text{sym}}\| \le C\,\|\mathbf{b}\|_{3,p}.$$

Proof: We have

$$\begin{aligned}|T_{\text{sym}}h(x)| &\le \frac{|\mathbf{b}(x)|^{1/2}}{2\pi}\int_{B_{1/4}} \frac{|\mathbf{b}(y)|^{1/2}|h(y)|}{|x-y|^2}\,dy \\ &\le \frac{|\mathbf{b}(x)|^{1/2}}{2\pi}\left[\int_{B_{1/4}} \frac{|\mathbf{b}(x)|^{q'/2}}{|x-y|^2}\,dy\right]^{1/q'} \left[\int_{B_{1/4}} \frac{|h(y)|^q}{|x-y|^2}\,dy\right]^{1/q},\end{aligned}$$

where $1/q + 1/q' = 1$. We choose q' in the range $2 < q' < 2p$ whence $q < 2$. Consider now the function $g_1(x)$ defined by

$$g_1(x) = \int_{B_{1/4}} \frac{|\mathbf{b}(x)|^{q'/2}}{|x-y|^2}\,dy.$$

Now $|\mathbf{b}|^{q'/2}$ is in the Morrey space $M_{2p/q'}^{6/q'}$. Hence by Theorem 9 of [**O**] g_1 is in M_s^t where

$$1/t = q'/6 - 1/3, \quad s = t(2p/q')/(6/q') = tp/3,$$

and $\|g_1\|_{t,s} \le C\|\mathbf{b}\|_{3,p}^{q'/2}$ for some constant C depending only on p, q'. Next let $g(x)$ be given by

$$g(x) = \frac{|\mathbf{b}(x)|^{q/2}}{(2\pi)^q}\,g_1(x)^{q/q'}.$$

Then

$$|T_{\rm sym}h(x)| \le \Big[T_g|h|^q(x)\Big]^{1/q}. \tag{2.5}$$

By Holder's inequality for Morrey spaces, Lemma 11 of [**O**], we have that g is in M^3_p since

$$q/6 + q/q't = q/6 + (q/q')[q'/6 - 1/3] = 1/3,$$
$$q/2p + q/q's = q/2p + (q/q')(3/p)[q'/6 - 1/3] = 1/p.$$

Furthermore,

$$\|g\|_{3,p} \le \|\mathbf{b}\|^{q/2}_{3,p}\, \|g_1\|^{q/q'}_{t,s} \le C^{q-1}\, \|\mathbf{b}\|^q_{3,p}.$$

Now $|h|^q$ is in the space $L^{2/q}(B_{1/4})$. Hence by Corollary 3 of [**O**] $T_g\,|h|^q$ is also in the space $L^{2/q}(B_{1/4})$ provided $2/q < p$. Furthermore

$$\|T_g\,|h|^q\|_{2/q} \le C_1\,\|g\|_{3,p}\|\,|h|^q\,\|_{2/q} \le C_1\,C^{q-1}\,\|\mathbf{b}\|^q_{3,p}\,\|h\|^q_2,$$

for some constant C_1 depending only on q, p. Now from (2.5) and the previous inequality we conclude that

$$\|T_{\rm sym}h\|_2 \le C_1^{1/q}\,C^{1/q'}\,\|\mathbf{b}\|_{3,p}\,\|h\|_2.$$

The result follows since if $p>1$ the two inequalities $2<q'<2p$, $2/q<p$ can be simultaneously satisfied. In fact q can be chosen in the range

$$\frac{1}{2} < \frac{1}{q} < \min[p/2, 1-1/2p] = 1 - 1/2p. \quad \blacksquare$$

Remark 2.2. Theorem 9 of [**O**] was originally proved in [**A**]. A different proof was given in [**CF**].

The proof of Proposition 2.3 gives us an important insight into how the proof of Proposition 2.2 should go. We shall follow the lines of the proof of Theorem 1.2 of [**CR**] and Proposition 2.1 of [**CO**], but the Calderon-Zygmund decomposition will be based on taking averages of $|h(y)|^q$. Let Q_0 be a unit cube and for $n = 1, 2, \ldots$ Q_n be the dyadic subcubes of Q_0 with side of length 2^{-n}. Let $d(Q_n)$ be given by

$$d(Q_n) = \sup\{d(x, \partial B_{1/4}) : x \in Q_n\}.$$

For $n = 0, 1, 2, \ldots$ define an operator S_n on functions h with domain $B_{1/4}$ by

$$S_n h(x) = \frac{2^{2n}}{d(Q_n)} \int_{Q_n \cap B_{1/4}} (1/4 - |y|)\, |\mathbf{b}(y)|^{1/2}\, |h(y)|\, dy, \quad x \in Q_n \cap B_{1/4}.$$

Let T_{Q_0} be the operator given by

$$T_{Q_0}\, h(x) = \sum_{n=0}^{\infty} |\mathbf{b}(x)|^{1/2}\, S_n h(x), \quad x \in B_{1/4}.$$

Then Jensen's inequality implies there is a universal constant C such that if $Q_0(\xi)$ is the unit cube centered at ξ,

$$\text{(2.6)} \quad \int_{B_{1/4}} (1/4 - |x|)^2 |T_{\mathrm{sym}} h(x)|^2\, dx \\ \leq C \int_{B_{1/4}} d\xi \int_{Q_0(\xi) \cap B_{1/4}} (1/4 - |x|)^2 \left| T_{Q_0(\xi)} h(x) \right|^2 dx.$$

Hence it is sufficient to prove Proposition 2.2 with the operator T_{sym} replaced by T_{Q_0}. The key lemma analagous to Lemma 2.4 of [**CR**] is

Lemma 2.5. *Let $Q' \subset Q_0$ be an arbitrary dyadic subcube of Q_0 with side of length $2^{-n_{Q'}}$. Suppose $1 < p \leq 3$ and $q > 1$ satisfies the inequality $1/2 < 1/q < 1 - 1/2p$. For Q a dyadic subcube of Q_0 and h a function with domain $B_{1/4}$ let h_Q be given by*

$$h_Q = \left[\frac{1}{|Q|} \int_{Q \cap B_{1/4}} (1/4 - |y|)^q\, |h(y)|^q\, dy \right]^{1/q}.$$

Then there are constants C, $\varepsilon > 0$ depending only on p and q such that

$$\text{(2.7)} \quad |Q|^{1/6+\varepsilon}\, h_Q \leq |Q'|^{1/6+\varepsilon}\, h_{Q'}$$

for all dyadic subcubes Q of Q' implies the inequality

$$\int_{Q' \cap B_{1/4}} (1/4 - |x|)^2 \left[\sum_{n=n_{Q'}}^{\infty} |\mathbf{b}(x)|^{1/2}\, S_n h(x) \right]^2 dx \leq C^2 \|\mathbf{b}\|_{3,p}^2\, |Q'|\, h_{Q'}^2.$$

Proof: We have

$$\left[\sum_{n=n_{Q'}}^{\infty} S_n h(x)\right]^2 \le 2\sum_{k=n_{Q'}}^{\infty} S_k h(x) \sum_{n=n_{Q'}}^{k} S_n h(x).$$

It follows from Holder's inequality that

$$S_n h(x) \le \frac{2^{2n}}{d(Q_n)}\left[\int_{Q_n\cap B_{1/4}} (1/4-|y|)^q\,|h(y)|^q\,dy\right]^{1/q}\left[\int_{Q_n}|\mathbf{b}(y)|^{q'/2}\,dy\right]^{1/q'},$$

where $1/q+1/q'=1$. Since $1/q<1-1/2p$ implies $q'/2<p$ we conclude that

$$S_n h(x) \le \|\mathbf{b}\|_{3,p}^{1/2}\,|Q_n|^{1/6}\,h_{Q_n}\Big/ d(Q_n), \quad x\in Q_n,$$

whence

$$(1/4-|x|)S_n h(x) \le \|\mathbf{b}\|_{3,p}^{1/2}\,|Q_n|^{1/6}\,h_{Q_n}, \quad x\in Q_n. \tag{2.8}$$

Now if we use (2.7) we have from the previous inequality

$$\begin{aligned}(1/4-|x|)\sum_{n=n_{Q'}}^{k} S_n h(x) &\le \|\mathbf{b}\|_{3,p}^{1/2}\sum_{n=n_{Q'}}^{k}|Q_n|^{1/6}\,h_{Q_n},\\ &\le \|\mathbf{b}\|_{3,p}^{1/2}\,|Q'|^{1/6}h_{Q'}\sum_{n=n_{Q'}}^{k} 2^{3\varepsilon(n-n_{Q'})}\\ &\le C_\varepsilon\|\mathbf{b}\|_{3,p}^{1/2}\,|Q'|^{1/6}\,h_{Q'}2^{3\varepsilon(k-n_{Q'})}\end{aligned}$$

for some constant C_ε depending only on $\varepsilon>0$. Hence

$$\int_{Q'\cap B_{1/4}}(1/4-|x|)^2\left[\sum_{n=n_{Q'}}^{\infty}|\mathbf{b}(x)|^{1/2}\,S_n h(x)\right]^2 dx \tag{2.9}$$

$$\le 2C_\varepsilon\|\mathbf{b}\|_{3,p}^{1/2}\,|Q'|^{1/6}h_{Q'}\sum_{k=n_{Q'}}^{\infty}2^{3\varepsilon(k-n_{Q'})}\int_{Q'\cap B_{1/4}}(1/4-|x|)\,|\mathbf{b}(x)|S_k h(x)\,dx.$$

For m an integer let E_m be the set

$$E_m=\Big\{x\in\mathbb{R}^3 : 2^{m-1}<|\mathbf{b}(x)|\le 2^m\Big\}.$$

For m, k integers and $k \geq n_{Q'}$ let $a_{m,k}$ be given by

$$a_{m,k} = \sum_{Q_k \subset Q'} |E_m \cap Q_k| \int_{Q_k} (1/4 - |y|)|\mathbf{b}(y)|^{1/2} \, |h(y)| \, dy.$$

