Publicacions Matemàtiques, Vol **42** (1998), 319–338.

REGULARITY OF SOME NONLINEAR QUANTITIES ON SUPERHARMONIC FUNCTIONS IN LOCAL HERZ-TYPE HARDY SPACES

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Abstract

In this paper, the authors introduce a kind of local Hardy spaces in $\mathbb{R}^n$ associated with the local Herz spaces. Then the authors investigate the regularity in these local Hardy spaces of some nonlinear quantities on superharmonic functions on $\mathbb{R}^2$. The main results of the authors extend the corresponding results of Evans and Müller in a recent paper.

1. Introduction

In recent years, Hardy space methods have lead to remarkable progress on nonlinear partial differential equations with critical growth; see [**3**], [**15**], [**1**] and [**16**]. The main reason is as follows: a typical difficulty when studying such equations is that the nonlinear term is a priori only known to be in $L^1(\mathbb{R}^n)$ in which there is no good elliptic theory; however, there is a well-established regularity theory in a particular subspace of $L^1(\mathbb{R}^n)$, namely the standard Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. In other words, Calderón-Zygmund singular integral operators are not bounded on $L^1(\mathbb{R}^n)$; but, they are indeed bounded on $\mathcal{H}^1(\mathbb{R}^n)$ if they satisfy the vanishing moment condition; see [**5**].

Keywords. Hardy spaces, Herz space, superharmonic function.
1991 *Mathematics subject classifications*: Primary 42B30; Secondary 35B65.
*The second and the third authors were supported in part by the NNSF and the SECF of China.

Moreover, Coifman, Meyer, Lions and Semmes in [**1**] proved that certain nonlinear quantities mentioned above are in fact in $\mathcal{H}^1(\mathbb{R}^n)$. In particular, L. C. Evans and S. Müller in [**3**] established the local boundedness in standard Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ of some nonlinear quantities on the weakly superharmonic functions. Applying their results to the two-dimensional Euler equations with nonnegative vorticity, Evans and Müller gave a new proof of Delort's results in [**2**].

On the other hand, a theory of the Hardy spaces associated with Herz spaces has been developed considerably in recent years; see [**6**], [**11**] and [**12**]. These Herz-type Hardy spaces are good substitutes of the usual Hardy spaces when studying boundedness of non-translation invariant operators (see [**13**]) and can be regarded as the local version at the origin of the usual Hardy spaces. There is also a good regularity theory in these spaces; see [**12**] and [**14**]. However, it is still not clear how to apply the theory on these spaces to partial differential equations. The main purpose of this paper is to establish a local version at the origin of Evans and Müller result in [**3**]. That is, we will study the local boundedness on the Hardy spaces associated with Herz spaces of some nonlinear quantities on weakly superharmonic functions. Our main results also extend the corresponding results of Evans and Müller, Theorem 1.1 and Theorem 5.4 in [**3**]. We hope this will be helpful to finding more applications of the Herz-type Hardy spaces in the study on partial differential equations; see also [**14**].

Let us first introduce some definitions.

For $k \in \mathbb{Z}$, let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$. We denote the characteristic function of A_k by χ_k.

Definition 1 ([8]). Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ is said to belong to Herz space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$, if

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} \equiv \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p} < \infty.$$

We introduce the local Herz spaces as follows.

Definition 2. Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ is said to belong to local Herz space $\dot{K}^{\alpha,p}_{q,\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ if for every $\phi \in C^\infty_c(\mathbb{R}^n)$, $\phi f \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$.