Then we have

$$\sum_{k=n_{Q'}}^{\infty} 2^{3\varepsilon(k-n_{Q'})} \int_{Q' \cap B_{1/4}} (1/4 - |x|) \, |\mathbf{b}(x)| S_k h(x) \, dx \leq \sum_{m=-\infty}^{\infty} \sum_{k=n_{Q'}}^{\infty} 2^{3\varepsilon(k-n_{Q'})} 2^{m+2k} \, a_{m,k}. \tag{2.10}$$

There are two estimates on $a_{m,k}$ which we use. The first follows from (2.7). Thus from (2.8) we have

$$a_{m,k} \leq \sum_{Q_k \subset Q'} |E_m \cap Q_k| \, \|\mathbf{b}\|_{3,p}^{1/2} \, 2^{-5k/2} \, h_{Q_k},$$

and then (2.7) implies that

$$a_{m,k} \leq |E_m \cap Q'| \, \|\mathbf{b}\|_{3,p}^{1/2} \, 2^{-5k/2} \, 2^{(1/2+3\varepsilon)(k-n_{Q'})} h_{Q'}.$$

The second estimate is obtained by using the fact that $|E_m \cap Q_k| \leq |Q_k|$. Thus

$$a_{m,k} \leq 2^{-3k} \int_{Q'} (1/4 - |y|) \, |\mathbf{b}(y)|^{1/2} \, |h(y)| \, dy.$$

If we again apply (2.8) we have that

$$a_{m,k} \leq 2^{-3k} \, \|\mathbf{b}\|_{3,p}^{1/2} \, |Q'|^{5/6} \, h_{Q'}.$$

If follows now that for any α, $0 < \alpha < 1$, we have

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} \sum_{k=n_{Q'}}^{\infty} 2^{3\varepsilon(k-n_{Q'})} \, 2^{m+2k} \, a_{m,k} \\ &\quad \leq \sum_{m=-\infty}^{\infty} \sum_{k=n_{Q'}}^{\infty} 2^{3\varepsilon(k-n_{Q'})} \, 2^{m+2k} \Big[2^{-3k} \|\mathbf{b}\|_{3,p}^{1/2} \, |Q'|^{5/6} \, h_{Q'} \Big]^{\alpha} \\ &\qquad\qquad \Big[|E_m \cap Q'| \, \|\mathbf{b}\|_{3,p}^{1/2} \, 2^{-5k/2} \, 2^{(1/2+3\varepsilon)(k-n_{Q'})} h_{Q'} \Big]^{1-\alpha}. \end{aligned}$$

For any $\alpha > 0$ one can find sufficiently small $\varepsilon > 0$ such that the sum with respect to k above converges. Thus there is a constant $C_\alpha > 0$ depending only on α, ε and

$$(2.11)\qquad \sum_{m=-\infty}^{\infty}\sum_{k=n_{Q'}}^{\infty} 2^{3\varepsilon(k-n_{Q'})}\,2^{m+2k}a_{m,k} \le C_\alpha \|\mathbf{b}\|_{3,p}^{1/2}|Q'|^{\alpha+1/6}\,h_{Q'}\sum_{m=-\infty}^{\infty}2^m|E_m\cap Q'|^{1-\alpha}.$$

Let m_0 be an arbitrary integer. Evidently one has

$$\sum_{m=-\infty}^{m_0}2^m|E_m\cap Q'|^{1-\alpha}\le |Q'|^{1-\alpha}\sum_{m=-\infty}^{m_0}2^m = 2|Q'|^{1-\alpha}2^{m_0}.$$

Using the fact that

$$2^{mp}\,|E_m\cap Q'|\le 2^p\,\|\mathbf{b}\|_{3,p}^p|Q'|^{1-p/3},$$

it follows that if $\alpha > 0$ satisfies $(1-\alpha)p > 1$ then

$$\sum_{m=m_0+1}^{\infty}2^m|E_m\cap Q'|^{1-\alpha}\le C_\alpha\|\mathbf{b}\|_{3,p}^{p(1-\alpha)}|Q'|^{(1-p/3)(1-\alpha)}2^{m_0(1+\alpha p-p)},$$

for some finite constant C_α. Hence, setting $\lambda = 2^{m_0}$, it follows that

$$\sum_{m=-\infty}^{\infty}2^m|E_m\cap Q'|^{1-\alpha} \le 2|Q'|^{1-\alpha}\lambda + C_\alpha\|\mathbf{b}\|_{3,p}^{p(1-\alpha)}|Q'|^{(1-p/3)(1-\alpha)}\lambda^{(1+\alpha p-p)}.$$

The RHS of the last inequality is minimised when $\lambda \sim \|\mathbf{b}\|_{3,p}|Q'|^{-1/3}$. We conclude therefore that

$$\sum_{m=-\infty}^{\infty}2^m\,|E_m\cap Q'|^{1-\alpha}\le C_\alpha\,\|\mathbf{b}\|_{3,p}\,|Q'|^{2/3-\alpha},$$

for some finite constant C_α. Putting this last inequality together with (2.9), (2.10), and (2.11) we conclude that

$$\begin{aligned}\int_{Q'\cap B_{1/4}}(1/4-|x|)^2\left[\sum_{n=n_{Q'}}^{\infty}|\mathbf{b}(x)|^{1/2}\,S_nh(x)\right]^2 dx\\ \le C^2\|\mathbf{b}\|_{3,p}^{1/2}\,|Q'|^{1/6}\,h_{Q'}\|\mathbf{b}\|_{3,p}^{1/2}\,|Q'|^{\alpha+1/6}\,h_{Q'}\,\|\mathbf{b}\|_{3,p}\,|Q'|^{2/3-\alpha}\\ = C^2\,\|\mathbf{b}\|_{3,p}^2|Q'|\,h_{Q'}^2,\end{aligned}$$

for some constant C depending only on p and q. ■

Proposition 2.2 follows now from (2.6) and Lemma 2.5 just in the same way as Theorem 1.2 of [**CR**] follows from Lemma 2.4 of [**CR**]. Next we consider the third stage in section 4 of the proof that the operator Q_ρ is bounded on L^2. Let K_ρ be an operator on functions $h : B_{1/4} \to \mathbb{C}$ defined by

$$K_\rho h(x) = \int_{B_{1/4}} dy\, G_D(x,y)|\mathbf{b}(y)|^{1/2}\, h(y), \quad |x| = \frac{1}{4}(1-\rho), \tag{2.12}$$

where $0 < \rho < 1/2$.

Proposition 2.4. *For $0 < \rho < 1/2$, K_ρ is a bounded operator from $L^2_{\text{weight}}(B_{1/4})$ to $L^2(\partial B_{(1-\rho)/4})$ and the norm of K_ρ satisfies an inequality $\|K_\rho\| \le C\, \|\mathbf{b}\|_{3,p}^{1/2}$, where the constant C depends only on $p > 1$.*

Proof: We write the operator K_ρ as a sum

$$K_\rho = \sum_{n=0}^{\infty} K_{\rho,n},$$

where

$$K_{\rho,0}h(x) = \int_{B_{1/4}\cap\left\{|x-y|<\frac{1}{5}\rho\right\}} dy\, G_D(x,y)|\mathbf{b}(y)|^{1/2}\, h(y),$$

$$K_{\rho,n}h(x) = \int_{B_{1/4}\cap\left\{\frac{1}{5}\rho 2^{n-1}\le|x-y|<\frac{1}{5}\rho 2^n\right\}} dy\, G_D(x,y)|\mathbf{b}(y)|^{1/2}\, h(y), \qquad n = 1,2,\dots .$$

Evidently

$$\begin{aligned} |K_{\rho,0}h(x)| &\le \int_{B_{1/4}\cap\left\{|x-y|<\frac{1}{5}\rho\right\}} \frac{|\mathbf{b}(y)|^{1/2}|h(y)|}{|x-y|}\, dy \\ &\le \left[\int_{\left\{|x-y|<\frac{1}{5}\rho\right\}} \frac{|\mathbf{b}(y)|}{|x-y|}\, dy\right]^{1/2} \left[\int_{B_{1/4}\cap\left\{|x-y|<\frac{1}{5}\rho\right\}} \frac{|h(y)|^2}{|x-y|}\, dy\right]^{1/2}, \end{aligned}$$

by the Schwarz inequality. Since $\mathbf{b} \in M_p^3$ there is a universal constant C such that

$$\int_{\left\{|x-y|<\frac{1}{5}\rho\right\}} \frac{|\mathbf{b}(y)|}{|x-y|}\, dy \le C\, \|\mathbf{b}\|_{3,p}\, \rho.$$

Hence

$$\int_{|x|=\frac14(1-\rho)} |K_{\rho,0}h(x)|^2\,dx \le C\|\mathbf{b}\|_{3,p}\rho \int_{\{|x|=\frac14(1-\rho)\}} dx \int_{\left\{|x-y|<\frac15\rho\right\}} \frac{|h(y)|^2}{|x-y|}\,dy$$

$$\le C_1\,\|\mathbf{b}\|_{3,p}\,\rho^2 \int_{\{1/4-|y|>\rho/20\}\cap B_{1/4}} |h(y)|^2\,dy$$

$$\le C_2\,\|\mathbf{b}\|_{3,p} \int_{B_{1/4}} (1/4-|y|)^2|h(y)|^2\,dy$$

$$= C_2\,\|\mathbf{b}\|_{3,p}\,\|h\|^2_{2,\text{weight}}$$

where C_1 and C_2 are universal constants. Hence we have shown that $K_{\rho,0}$ is a bounded operator and that

$$\|K_{\rho,0}\| \le C_2^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}.$$

Next we consider $K_{\rho,n}$ for $n \ge 1$. We use the fact that there is a universal constant C such that if $|x| = (1-\rho)/4$, then

$$G_D(x,y) \le C\rho(1/4-|y|)/(\rho 2^n)^3,\ y\in B_{1/4}\cap\left\{\frac15\rho 2^{n-1} \le |x-y| < \frac15\rho 2^n\right\}.$$

Applying the Schwarz inequality as we did before we have

$$|K_{\rho,n}h(x)|$$

$$\le \frac{C_1\rho}{(\rho 2^n)^3}\|\mathbf{b}\|_{3,p}^{1/2}\,\rho 2^n\left[\int_{B_{1/4}\cap\left\{\frac15\rho 2^{n-1}\le|x-y|<\frac15\rho 2^n\right\}} (1/4-|y|)^2|h(y)|^2\,dy\right]^{1/2},$$

for some universal constant C_1. Hence there is a universal constant C_2 such that

$$\int_{|x|=\frac14(1-\rho)} |K_{\rho,n}h(x)|^2\,dx \le C_2\|\mathbf{b}\|_{3,p}\,2^{-2n}\,\|h\|^2_{2,\text{weight}}.$$

It follows that $K_{\rho,n}$ is a bounded operator and

$$\|K_{\rho,n}\| \le C_2^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}\,2^{-n}, \quad n \ge 1.$$

Now the boundedness of K_ρ follows from the Minkowski inequality

$$\|K_\rho\| \le \sum_{n=0}^{\infty} \|K_{\rho,n}\| \le C_2^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}\sum_{n=0}^{\infty} 2^{-n} = 2C_2^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}. \quad \blacksquare$$

Proof of Theorem 1.4: Suppose g is in $L^2(\partial B_{1/4})$. Then from Proposition 2.1 and Harnack the function $h(y) = |\mathbf{b}(y)|^{1/2}\mathbf{n}(y) \cdot \nabla Pg(y)$ is in $L^2_{\text{weight}}(B_{1/4})$ and

$$\|h\|_{2,\text{weight}} \le C\, \|\mathbf{b}\|_{3,p}^{1/2}\, \|g\|_2,$$

where C is a constant depending only on $p > 1$. By Proposition 2.2 the function u defined by $u = (I - T_{\text{sym}})^{-1}h$ is in $L^2_{\text{weight}}(B_{1/4})$ and

$$\|u\|_{2,\text{weight}} \le C_1\, \|h\|_{2,\text{weight}},$$

for some constant C_1 depending only on $p > 1$ provided $\|\mathbf{b}\|_{3,p}$ is sufficiently small. Finally we have $Q_\rho g = K_\rho u$. Hence Proposition 2.4 and the previous two inequalities tells us that Q_ρ is a bounded operator from $L^2(\partial B_{1/4})$ to $L^2(\partial B_{(1-\rho)/4})$ provided $\|\mathbf{b}\|_{3,p}$ is sufficiently small and $\|Q_\rho\| \le C\|\mathbf{b}\|_{3,p}$ for some constant C depending only on $p > 1$. ■

Chapter 3. Fourier Space Localisation

Our first goal is to prove a version of Theorem 1.5 which takes account of the location of g in Fourier space.

Theorem 3.1. *Suppose f, g are in $L^2(S)$. Then there exists $\varepsilon > 0$ and a constant $C > 0$ depending only on $p > 1$ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies*

$$|\langle f, Q_\rho E_k g\rangle| \le C\|\mathbf{b}\|_{3,p}\, \|f\|_2\, \|E_k g\|_2\, \rho 2^k, \quad 0 \le k < \infty. \tag{3.1}$$

We can prove Theorem 3.1 by slightly modifying the proof of Theorem 1.4. The main point to observe is that the quantity 2^k in the estimate (3.1) replaces the weighting factor in the L^2 norm of Proposition 2.2. We shall therefore need to apply Proposition 2.3 here instead of Proposition 2.2. First we define an operator which is analogous to the operator A of (2.1). Thus for $0 < \rho < 1/2$ let $A_\rho f(y)$ be defined on functions f with domain $\partial B_{(1-\rho)/4}$ by

$$A_\rho f(y) = |\mathbf{b}(y)|^{1/2}\frac{4}{\rho}\int_{|x|=(1-\rho)/4} f(x) G_D(x,y)\, dx, \quad |y| < 1/4. \tag{3.2}$$

It is evident that

$$\lim_{\rho\to 0} A_\rho f(y) = Af(y), \quad |y| < 1/4.$$

The following proposition is therefore a generalization of Proposition 2.1.

Proposition 3.1. *For $0 < \rho < 1/2$, A_ρ is a bounded linear operator from $L^2(\partial B_{(1-\rho)/4})$ to $L^2(B_{1/4})$. There is a constant C depending only on $p > 1$ such that $\|A_\rho\| \le C\,\|\mathbf{b}\|_{3,p}^{1/2}$.*

Proof: Let $\mathcal{U}_\rho$ be the spherical shell

$$\mathcal{U}_\rho = \left\{ y : \frac{1}{4}(1 - 3\rho/2) < |y| < \frac{1}{4} \right\}$$

and χ_ρ be the characteristic function of the set $\mathcal{U}_\rho$. Let us first consider the operator K_ρ defined by

$$K_\rho f(y) = \chi_\rho(y)\, A_\rho f(y), \quad y \in B_{1/4}.$$

We show that K_ρ is bounded by arguing as in Proposition 2.4. Thus we write

$$K_\rho = \sum_{n=0}^{\infty} K_{\rho,n},$$

where

$$K_{\rho,0} f(y) = \chi_\rho(y) |\mathbf{b}(y)|^{1/2} \frac{4}{\rho} \int_{\left\{|x|=\frac{1}{4}(1-\rho),\, |x-y|<\rho\right\}} f(x) G_D(x,y)\, dx,$$

$$K_{\rho,n} f(y) = \chi_\rho(y) |\mathbf{b}(y)|^{1/2} \frac{4}{\rho} \int_{\left\{|x|=\frac{1}{4}(1-\rho),\, \rho 2^{n-1} \le |x-y| < \rho 2^n\right\}} f(x) G_D(x,y)\, dx, \quad n \ge 1.$$

We have now from the Schwarz inequality that

$$|K_{\rho,0} f(y)|^2 \le C\, \chi_\rho(y) |\mathbf{b}(y)|\, \rho^{-1} \int_{\left\{|x|=\frac{1}{4}(1-\rho),\, |x-y|<\rho\right\}} \frac{|f(x)|^2}{|x-y|}\, dx,$$

for some universal constant C. Hence

$$\begin{aligned} \|K_{\rho,0} f(y)\|_2^2 &\le C\, \rho^{-1} \int_{|x|=\frac{1}{4}(1-\rho)} dx |f(x)|^2 \int_{|x-y|<\rho} \frac{|\mathbf{b}(y)|}{|x-y|}\, dy \\ &\le C_1\, \|\mathbf{b}\|_{3,p}\, \|f\|_2^2 \end{aligned}$$

for some constant C_1. We conclude that $\|K_{\rho,0}\| \le C_1^{1/2}\, \|\mathbf{b}\|_{3,p}^{1/2}$. To estimate $K_{\rho,n}$ for $n \ge 1$ we use the bound

$$G_D(x,y) \le C\, \rho/(\rho 2^n)^2, \quad |x| = \frac{1}{4}(1-\rho), \quad \rho 2^{n-1} \le |x-y| < \rho 2^n,$$

where C is a universal constant. Hence we have

$$
\begin{aligned}
&|K_{\rho,n}f(y)|^2 \\
&\quad\le C_1\chi_\rho(y)|\mathbf{b}(y)|\frac{1}{(\rho 2^n)^2}\int_{\left\{|x|=\frac{1}{4}(1-\rho),\ \rho 2^{n-1}\le|x-y|<\rho 2^n\right\}}|f(x)|^2\,dx,
\end{aligned}
$$

for some universal constant C_1. Hence

$$
\|K_{\rho,n}f\|_2^2 \le \frac{C_1}{(\rho 2^n)^2}\int_{|x|=\frac{1}{4}(1-\rho)} dx|f(x)|^2\int_{\mathcal{U}_\rho\cap\{\rho 2^{n-1}\le|x-y|<\rho 2^n\}}|\mathbf{b}(y)|\,dy.
$$

We have now

$$
\begin{aligned}
&\int_{\mathcal{U}_\rho\cap\{\rho 2^{n-1}\le|x-y|<\rho 2^n\}}|\mathbf{b}(y)|\,dy \\
&\quad\le \operatorname{meas}\left[\mathcal{U}_\rho\cap\{\rho 2^{n-1}\le|x-y|<\rho 2^n\}\right]^{1-1/p}\left[\int_{|x-y|<\rho 2^n}|\mathbf{b}(y)|^p\,dy\right]^{1/p} \\
&\quad\le C(\rho^3 2^{2n})^{1-1/p}\,\|\mathbf{b}\|_{3,p}\,(\rho 2^n)^{3/p-1} \\
&\quad= C\,\|\mathbf{b}\|_{3,p}\,\rho^2 2^{n(1+1/p)},
\end{aligned}
$$

for some universal constant C. From the last two inequalities we conclude that there is a universal constant C_1 such that

$$
\|K_{\rho,n}\| \le C_1^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}\,2^{-n(1-1/p)/2}, \quad n\ge 1.
$$

Hence

$$
\|K_\rho\| \le C_1^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2}\sum_{n=0}^{\infty}2^{-n(1-1/p)/2} \le C_2^{1/2}\,\|\mathbf{b}\|_{3,p}^{1/2} \tag{3.3}
$$

for some constant C_2 depending only on $p>1$.