Obviously, by the above definition, $f \in \dot{K}^{\alpha,p}_{q,\mathrm{loc}}(\mathbb{R}^n)$ is equivalent to the fact that for any $k \in \mathbb{Z}$, $\chi_{B_k} f \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$. It is also easy to observe that $f \in \dot{K}^{\alpha,p}_{q,\mathrm{loc}}(\mathbb{R}^n)$ if and only if for any $k \in \mathbb{N}$, $\chi_{B_k} f \in \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$. We easily verify that $\dot{K}^{n(1/p-1/q),p}_{q,\mathrm{loc}}(\mathbb{R}^n) \subsetneqq L^p_{\mathrm{loc}}(\mathbb{R}^n)$ when $q > p \geq 1$ and $\dot{K}^{0,p}_{p,\mathrm{loc}}(\mathbb{R}^n) = L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Now choose $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}$ to be any smooth function satisfying

$$\operatorname{supp} \eta \subseteq B(0,1) \quad \text{and} \quad \int_{B(0,1)} \eta(x)\, dx = 1.$$

If $f \in L^1(\mathbb{R}^n)$, we write

$$f^*(x) \equiv \sup_{r \in (0,\infty)} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} f(y) \eta\left(\frac{x-y}{r}\right) dy \right|.$$

In [**11**] (and independently in [**6**]), S. Lu and D. Yang introduce the following Herz-type Hardy spaces, $\mathcal{H}\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)$, which can be regarded as the local version at the origin of the standard Hardy space, $\mathcal{H}^1(\mathbb{R}^n)$, studied by C. Fefferman and E. M. Stein in [**5**].

Definition 3 ([11]). Let $1 \leq q < \infty$. $f \in L^1(\mathbb{R}^n)$ is said to belong to the space $\mathcal{H}\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)$ if $f^* \in \dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)$. Moreover, we define

$$\|f\|_{\mathcal{H}\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)} \equiv \|f^*\|_{\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)}.$$

According to Lu-Yang [**12**], this definition does not depend on the particular choice of η. Obviously, when $q = 1$, $\mathcal{H}\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n) = \mathcal{H}^1(\mathbb{R}^n)$, the standard Hardy space studied by Fefferman and Stein in [**5**].

In [**4**], Fan and Yang introduce the local version, the space $h\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)$, of the space $\mathcal{H}\dot{K}^{n(1-1/q),1}_q(\mathbb{R}^n)$ in Goldberg's sense [**7**]. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we will hereafter set

$$f^{**}(x) = \sup_{0<r\leq 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} f(y) \eta\left(\frac{x-y}{r}\right) dy \right|.$$

Definition 4 ([4]). Let $1 \le q < \infty$. The space $h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ is defined by

$$h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n) \equiv \{f \in L^1(\mathbb{R}^n) : f^{**} \in \dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)\}.$$

Moreover, in this case, we set

$$\|f\|_{h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \equiv \|f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)}.$$

It is clearly that when $q = 1$, $h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$, the local Hardy space introduced by Goldberg in [**7**]. By the results in [**4**], we also know that the above definition does not depend on the special choice of η. Now, we introduce another local version $\mathcal{H}_{\mathrm{loc}}K_q^{n(1-1/q),1}(\mathbb{R}^n)$ of the space $\mathcal{H}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ as follows.

Definition 5. Let $1 \le q < \infty$. $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the space $\mathcal{H}_{\mathrm{loc}}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ if for any $\phi \in C_c^\infty(\mathbb{R}^n)$, $\phi f \in h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$.

In the next section, we will prove the following equivalent definition of the space $\mathcal{H}_{\mathrm{loc}}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$.

Proposition 1. *Let $1 \le q < \infty$. Then $f \in \mathcal{H}_{\mathrm{loc}}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ if and only if for any $k \in \mathbb{Z}$, $\chi_{B_k} f^{**} \in \dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$, which is also equivalent to the fact that $f^{**} \in \dot{K}_{q,\mathrm{loc}}^{n(1-1/q),1}(\mathbb{R}^n)$.*

We remark that when $q = 1$, $\mathcal{H}_{\mathrm{loc}}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n) = \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^n)$, the local Hardy space introduced by Evans and Müller in [**3**].

To state our main theorem, we still need to introduce the local Herz-type Sobolev space $H^1_{\mathrm{loc}}\dot{K}_q^{n(1/2-1/q),2}(\mathbb{R}^n)$; see also [**14**].

Definition 6. Let $1 \le q < \infty$. We call $u \in H^1_{\mathrm{loc}}\dot{K}_q^{n(1/2-1/q),2}(\mathbb{R}^n)$ if u and its distributional first partial derivatives $u_{x_1}, \dots, u_{x_n}$ belong to the space $\dot{K}_{q,\mathrm{loc}}^{n(1/2-1/q),2}(\mathbb{R}^n)$.