Next we consider the operator A_0 defined by

$$
A_0\,f(y) = [1-\chi_\rho(y)]\,A_\rho\,f(y), \quad y\in B_{1/4}.
$$

It is easy to see that there is a universal constant C such that

$$
G_D(x,y) \le C\,\rho(1/4-|y|)/|x-y|^3, \quad y\in B_{1/4}\backslash\mathcal{U}_\rho, \quad |x| = \frac{1}{4}(1-\rho).
$$

Furthermore for $y \in B_{1/4} \backslash \mathcal{U}_\rho$ one has $1/4 - |y| \le C_1 \left[\frac{1}{4}(1-\rho) - |y|\right]$ for a suitable constant C_1. Hence the operator A_0 has a kernel which is bounded by a constant times the kernel of A. Applying Proposition 2.1 we conclude that A_0 is bounded and $\|A_0\| \le C_2^{1/2} \|\mathbf{b}\|_{3,p}^{1/2}$. Since $A_\rho = A_0 + K_\rho$ the result follows from this last inequality and (3.3). ∎

Proposition 3.1 enables us to pull out the factor ρ in the inequality (3.1). Next we address the problem of how to pull out the factor 2^k in (3.1). The factor occurs due to the effect of the gradient in the expression $\nabla(PE_k g)$. Suppose $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We shall want to show that

$$|y| \frac{\partial}{\partial y_i} PE_k g(y) \simeq 2^k \, P \, h_i(y), \quad |y| < 1/4, \quad 1 \le i \le 3, \tag{3.4}$$

where h_i is an L^2 function on $\partial B_{1/4}$ with norm comparable to g. Furthermore we shall need to show that h_i is approximately concentrated in Fourier space on the range of E_k. To do this we introduce polar coordinates (r, θ, φ) on the ball, $0 \le r < 1/4$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$. We may also assume that $i = 3$ in (3.4) and the representation

$$\frac{\partial}{\partial y_3} = \cos\theta \frac{\partial}{\partial r} - \sin\theta \frac{1}{r} \frac{\partial}{\partial \theta}.$$

Let $Y_{\ell,m}(\theta, \varphi)$ be the spherical harmonics on the unit sphere. Thus ℓ is a nonnegative integer, m is an integer satisfying $-\ell \le m \le \ell$ and

$$-\Delta_S \, Y_{\ell,m} = \ell(\ell+1) \, Y_{\ell,m},$$
$$\frac{1}{i} \frac{\partial}{\partial \varphi} \, Y_{\ell,m} = m \, Y_{\ell,m}.$$

Now the Poisson kernel applied to the boundary data $Y_{\ell,m}$ yields

$$PY_{\ell,m}(r, \theta, \varphi) = (4r)^\ell \, Y_{\ell,m}(\theta, \varphi).$$

Thus

$$|y| \frac{\partial}{\partial y_3} \, P \, Y_{\ell,m}(r, \theta, \varphi) = \ell (4r)^\ell \cos\theta \, Y_{\ell,m}(\theta, \varphi) - (4r)^\ell \sin\theta \frac{\partial}{\partial \theta} \, Y_{\ell,m}(\theta, \varphi).$$

Let $P_{\ell,m}(z)$, $\ell = 0, 1, 2, \ldots$, $0 \le m \le \ell$ be the associated Legendre functions. Then one has [**M**, p. 495], for $m \ge 0$,

$$Y_{\ell,\pm m}(\theta, \varphi) = \sigma_{\pm m,\ell} \left[\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!} \right]^{1/2} P_{\ell,m}(\cos\theta) e^{\pm i m \varphi},$$

where $|\sigma_{\pm m,\ell}| = 1$. If we use the relations

$$(2\ell+1)z\,P_{\ell,m}(z) = (\ell+m)P_{\ell-1,m}(z) + (\ell+1-m)P_{\ell+1,m}(z),$$

$$(z^2-1)\frac{d}{dz}\,P_{\ell,m}(z) = (\ell-m+1)P_{\ell+1,m}(z) - (\ell+m+1)zP_{\ell,m}(z),$$

to be found in [**H**, p. 289,290], we may conclude that

$$\cos\theta Y_{\ell,m}(\theta,\varphi) = \left[\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1,m}(\theta,\varphi) + \left[\frac{(\ell+1+m)(\ell+1-m)}{(2\ell+3)(2\ell+1)}\right]^{1/2} Y_{\ell+1,m}(\theta,\varphi),$$

$$\sin\theta\frac{\partial}{\partial\theta}Y_{\ell,m}(\theta,\varphi) = -(\ell+m+1)\left[\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1,m}(\theta,\varphi) + (\ell-m)\left[\frac{(\ell+1+m)(\ell+1-m)}{(2\ell+3)(2\ell-1)}\right]^{1/2} Y_{\ell+1,m}(\theta,\varphi).$$

Hence we have shown that

$$|y|\frac{\partial}{\partial y_3}PY_{\ell,m}(r,\theta,\varphi) = \ell\left[ar\,PY_{\ell-1,m}(r,\theta,\varphi) + br^{-1}PY_{\ell+1,m}(r,\theta,\varphi)\right],$$

where a, b are constants which are bounded by 16 in absolute value. It is easy now to state a rigorous version of (3.4).

Lemma 3.1. *Let g be square integrable on the sphere $\partial B_{1/4}$, such that $E_k g = g$ for some $k \geq 1$. Then for any i, $1 \leq i \leq 3$, there exist functions h_+, h_- on $\partial B_{1/4}$ such that*

$$|y|\frac{\partial}{\partial y_i}Pg(y) = 2^k\,(4|y|)Ph_-(y) + 2^k(1/4|y|)\,Ph_+(y), \quad |y| < 1/4,$$

and the functions h_+, h_- satisfy

$$(E_k + E_{k+1})h_+ = h_+, \quad (E_k + E_{k-1})h_- = h_-,$$
$$\|h_+\|_2 \leq C\,\|g\|_2, \qquad \|h_-\|_2 \leq C\,\|g\|_2,$$

where C is a universal constant.

Proof of Theorem 3.1: Consider the function h_k on $B_{1/4}$ defined by

$$h_k(y) = 2^{-k}\,|\mathbf{b}(y)|^{1/2}\,\mathbf{n}(y)\cdot\nabla P E_k g(y).$$

In view of Lemma 3.1 and Proposition 2.1 the function h_k is in $L^2(B_{1/4})$ and

$$\|h_k\|_2 \le C\,\|\mathbf{b}\|_{3,p}^{1/2}\,\|E_k g\|_2.$$

Next by Proposition 2.3 if $\|\mathbf{b}\|_{3,p}$ is sufficiently small then $u_k = (I - T_{\rm sym})^{-1}\,h_k$ is also in $L^2(B_{1/4})$ and

$$\|u_k\|_2 \le C_1\,\|h_k\|_2$$

for some constant $C_1 \le 2$. Observe now that

$$\langle f, Q_\rho E_k g\rangle = \rho 2^k\,\langle A_\rho f, u_k\rangle,$$

where A_ρ is the operator (3.2). The result follows now from the last two inequalities and Proposition 3.1. ■

Next we wish to prove a version of Theorem 1.5 which takes account of both the location of f and g in Fourier space.

Theorem 3.2. *Suppose f, g are in $L^2(S)$. Then there exists $\varepsilon > 0$ and constants η, $C > 0$ depending only on $p > 1$ such that $\|\mathbf{b}\|_{3,p} < \varepsilon$ implies*

$$|\langle E_{k'} f, Q_\rho E_k g\rangle| \le C\|\mathbf{b}\|_{3,p}\,\|E_{k'} f\|_2\,\|E_k g\|_2\,2^{\eta(k'-k)}, \quad 0 \le k,\, k' < \infty.$$

Evidently Theorem 3.2 is implied by Theorem 1.4 if $k \le k'$ so we shall assume that $k > k'$. The basic fact we want to use is that the function $P(E_k g)(y)$, $y \in B_{1/4}$ falls off rapidly from the boundary, $|y| = 1/4$. In fact a simple computation shows that the function is essentially concentrated on the shell $\frac{1}{4}(1-2^{-k}) < |y| < \frac{1}{4}$. We need to define a norm which is sensitive to this fact. For $h : B_{1/4} \to \mathbb{C}$, $k = 0, 1, 2, \ldots$ and $\delta > 0$ we define a norm $\|h\|_{2,k,\delta}$ by

$$\|h\|_{2,k,\delta} = \sup_{0\le r\le k}\left[2^{2r\delta}\int_{\mathcal{U}_{r,k}} |h(y)|^2\,dy\right]^{1/2},$$

where $\mathcal{U}_{r,k}$ are spherical shells given by

$$\mathcal{U}_{0,k} = \left\{y : \frac{1}{4}(1-2^{-k}) < |y| < \frac{1}{4}\right\},$$

$$\mathcal{U}_{r,k} = \left\{y : \frac{1}{4}(1-2^r\,2^{-k}) < |y| < \frac{1}{4}(1-2^{r-1}\,2^{-k})\right\}, \quad 1 \le r \le k.$$

Observe that if $k = 0$ then one just gets back the L^2 norm of h.

Lemma 3.2. *Suppose g is square integrable on $\partial B_{1/4}$ and A is the operator defined by (2.1). Then for $k = 0, 1, 2, \dots,$ and any $\delta > 0$ the function $AE_k g(y)$ has norm satisfying*

$$\|AE_k g\|_{2,k,\delta} \le C_{\delta,p}\, \|\mathbf{b}\|_{3,p}^{1/2} \|E_k g\|_2,$$

where the constant $C_{\delta,p}$ depends only on $\delta \ge 0$, $p > 1$.

Proof: Evidently

$$\int_{U_{0,k}} |AE_k g(y)|^2\, dy \le \int_{B_{1/4}} |AE_k g(y)|^2\, dy \le C_p\, \|g\|_2^2 \tag{3.5}$$

by Proposition 2.1. Next for $1 \le r \le k$ observe that if $y \in \mathcal{U}_{r,k}$ then

$$Pg(y) = P_r g_r(y)$$

where the operator P_r acts on functions with domain $\left\{|y| = \frac{1}{4}(1 - 2^{r-1}2^{-k})\right\}$ and gives the solution of the Dirichlet problem. Thus $u(y) = P_r h(y)$ satisfies

$$\Delta u(y) = 0, \qquad |y| < \frac{1}{4}(1 - 2^{r-1}\, 2^{-k}),$$

$$u(y) = h(y), \quad |y| = \frac{1}{4}(1 - 2^{r-1}\, 2^{-k}).$$

It is easy to see that the function g_r has L^2 norm bounded as $\|g_r\|_2 \le \exp[-c2^r]\, \|g\|_2$ for some positive constant $c > 0$. Now applying Theorem 1.4 again we have that

$$\int_{\mathcal{U}_{r,k}} |AE_k g(y)|^2\, dy \le \|g_r\|_2^2 \le C_p \exp[-c2^{r+1}]\, \|g\|_2^2.$$

The result follows now from this last inequality and (3.5). ∎

Next we need to show that the operator T_{sym} is a bounded operator on the space determined by $\|\ \|_{2,k,\delta}$. As before we shall define two spaces associated with this norm. First the space $L^2_{k,\delta}(B_{1/4})$ is defined by $h \in L^2_{k,\delta}(B_{1/4})$ if $\|h\|_{2,k,\delta} < \infty$, with norm given by $\|\ \|_{2,k,\delta}$. The weighted space $L^2_{k,\delta,\mathrm{weight}}(B_{1/4})$ is defined as all h such that the function $f(y) = (1/4 - |y|)h(y)$ has finite norm $\|f\|_{2,k,\delta} < \infty$. The weighted norm of h is then given by $\|h\|_{2,k,\delta,\mathrm{weight}} = \|f\|_{2,k,\delta}$. We have now two theorems analagous to Propositions 2.2, 2.3.

Proposition 3.2. *There exists $\delta > 0$ such that the operator $T_{\rm sym}$ is bounded on the space $L^2_{k,\delta,{\rm weight}}(B_{1/4})$ and there exists a constant $C > 0$ depending only on δ, $p > 1$ such that $\|T_{\rm sym}\| \le C\,\|\mathbf{b}\|_{3,p}$.*

Proposition 3.3. *There exists $\delta > 0$ such that the operator $T_{\rm sym}$ is bounded on the space $L^2_{k,\delta}(B_{1/4})$ and there exists a constant $C > 0$ depending only on δ, $p > 1$ such that $\|T_{\rm sym}\| \le C\,\|\mathbf{b}\|_{3,p}$.*

Proof: We shall first prove Proposition 3.3. Let us suppose $x \in \mathcal{U}_{k,k}$. Then

$$
\begin{aligned}
|T_{\rm sym}h(x)| &\le \frac{|\mathbf{b}(x)|^{1/2}}{2\pi}\sum_{s=0}^{k}\int_{\mathcal{U}_{s,k}} \frac{|\mathbf{b}(y)|^{1/2}|h(y)|}{|x-y|^2}\,dy \\
&\le T_{\rm sym}h_1(x) + C\,|\mathbf{b}(x)|^{1/2}\int_{B_{1/4}} |\mathbf{b}(y)|^{1/2}\,|h(y)|\,dy
\end{aligned}
$$

where C is a universal constant and

$$
\begin{aligned}
h_1(y) &= h(y), \quad y \in \mathcal{U}_{k-1,k}\cup\mathcal{U}_{k,k}, \\
&= 0, \qquad \text{otherwise.}
\end{aligned}
$$

By the Minkowski inequality we have then

$$
\begin{aligned}
\left[\int_{\mathcal{U}_{k,k}} |T_{\rm sym}h(x)|^2\,dx\right]^{1/2} &\le \|T_{\rm sym}h_1\|_2 \\
&\quad + C\left[\int_{B_{1/4}} |\mathbf{b}(x)|\,dx\right]^{1/2}\left[\int_{B_{1/4}} |\mathbf{b}(y)|^{1/2}|h(y)|\,dy\right].
\end{aligned}
$$

Evidently $\|h_1\|_2 \le 2^{-(k-2)\delta}\,\|h\|_{2,k,\delta}$. On the other hand

$$
\begin{aligned}
\int_{B_{1/4}} |\mathbf{b}(y)|^{1/2}\,|h(y)|\,dy &\le \sum_{s=0}^{k}\left[\int_{\mathcal{U}_{s,k}} |\mathbf{b}(y)|\,dy\right]^{1/2}\left[\int_{\mathcal{U}_{s,k}} |h(y)|^2\,dy\right]^{1/2} \\
&\le \sum_{s=0}^{k}\left[\int_{\mathcal{U}_{s,k}} |\mathbf{b}(y)|\,dy\right]^{1/2} 2^{-s\delta}\,\|h\|_{2,k,\delta} \\
&\le \sum_{s=0}^{k} \mathrm{meas}(\mathcal{U}_{s,k})^{(1-1/p)/2}\,\|\mathbf{b}\|_{3,p}^{1/2}\,2^{-s\delta}\,\|h\|_{2,k,\delta} \\
&\le C\sum_{s=0}^{k} \|\mathbf{b}\|_{3,p}^{1/2}\,\|h\|_{2,k,\delta}\,2^{(s-k)(1-1/p)/2}\,2^{-s\delta} \\
&\le C_1\,\|\mathbf{b}\|_{3,p}^{1/2}\,\|h\|_{2,k,\delta}\,2^{-k\delta}
\end{aligned}
$$

provided $\delta < (1-1/p)/2$. Now from Proposition 2.3 and the last four inequalities we conclude that

$$\left[\int_{\mathcal{U}_{k,k}} |T_{\text{sym}}h(x)|^2\,dx\right]^{1/2} \le C\,\|\mathbf{b}\|_{3,p}\,\|h\|_{2,k,\delta}\,2^{-k\delta},$$

where the constant C depends only on $p > 1$ and $\delta > 0$.

We can easily deal with the case of the integral of $T_{\text{sym}}h$ over $\mathcal{U}_{0,k}$ by observing that

$$\left[\int_{\mathcal{U}_{0,k}} |T_{\text{sym}}h(x)|^2\,dx\right]^{1/2} \le \|T_{\text{sym}}h\|_2 \le C\|\mathbf{b}\|_{3,p}\,\|h\|_2 \le C_\delta\,\|\mathbf{b}\|_{3,p}\,\|h\|_{2,k,\delta}$$

by Proposition 2.3. Thus we are left to deal with integrals over $\mathcal{U}_{r,k}$ with $1 \le r \le k-1$. Define the function $h_1(y)$ by

$$\begin{aligned} h_1(y) &= h(y), \quad y \in \mathcal{U}_{s,k}, \quad s \ge r-1, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

Next for $s \le r-2$ and integer n satisfying $2 \le n \le k-r+4$ let $g_{s,n}(x)$ be the function

$$g_{s,n}(x) = \frac{|\mathbf{b}(x)|^{1/2}}{2\pi} 2^{2n} \int_{\mathcal{U}_{s,k}\cap\{2^{-n}<|x-y|<2^{-n+1}\}} |\mathbf{b}(y)|^{1/2}\,|h(y)|\,dy.$$

Then we have the inequality

$$|T_{\text{sym}}h(x)| \le |T_{\text{sym}}h_1(x)| + \sum_{s=0}^{r-2}\sum_{n=2}^{k-r+4} g_{s,n}(x), \quad x \in \mathcal{U}_{r,k}.$$

It follows by the Minkowski inequality that

$$\begin{aligned} \text{(3.6)} \quad \left[\int_{\mathcal{U}_{r,k}} |T_{\text{sym}}h(x)|^2\,dx\right]^{1/2} &\le \left[\int_{\mathcal{U}_{r,k}} |T_{\text{sym}}h_1(x)|^2\,dx\right]^{1/2} \\ &+ \sum_{s=0}^{r-2}\sum_{n=2}^{k-r+4}\left[\int_{\mathcal{U}_{r,k}} g_{s,n}(x)^2\,dx\right]^{1/2}. \end{aligned}$$

By Proposition 2.3 we have

$$\left[\int_{\mathcal{U}_{r,k}} |T_{\rm sym}h_1(x)|^2\,dx\right]^{1/2} \le \|T_{\rm sym}h_1\|_2 \tag{3.7}$$
$$\le C\|\mathbf{b}\|_{3,p}\,\|h_1\|_2$$
$$\le C_1\|\mathbf{b}\|_{3,p}\,\|h\|_{2,k,\delta}\,2^{-r\delta}.$$

Observe next that for any z,

$$\int_{\mathcal{U}_{r,k}\cap\{|x-z|<2^{-n}\}} g_{s,n}(x)^2\,dx \le \frac{2^{4n}}{(2\pi)^2}\int_{\mathcal{U}_{r,k}\cap\{|x-z|<2^{-n}\}} |\mathbf{b}(x)|\,dx$$
$$\left[\int_{\mathcal{U}_{s,k}\cap\{|z-y|<2^{-n+2}\}} |\mathbf{b}(y)|^{1/2}\,|h(y)|\,dy\right]^2.$$

From Holder's inequality we have

$$\int_{\mathcal{U}_{r,k}\cap\{|x-z|<2^{-n}\}} |\mathbf{b}(x)|\,dx$$
$$\le \text{meas}\left[\mathcal{U}_{r,k}\cap\{|x-z|<2^{-n}\}\right]^{1-1/p}\left[\int_{|x-z|<2^{-n}} |\mathbf{b}(x)|^p\,dx\right]^{1/p}$$
$$\le C\,\|\mathbf{b}\|_{3,p}\,2^{(r-k-2n)(1-1/p)}\,2^{-n(3/p-1)}.$$