Obviously, when $q = 2$, $H^1_{\mathrm{loc}}\dot{K}_q^{n(1/2-1/q),2}(\mathbb{R}^n) = H^1_{\mathrm{loc}}(\mathbb{R}^n)$, the standard local Sobolev space.

Now, it is a position to state our main theorem.

Theorem 1. *Let* $1 < q < \infty$, $u \in H^1_{\mathrm{loc}}\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)$ *be a weak solution of the partial differential equation*

$$-\triangle u = w \quad in \quad \mathbb{R}^2.$$

where $w \in \dot{K}^{2(1-1/q),1}_{q,\mathrm{loc}}(\mathbb{R}^2)$ *and*

$$w \geq 0. \tag{1.1}$$

Then

$$u_{x_1}u_{x_2}, u^2_{x_1} - u^2_{x_2} \in \mathcal{H}_{\mathrm{loc}}\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2).$$

In addition, for each $\phi \in C^\infty_c(\mathbb{R}^2)$, *we have the estimate*

$$\begin{aligned}\|\phi u_{x_1}u_{x_2}\|_{h\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2)} &+ \|\phi(u^2_{x_1}u^2_{x_2})\|_{h\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2)} \\ &\leq c\|\chi_{B(0,R)}|Du|\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)}\end{aligned} \tag{1.2}$$

for some constant c and some radius R depending only on ϕ.

If we take $q = 1$, then Theorem 1 is just Theorem 1.1 in [**3**]. Thus, Theorem 1 is a generalization of Theorem 1.1 in [**3**]. In fact, Theorem 1 can be regarded as some kind local version of that theorem at the origin. See also Semmes [**16**] for several different versions of Evans and Müller's theorem.

It has been pointed by Evans and Müller in [**3**] that without the sign condition (1.1), Theorem 1 is false. However, in the radical case, the nonnegativity of w is not required. To be precise, we have

Theorem 2. *Let* $1 < q$, q_1, $q_2 < \infty$, $u \in H^1_{\mathrm{loc}}\dot{K}^{2(1/2-1/q_1),2}_q(\mathbb{R}^2)$ *be a weak solution of the partial differential equation*

$$-\triangle\, u = w \quad in \quad \mathbb{R}^2,$$

with $w \in \dot{K}^{2(1-1/q_2),1}_{q_2,\mathrm{loc}}(\mathbb{R}^2)$ *and* $u(x) = u(r)$, $w(x) = w(r)$ *for* $r = |x|$. *Then* $u_{x_1}u_{x_2}$, $u^2_{x_1} - u^2_{x_2} \in \mathcal{H}_{\mathrm{loc}}\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2)$. *Furthermore, for each* $\phi \in C^\infty_c(\mathbb{R}^2)$, *we have the estimate*

$$\begin{aligned}\|\phi u_{x_1}u_{x_2}\|_{h\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2)} &+ \|\phi(u^2_{x_1} - u^2_{x_2})\|_{h\dot{K}^{2(1-1/q),1}_q(\mathbb{R}^2)} \\ &\leq c\big(\|\chi_{B_k}|Du|\|^2_{\dot{K}^{2(1/2-1/q_1),2}_{q_1}(\mathbb{R}^2)} + \|\chi_{B_k}w\|^2_{\dot{K}^{2(1-1/q_2),1}_{q_2}(\mathbb{R}^2)}\big),\end{aligned} \tag{1.3}$$

where $k \in \mathbb{N}$ *depends only on* ϕ.

If we choose $q = 1$, $q_1 = 2$ and $q_2 = 1$, we recover Theorem 5.4 in [**3**].

Finally, we point that we also obtain a local version on Herz-type Hardy spaces $h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ of Jones-Journé's result on convergence a.e. and convergence in the distribution sense for a sequence bounded in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$; see [**9**], [**1**] and see also [**3**] for the local version on $h^1(\mathbb{R}^n)$.