Similarly we have

$$\left[\int_{\mathcal{U}_{s,k}\cap\{|z-y|<2^{-n+2}\}}|\mathbf{b}(y)|^{1/2}|h(y)|\,dy\right]^2 \le \left[\int_{\mathcal{U}_{s,k}\cap\{|z-y|<2^{-n+2}\}} |\mathbf{b}(y)|\,dy\right]$$
$$\left[\int_{\mathcal{U}_{s,k}\cap\{|z-y|<2^{-n+2}\}}|h(y)|^2\,dy\right] \le C\|\mathbf{b}\|_{3,p}2^{(s-k-2n)(1-1/p)}2^{-n(3/p-1)}$$
$$\int_{\mathcal{U}_{s,k}\cap\{|z-y|<2^{-n}\}} |h(y)|^2\,dy.$$

The last three inequalities imply then that

$$\int_{\mathcal{U}_{r,k}} g_{s,n}(x)^2\,dx \le C\,\|\mathbf{b}\|_{3,p}^2\,2^{4n}\,2^{(r-k-2n)(1-1/p)}\,2^{-n(3/p-1)}$$
$$2^{(s-k-2n)(1-1/p)}\,2^{-n(3/p-1)}\int_{\mathcal{U}_{s,k}} |h(y)|^2\,dy$$
$$\le C\|\mathbf{b}\|_{3,p}^2 2^{-2r\delta}\|h\|_{2,k,\delta}^2 2^{-2(k-r-n)(1-1/p)}2^{(s-r)(1-1/p-2\delta)}.$$

It follows then from the previous inequality and (3.6), (3.7) that if $\delta < (1-1/p)/2$ then

$$\left[\int_{\mathcal{U}_{r,k}} |T_{\mathrm{sym}} h(x)|^2\, dx\right]^{1/2} \le C\|\mathbf{b}\|_{3,p}\, \|h\|_{2,k,\delta}\, 2^{-r\delta}$$

for some constant C depending only on δ and $p > 1$. This completes the proof of Proposition 3.3. Proposition 3.2 follows in an exactly analogous way from Proposition 2.2. ■

Next let us consider the operator K_ρ defined by (2.12). Let f be a function in $L^2(\partial B_{(1-\rho)/4})$. We define the function $M_\rho f(y)$, $y \in B_{1/4}$ by

$$\int_{|x|=\frac{1}{4}(1-\rho)} f(x) K_\rho h(x)\, dx = \int_{B_{1/4}} \left(\frac{1}{4} - |y|\right) h(y) M_\rho f(y)\, dy,$$

for $h \in L^2_{\mathrm{weight}}(B_{1/4})$. Explicitly we have

$$M_\rho f(y) = \int_{|x|=\frac{1}{4}(1-\rho)} dx f(x) G_D(x,y) |\mathbf{b}(y)|^{1/2} \left(\frac{1}{4} - |y|\right)^{-1}.$$

In view of Proposition 2.4 we see that M_ρ is a bounded operator from $L^2(\partial B_{(1-\rho)/4})$ to $L^2(B_{1/4})$. Furthermore there is a constant C depending only on $p > 1$ such that

$$\|M_\rho\| \le C\, \|\mathbf{b}\|_{3,p}^{1/2} \tag{3.8}$$

provided $0 < \rho < 1/2$. We shall need to more accurately estimate the effect of M_ρ on a function f which is concentrated in a band of Fourier space. In particular we have the following:

Proposition 3.4. *Suppose k, k' are nonnegative integers and $0 < \rho < 1/2$. Then there exists $\delta > 0$ and a constant C depending only on $p > 1$ such that for $f \in L^2(\partial B_{(1-\rho)/4})$,*

$$\int_{\mathcal{U}_{0,k}} |M_\rho E_{k'} f(y)|^2\, dy \le C\, \|\mathbf{b}\|_{3,p}\, \|E_{k'} f\|_2^2\, 2^{-2(k-k')\delta}.$$

Proof: In view of the inequality (3.8) we may assume that $k > k'$. We shall first show that we may also assume $2^{-k'} > \rho$. This will follow from the inequality

$$\int_{\mathcal{U}_{0,k}} |M_\rho f(y)|^2\, dy \le C\, \|\mathbf{b}\|_{3,p}\, \|f\|_2^2\, (2^{-k}/\rho)^{2\delta}. \tag{3.9}$$

Observe that it is sufficient to prove (3.9) under the condition $2^{-k} < \rho/2$. In that case we write

$$M_\rho f(y) = \sum_{n=0}^{\infty} g_n(y), \quad y \in \mathcal{U}_{0,k}, \tag{3.10}$$

where

$$g_0(y) = \int_{\{|x|=\frac{1}{4}(1-\rho), |x-y|<\rho\}} dx f(x) G_D(x,y) |\mathbf{b}(y)|^{1/2} (1/4 - |y|)^{-1},$$

$$g_n(y) = \int_{\{|x|=\frac{1}{4}(1-\rho), 2^{n-1}\rho<|x-y|<2^n\rho\}} dx f(x) G_D(x,y) |\mathbf{b}(y)|^{1/2} (1/4 - |y|)^{-1}.$$

Since $2^{-k} < \rho/2$ we have

$$\begin{aligned} |g_n(y)| &\le \frac{C\rho}{(\rho 2^n)^3} |\mathbf{b}(y)|^{1/2} \int_{\{|x|=\frac{1}{4}(1-\rho), |x-y|<2^n\rho\}} |f(x)|\, dx \\ &\le \frac{C_1\rho}{(\rho 2^n)^2} |\mathbf{b}(y)|^{1/2} \left[\int_{\{|x|=\frac{1}{4}(1-\rho), |x-y|<2^n\rho\}} |f(x)|^2\, dx \right]^{1/2}, \\ &\qquad n = 0, 1, 2, \dots, \end{aligned} \tag{3.11}$$

for some universal constants C, C_1 by the Schwarz inequality. This last inequality implies

$$\begin{aligned} &\int_{\mathcal{U}_{0,k}} |g_n(y)|^2\, dy \\ &\quad \le \frac{C_1^2 \rho^2}{(\rho 2^n)^4} \int_{\{|x|=\frac{1}{4}(1-\rho)\}} dx |f(x)|^2 \int_{\mathcal{U}_{0,k} \cap \{|x-y|<2^n\rho\}} |\mathbf{b}(y)|\, dy. \end{aligned}$$

Now if we estimate

$$\begin{aligned} &\int_{\mathcal{U}_{0,k} \cap \{|x-y|<2^n\rho\}} |\mathbf{b}(y)|\, dy \\ &\quad \le \text{meas} \left[\mathcal{U}_{0,k} \cap \{|x-y| < 2^n\rho\} \right]^{1-1/p} \|\mathbf{b}\|_{3,p} \, (2^n\rho)^{3/p-1} \\ &\quad \le C \, \|\mathbf{b}\|_{3,p} \, (2^n\rho)^{1+1/p} \, 2^{-k(1-1/p)}, \end{aligned}$$

we can conclude that

$$\text{(3.12)}\qquad \left[\int_{\mathcal{U}_{0,k}} |g_n(y)|^2\,dy\right]^{1/2} \le C_1\,\|\mathbf{b}\|_{3,p}^{1/2}\,\|f\|_2\,(2^{-k}/\rho)^{(1-1/p)/2}\,2^{-n(3-1/p)/2},$$

for some universal constant C_1. Hence from (3.10) and the Minkowski inequality it follows that (3.9) holds with $\delta = (1-1/p)/2$.

We can assume now that $2^{-k'} > \rho$, $k > k'$. Let $f_{k'}(x)$ be the function

$$f_{k'} = 2^{-4k'}(-\Delta_S + 2^{2k'})^2 E_{k'} f,$$

where Δ_S is the Laplacian on the unit sphere. Evidently there is a universal constant C such that $\|f_{k'}\|_2 \le C\|E_{k'}f\|_2$. Furthermore $E_{k'}f$ can be written in terms of $f_{k'}$ by

$$E_{k'}f(x) = \int_{|x'|=\frac{1}{4}(1-\rho)} H_{k'}(x,x')f_{k'}(x')\,dx',$$

where $H_{k'}(x,x')$ is the kernel of the operator $2^{4k'}(-\Delta_S + 2^{2k'})^{-2}$. It is well known [**CH**] that there are constants C, $c > 0$ such that

$$0 \le H_{k'}(x,x') \le C\,2^{2k'}\exp[-c|x-x'|/2^{-k'}].$$

Again we write

$$M_\rho E_{k'}f(y) = \sum_{n=0}^{\infty} g_n(y), \qquad y \in \mathcal{U}_{0,k},$$

where

$$g_0(y) = \int_{\left\{|x|=\frac{1}{4}(1-\rho),|x'|=\frac{1}{4}(1-\rho),|x'-y|<2^{-k'}\right\}} dx\,dx'\,H_{k'}(x,x')f_{k'}(x')G_D(x,y)|\mathbf{b}(y)|^{1/2}\,(1/4-|y|)^{-1},$$

$$g_n(y) = \int_{\left\{|x|=\frac{1}{4}(1-\rho),|x'|=\frac{1}{4}(1-\rho),2^{n-1-k'}<|x'-y|<2^{n-k'}\right\}} dx\,dx'\,H_{k'}(x,x')f_{k'}(x')G_D(x,y)|\mathbf{b}(y)|^{1/2}\,(1/4-|y|)^{-1},$$

if $n \geq 1$. Observe now that

$$\int_{|x|=\frac{1}{4}(1-\rho)} dx H_{k'}(x,x')G_D(x,y)(1/4-|y|)^{-1} \leq C\, 2^{2k'} \int_{|x|=\frac{1}{4}(1-\rho)} dx\; G_D(x,y)(1/4-|y|)^{-1} \leq C_1\, 2^{2k'},$$

for some universal constant C_1. Hence

$$|g_0(y)| \leq C_1\, 2^{2k'} \int_{\left\{|x'|=\frac{1}{4}(1-\rho),|x'-y|<2^{-k'}\right\}} dx'|f_{k'}(x')|\,|\mathbf{b}(y)|^{1/2}. \tag{3.13}$$

Now if $2^{n-1-k'} < |x'-y| < 2^{n-k'}$, $n \geq 1$, then one easily sees that

$$\int_{|x|=\frac{1}{4}(1-\rho)} dx H_{k'}(x,x')G_D(x,y)(1/4-|y|)^{-1} \leq C\, 2^{2k'-3n},$$

for some universal constant C_2. Hence if $n \geq 1$ we have the inequality

$$|g_n(y)| \leq C_2\, 2^{2k'-3n} \int_{\left\{|x'|=\frac{1}{4}(1-\rho),|x'-y|<2^{n-k'}\right\}} dx'|f_{k'}(x')|\,|\mathbf{b}(y)|^{1/2}. \tag{3.14}$$

We can now bound the integrals of $|g_n(y)|^2$ over $\mathcal{U}_{0,k}$ exactly as in (3.12) by using the inequalities (3.13), (3.14). There is therefore a universal constant C_1 such that

$$\left[\int_{\mathcal{U}_{0,k}} |g_n(y)|^2\, dy\right]^{1/2} \leq C_1\, \|\mathbf{b}\|_{3,p}^{1/2}\, \|f_{k'}\|_2\, 2^{(k'-k)(1-1/p)/2}\, 2^{-n(3-1/p)/2}, \quad n \geq 0.$$

Since $\|f_{k'}\|_2 \leq C\, \|E_{k'}f\|_2$ the result follows as before. ■

Proof of Theorem 3.2: Let $h(y) = |\mathbf{b}(y)|^{1/2}\, \mathbf{n}(y) \cdot \nabla P E_k g(y)$, $y \in B_{1/4}$. Then from the Harnack principle and Lemma 3.2 it is easy to see that h is in the space $L^2_{k,\delta,\mathrm{weight}}(B_{1/4})$ for every $\delta > 0$ and

$$\|h\|_{2,k,\delta,\mathrm{weight}} \le C_{\delta,p} \|\mathbf{b}\|_{3,p}^{1/2}\, \|E_k g\|_2,$$

where the constant $C_{\delta,p}$ depends only on δ, $p > 1$. Now by Proposition 3.2 one sees that if $\|\mathbf{b}\|_{3,p}$ is sufficiently small then the function $\xi(y) = (1/4 - |y|)(I - T_{\mathrm{sym}})^{-1} h(y)$ is in $L^2_{k,\delta}(B_{1/4})$ and

$$\|\xi\|_{2,k,\delta} \le C_{\delta,p}\, \|\mathbf{b}\|_{3,p}^{1/2}\, \|E_k g\|_2,$$

for some suitable constant $C_{\delta,p}$. Observe next that

$$\begin{aligned}\left|\left\langle E_{k'} f, Q_\rho E_k g \right\rangle\right| &= \left| \int_{B_{1/4}} \xi(y) M_\rho E_{k'} f(y)\, dy \right| \\ &\le \sum_{r=0}^{k} \left[\int_{\mathcal{U}_{r,k}} |\xi(y)|^2\, dy \right]^{1/2} \left[\int_{\mathcal{U}_{r,k}} |M_\rho E_{k'} f(y)|^2\, dy \right]^{1/2}.\end{aligned}$$

If we use now Proposition 3.4 we have that

$$\begin{aligned}\left|\left\langle E_{k'} f, Q_\rho E_k g \right\rangle\right| &\le \sum_{r=0}^{k} 2^{-r\delta} \|\xi\|_{2,k,\delta} \left[\int_{\mathcal{U}_{r,k}} |M_\rho E_{k'} f(y)|^2\, dy \right]^{1/2} \\ &\le \sum_{r=0}^{k-k'} 2^{-r\delta} \|\xi\|_{2,k,\delta}\, C^{1/2} \|\mathbf{b}\|_{3,p}^{1/2}\, \|E_{k'} f\|_2\, 2^{-(k-k'-r)\delta'} \\ &\quad + \sum_{r=k-k'+1}^{k} 2^{-r\delta} \|\xi\|_{2,k,\delta}\, C^{1/2} \|\mathbf{b}\|_{3,p}^{1/2}\, \|E_{k'} f\|_2 \\ &\le C_1\, \|\mathbf{b}\|_{3,p}^{1/2}\, \|\xi\|_{2,k,\delta}\, \|E_{k'} f\|_2\, 2^{-(k-k')\delta} \\ &\le C_2\, \|\mathbf{b}\|_{3,p}\, \|E_k g\|_2\, \|E_{k'} f\|_2\, 2^{-(k-k')\delta}\end{aligned}$$

for constants C_1, C_2 depending only on $p > 1$ provided we choose $0 < \delta < \delta'$ appropriately. ■

The proof of Theorem 1.5 will be complete if we can prove:

Theorem 3.3. *Suppose f, g are in $L^2(S)$. Then there exists $\varepsilon > 0$ and constants η, $C > 0$ depending only on $p > 1$ such that if $\|\mathbf{b}\|_{3,p} < \varepsilon$ then*

$$\left|\left\langle E_{k'}f, Q_\rho E_k g\right\rangle\right| \le C\|\mathbf{b}\|_{3,p}\,\|E_{k'}f\|_2\,\|E_k g\|_2\,\rho 2^k\,2^{\eta(k-k')}, \quad 0 \le k,\, k' < \infty.$$

The main work to be done to prove this last proposition is to show that Proposition 3.4 also holds for the operator A_ρ defined by (3.2). Thus we have the following:

Proposition 3.5. *Suppose k, k' are nonnegative integers and $0 < \rho < 1/2$. Then there exists $\delta > 0$ and a constant C depending only on $p > 1$ such that for $f \in L^2(\partial B_{(1-\rho)/4})$,*

$$\int_{\mathcal{U}_{0,k}} \left|A_\rho E_{k'} f(y)\right|^2 dy \le C\,\|\mathbf{b}\|_{3,p}\,\|E_{k'}f\|_2^2\,2^{-2(k-k')\delta}.$$

Proof: We proceed in the same way as in Proposition 3.4. By Proposition 3.1 we can assume that $k \ge k'$. Next we show that one may also assume $2^{-k'} > \rho$. This follows from the inequality

$$\int_{\mathcal{U}_{0,k}} \left|A_\rho f(y)\right|^2 dy \le C\,\|\mathbf{b}\|_{3,p}\,\|f\|_2^2\,(2^{-k}/\rho)^{2\delta}. \tag{3.15}$$

Observe that it is sufficient to prove (3.15) under the condition $2^{-k} < \rho/2$. In that case we write

$$A_\rho f(y) = \sum_{n=0}^{\infty} g_n(y), \quad y \in \mathcal{U}_{0,k},$$

where

$$g_0(y) = \frac{4}{\rho}\int_{\left\{|x|=\frac14(1-\rho),|x-y|<\rho\right\}} dx f(x) G_D(x,y)|\mathbf{b}(y)|^{1/2},$$

$$g_n(y) = \frac{4}{\rho}\int_{\left\{|x|=\frac14(1-\rho),2^{n-1}\rho<|x-y|<2^n\rho\right\}} dx f(x) G_D(x,y)|\mathbf{b}(y)|^{1/2}, \quad n \ge 1.$$

Since $2^{-k} < \rho/2$ we have

$$|g_n(y)| \le \frac{C\rho}{(\rho 2^n)^3}|\mathbf{b}(y)|^{1/2}\int_{\left\{|x|=\frac14(1-\rho),|x-y|<2^n\rho\right\}} |f(x)|\,dx.$$

Since this last inequality is exactly the same as (3.11) we can conclude that the theorem holds in the case of $2^{-k'} \le \rho$. To deal with the case of $2^{-k'} > \rho$ we proceed again as in Proposition 3.4. The functions g_n are now defined by

$$g_0(y) = \frac{4}{\rho} \int_{\{|x|=\frac{1}{4}(1-\rho),|x'|=\frac{1}{4}(1-\rho),|x'-y|<2^{-k'}\}} dx\, dx' H_{k'}(x,x') f_{k'}(x') G_D(x,y) |\mathbf{b}(y)|^{1/2},$$

$$g_n(y) = \frac{4}{\rho} \int_{\{|x|=\frac{1}{4}(1-\rho),|x'|=\frac{1}{4}(1-\rho),2^{n-1-k'}<|x'-y|<2^{n-k'}\}} dx\, dx' H_{k'}(x,x') f_{k'}(x') G_D(x,y) |\mathbf{b}(y)|^{1/2},$$

if $n \ge 1$. Evidently one has

$$\frac{4}{\rho} \int_{|x|=\frac{1}{4}(1-\rho)} dx H_{k'}(x,x') G_D(x,y) \le \frac{4C2^{2k'}}{\rho} \int_{|x|=\frac{1}{4}(1-\rho)} dx G_D(x,y) \le C_1\, 2^{2k'},$$

for some universal constant C_1. Similarly

$$\frac{4}{\rho} \int_{|x|=\frac{1}{4}(1-\rho)} dx H_{k'}(x,x') G_D(x,y) \le C_2\, 2^{2k'-3n}$$

for some universal constant C_2 if $2^{n-1-k'} < |x'-y| < 2^{n-k'}$, $n \ge 1$. Now, using these last two estimates the proof of the theorem is identical to the proof of Proposition 3.4. ■