Proposition 2. *Let $1 < q < \infty$. Suppose $\{f_k\}_{k=1}^{\infty}$ is bounded in $h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ and $f_k \to f$, a.e., as $k \to \infty$, for $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then $f \in h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ and*

$$f_k \to f \tag{1.4}$$

in the sense of distributions.

If we take $q = 1$, Proposition 2 is just Theorem 3.1 in [**3**].

2. Proofs of Theorems

We begin with the proof of Proposition 1.

Proof of Proposition 1: When $q = 1$, this proposition is just Lemma 5.1 in [**3**]. Now we restrict that $1 < q < \infty$. Suppose $f \in \mathcal{H}_{\rm loc}\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$. For any $k \in \mathbb{Z}$, we choose $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ when $x \in B(0, 2^k + 1)$. Then, if $x \in B_k$,

$$\begin{aligned}
(\phi f)^{**}(x) &= \sup_{0<r\le 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{r}\right) \phi(y) f(y)\, dy \right| \\
&= \sup_{0<r\le 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{r}\right) f(y)\, dy \right| \\
&= f^{**}(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\chi_{B_k} f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} &= \|\chi_{B_k} (\phi f)^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \\
&\le \|(\phi f)^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \\
&< \infty.
\end{aligned}$$

That is, $\chi_{B_k} f^{**} \in \dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$.

Now suppose $\chi_{B_k} f^{**} \in \dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ for each $k \in \mathbb{Z}$. Fix any $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subseteq B(0,R)$. Choose $l_0 \in \mathbb{N}$ such that $2^{l_0-1} \le R+1 < 2^{l_0}$. Observe first that

$$(\phi f)^{**}(x) = \sup_{0<r\le 1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{r}\right) \phi(y) f(y)\, dy \right|$$

vanishes if $x \in \mathbb{R}^n \setminus B(0,R+1)$. If $x \in B(0,R+1)$, by (5.2) in [**3**, p. 215], we have

$$\begin{aligned}(\phi f)^{**}(x) &\le c\left(f^{**}(x) + \sup_{0<r\le 1} \frac{1}{r^{n-1}} \int_{B(x,r)} |f(y)|\, dy\right)\\ &\le c\left(f^{**}(x) + \sum_{l=-\infty}^{l_0+1} \int_{\mathbb{R}^n} f_l(x-y)|y|^{1-n}\chi_{B_0}(y)\, dy\right),\end{aligned}$$

where $f_l(x) = |f(x)|\chi_l(x)$. Let $\overline{g}_l(x) \equiv \int_{\mathbb{R}^n} f_l(x-y)|y|^{1-n}\chi_{B_0}(y)\, dy$ and

$$g(x) = \sum_{l=-\infty}^{l_0+1} \overline{g}_l(x).$$

Then,

$$\|(\phi f)^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \le c\|\chi_{B_{l_0}} f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} + c\|\chi_{B_{l_0}} g\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)}.$$

By the hypothesis, we know that $\|\chi_{B_{l_0}} f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} < \infty$. We still need to show that $\|\chi_{B_{l_0}} g\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} < \infty$. In fact, we have

$$\begin{aligned}\|\chi_{B_{l_0}} g\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} &\le \sum_{l=-\infty}^{l_0+1} \sum_{k=-\infty}^{l_0} 2^{kn(1-1/q)} \|\overline{g}_l \chi_k\|_{L^q(\mathbb{R}^n)}\\ &= \sum_{l=-\infty}^{l_0+1} \sum_{k=-\infty}^{\min\{l+2,l_0\}} \cdots + \sum_{l=-\infty}^{l_0+1} \sum_{k=l+3}^{l_0} \cdots\\ &\equiv I_1 + I_2.\end{aligned}$$