Proof of Theorem 3.3: Let $h(y) = |\mathbf{b}(y)|^{1/2} \mathbf{n}(y) \cdot \nabla P E_k g(y)$, $y \in B_{1/4}$. From Lemmas 3.1, 3.2 it follows that h is in the space $L^2_{k,\delta}(B_{1/4})$ for every $\delta > 0$ and

$$\|h\|_{2,k,\delta} \le C_{\delta,p} \|\mathbf{b}\|_{3,p}^{1/2}\, 2^k\, \|E_k g\|_2,$$

where the constant $C_{\delta,p}$ depends only on δ, $p > 1$. Now by Proposition 3.3 one sees that if $\|\mathbf{b}\|_{3,p}$ is sufficiently small then the function $\xi(y) = (I - T_{\mathrm{sym}})^{-1} h(y)$ is in $L^2_{k,\delta}(B_{1/4})$ and

$$\|\xi\|_{2,k,\delta} \le C_{\delta,p} \|\mathbf{b}\|_{3,p}^{1/2}\, 2^k\, \|E_k g\|_2, \tag{3.16}$$

for some suitable constant $C_{\delta,p}$. Observe next that

$$\left|\left\langle E_{k'}f, Q_\rho E_k g\right\rangle\right| = \left|\rho\int_{B_{1/4}} \xi(y) A_\rho E_{k'} f(y)\, dy\right|.$$

The rest of the proof follows now from (3.16) and Proposition 3.5 in exactly the same way as Theorem 3.2 follows from Proposition 3.4. ■

Chapter 4. Proof of Theorem 1.3

We first define the density $f_{\rho,\mathbf{b}}$ in terms of the density f. To do this we consider the Dirichlet problem

$$\begin{aligned}(\Delta + \mathbf{b}(y)\cdot\nabla)v(y) = 0, &\qquad y \in B_{1/4},\\ v(y) = g(y), &\quad y \in \partial B_{1/4}.\end{aligned}$$

Formally $v(y)$ is given by the formula

$$v(y) = Pg(y) + Qg(y), \quad y \in B_{1/4},$$

where P is the Poisson integral (1.4) and Qg is defined by (1.5). Now v can be represented in terms of the diffusion process $X_{\mathbf{b}}(t)$ by the expression

$$v(y) = E_y[g(X_{\mathbf{b}}(\tau))], \quad y \in B_{1/4},$$

where τ is the first hitting time on $\partial B_{1/4}$ for the process started at $X_{\mathbf{b}}(0) = y$. It is clear then that if we regard the density f as a function on $\partial B_{(1-\rho)/4}$ and the density $f_{\rho,\mathbf{b}}$ as a function on $\partial B_{1/4}$, then

$$\begin{aligned}&\int_{\partial B_{(1-\rho)/4}} f(y)v(y)\, dy \,/\text{normalisation}\\ &\qquad\qquad = \int_{\partial B_{1/4}} f_{\rho,\mathbf{b}}(y)g(y)\, dy \,/\text{normalisation},\end{aligned}$$

where the normalisations are chosen so that the measures are probability measures. It follows therefore, on going back to regarding f and $f_{\rho,\mathbf{b}}$ as functions on the unit sphere S that

$$\langle f_{\rho,\mathbf{b}}, g\rangle = \langle f, P_\rho g + Q_\rho g\rangle, \quad g \in L^2(S),$$

where $P_\rho g(y) = Pg(y)$, $y \in \partial B_{(1-\rho)/4}$. Hence $f_{\rho,\mathbf{b}} = P_\rho^* f + Q_\rho^* f$, where P_ρ^* and Q_ρ^* are the formal adjoints of P_ρ, Q_ρ respectively.

We can analyse the operator P_ρ precisely since we know its eigenfunctions. In fact if $Y_{l,m}(\theta,\phi)$, $0 \le \theta < \pi$, $0 \le \phi < 2\pi$, is a spherical harmonic and we take $g = Y_{l,m}$ then $Pg(y) = (4|y|)^l Y_{l,m}(\theta,\phi)$, $y \in B_{1/4}$. Hence

$$P_\rho Y_{l,m} = (1-\rho)^l Y_{l,m}. \tag{4.1}$$

It follows in particular that P_ρ is selfadjoint, whence $P_\rho = P_\rho^*$. We also have that $P_\rho 1 = P_\rho Y_{0,0} = 1$. Hence for any $f \in L^2(S)$ we have that

$$\|P_\rho f - Avf\|_2 \le (1-\rho)\|f - Avf\|_2. \tag{4.2}$$

Theorem 1.2 follows from (4.2) and the fact that $\|Q_\rho\| \le C\|\mathbf{b}\|_{3,p}$ where C depends only on p, ρ. In fact

$$\begin{aligned}\|f_{\rho,\mathbf{b}} - Avf_{\rho,\mathbf{b}}\|_2 &= \|P_\rho f + Q_\rho^* f - Avf\|_2 \\ &\le \|P_\rho f - Avf\|_2 + \|Q_\rho^* f\|_2 \\ &\le (1-\rho)\|f - Avf\|_2 + C\|\mathbf{b}\|_{3,p}\|f\|_2.\end{aligned}$$

Suppose now $\|f - Avf\|_2 \le \delta|Avf|$. Then $\|f\|_2 \le (1+\delta)|Avf|$. Hence the last inequality yields

$$\begin{aligned}\|f_{\rho,\mathbf{b}} - Avf_{\rho,\mathbf{b}}\|_2 &\le (1-\rho)\delta|Avf| + C\|\mathbf{b}\|_{3,p}(1+\delta)|Avf| \\ &= [1-\rho + C\|\mathbf{b}\|_{3,p}(1+\delta^{-1})]\delta|Avf_{\rho,\mathbf{b}}|,\end{aligned}$$

since $Avf = Avf_{\rho,\mathbf{b}}$. Theorem 1.2 follows from the last inequality by choosing $\|\mathbf{b}\|_{3,p}$ sufficiently small.

Theorem 1.3 follows by a similar argument from Theorem 1.5. Since $Y_{l,m}$ is an eigenfunction of $-\Delta_S$ with eigenvalue $l(l+1), l = 0,1,2,\ldots,$ it follows from (4.1) that there is a universal constant $c > 0$ with

$$\|E_k P_\rho f\|_2 \le \{1 - c\min[\rho 2^k, 1]\}\|E_k f\|_2, \quad k \ge 1. \tag{4.3}$$

The inequality (4.3) plays the same role in the proof of Theorem 1.3 as (4.2) plays in the proof of Theorem 1.2. By the Minkowski inequality we have

$$\begin{aligned}\|E_k f_{\rho,\mathbf{b}}\|_2 &= \|E_k P_\rho f + E_k Q_\rho^* f\|_2 \\ &\le \|E_k P_\rho f\|_2 + \|E_k Q_\rho^* f\|_2.\end{aligned}$$

Theorem 1.5 yields an appropriate estimate on $\|E_k Q_\rho^* f\|_2$, which when combined with (4.3) proves Theorem 1.3. To see this let us assume $\|f - Avf\|_{2,\nu} \le \delta |Avf|$. Then

$$\|E_k f\|_2 \le \delta |Avf| / 2^{\nu k}, \quad k = 1, 2, \ldots . \tag{4.4}$$

Now, for some g satisfying $\|E_k g\|_2 = 1$,

$$\|E_k Q_\rho^* f\|_2 = \langle f, Q_\rho E_k g\rangle \le \sum_{k'=0}^{\infty} |\langle E_{k'} f, Q_\rho E_k g\rangle|. \tag{4.5}$$

Hence from Theorem 1.5 and (4.4) we have,

$$\begin{aligned}\|E_k Q_\rho^* f\|_2 \le C\, \|\mathbf{b}\|_{3,p} |Avf| \min[\rho 2^k,\, 1] 2^{-\eta k} \\ + \sum_{k'=1}^{k} C\, \|\mathbf{b}\|_{3,p}\, \delta\, |Avf| 2^{-\nu k'} \min[\rho 2^k,\, 1] 2^{\eta(k-k')} \\ + \sum_{k'=k+1}^{\infty} C\, \|\mathbf{b}\|_{3,p} \delta\, |Avf| 2^{-\nu k'} \min[\rho 2^k,\, 1].\end{aligned}$$

Note that the first term on the right in the last inequality comes from the $k' = 0$ term on the right in (4.5). Hence if $\eta > \nu$ we have that

$$\begin{aligned}&\|E_k Q_\rho^* f\|_2 \le C \|\mathbf{b}\|_{3,p} |Avf| \min[\rho 2^k,\, 1] \\ &\quad \left\{ 2^{-\eta k} + \delta 2^{(\eta-\nu)(k+1)} 2^{-\eta k} / [2^{(\eta-\nu)} - 1] + \delta 2^{-\nu(k+1)} / [1 - 2^{-\nu}] \right\}.\end{aligned}$$

We conclude that

$$\|E_k Q_\rho^* f\|_2 \le C(\delta) \|\mathbf{b}\|_{3,p} \delta |Avf| \min[\rho 2^k,\, 1] 2^{-\nu k}, \quad k \ge 1,$$

where the constant $C(\delta)$ depends only on δ. Combining this last inequality with (4.3) we have that

$$\|E_k f_{\rho,\mathbf{b}}\|_2 \le \{1 + [C(\delta)\|\mathbf{b}\|_{3,p} - c] \min[\rho 2^k, 1]\} \delta |Avf| 2^{-\nu k}, \quad k \ge 1.$$

The theorem follows now by choosing $\|\mathbf{b}\|_{3,p}$ sufficiently small so that $C(\delta)\|\mathbf{b}\|_{3,p} < c$.

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