For I_1, by the trivial estimate $\|\overline{g}_l\chi_k\|_{L^q(\mathbb{R}^n)} \le c\|f_l\|_{L^q(\mathbb{R}^n)}$ and the hypothesis, we obtain

$$\begin{aligned} I_1 &\le c\sum_{l=-\infty}^{l_0+1}\|f_l\|_{L^q(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{l+2}2^{kn(1-1/q)}\right)\\ &\le c\sum_{l=-\infty}^{l_0+1}2^{ln(1-1/q)}\|f_l\|_{L^q(\mathbb{R}^n)}\\ &\le c\|\chi_{B_{l_0+1}}f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)}\\ &< \infty. \end{aligned}$$

To estimate I_2, we first note that if $l \ge 0$ and $k \ge l+2$, then $\overline{g}_l(x) \equiv 0$ when $x \in A_k$. Thus,

$$I_2 = \sum_{l=-\infty}^{-1}\sum_{k=l+3}^{2}2^{kn(1-1/q)}\|\overline{g}_l\chi_k\|_{L^q(\mathbb{R}^n)}.$$

Now, since $l \le -1$ and $k \ge l+3$, for $x \in A_k$, we have

$$\begin{aligned} \overline{g}_l(x) &= \int_{\mathbb{R}^n}|f_l(x-y)||y|^{1-n}\chi_{B_0}(y)\,dy\\ &\le \int_{3|x|\le 4|y|\le 5|x|}|f_l(x-y)||y|^{1-n}\chi_{B_0}(y)\,dy\\ &\le c2^{k(1-n)}\|f_l\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

By substituting this into I_2, we obtain

$$\begin{aligned} I_2 &\le c\sum_{l=-\infty}^{-1}\sum_{k=l+3}^{2}2^k\|f_l\|_{L^1(\mathbb{R}^n)}\\ &\le c\sum_{l=-\infty}^{-1}2^{ln(1-1/q)}\|f\chi_l\|_{L^q(\mathbb{R}^n)}\\ &\le c\|\chi_{B_{-1}}f^{**}\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)}\\ &< \infty. \end{aligned}$$

This finishes the proof of Proposition 1. ■

Proof of Theorem 1: Fix $R > 8$. Let $B(0,R)$ denote the closed ball with center 0 and radius R, and set

$$v(x) = -\frac{1}{2\pi}\int_{B(0,R)} w(y)\log(|x-y|)\,dy.$$

Then

$$\theta = u - v \quad \text{is harmonic within} \quad B(0,R), \tag{2.1}$$

and

$$v_{x_i}(x) = \frac{1}{2\pi}\int_{B(0,R)} w(y)\frac{x_i - y_i}{|x-y|^2}\,dy, \quad i = 1,2. \tag{2.2}$$

Choosing $\eta \in C_c^\infty(\mathbb{R}^2)$ satisfying

$$\operatorname{supp}\eta \subseteq B(0,1), \qquad \int_{B(0,1)} \eta(x)\,dx = 1, \quad \eta \geq 0.$$

Let us also fix any point $x_0 \in \mathbb{R}^2$. Consider then for $0 < r \leq 1$, the expression

$$\begin{aligned} A &= \frac{1}{r^2}\int_{\mathbb{R}^2} v_{x_1}(x)v_{x_2}(x)\eta\left(\frac{x-x_0}{r}\right)\,dx \\ &= \frac{1}{4\pi^2 r^2}\int_{B(0,R)}\int_{B(0,R)} w(y)w(z) \\ &\qquad\qquad \left(\int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2}\frac{x_2 - z_2}{|x-z|^2}\eta\left(\frac{x-x_0}{r}\right)\,dx\right)\,dy\,dz, \end{aligned}$$

by (2.2). We change variables by replacing $(x - x_0)/r$ with x, $(y - x_0)/r$ with y, and $(z - x_0)/r$ with z to discover

$$\begin{aligned} A = \frac{r^2}{4\pi^2}&\int_{B(-x_0/r,R/r)}\int_{B(-x_0/r,R/r)} w(x_0 + ry)w(x_0 + rz) \\ &\times\left(\int_{\mathbb{R}^2}\frac{x_1 - y_1}{|x-y|^2}\frac{x_2 - z_2}{|x-z|^2}\eta(x)\,dx\right)\,dy\,dz. \end{aligned} \tag{2.3}$$

For fixed y, $z \in \mathbb{R}^2$ and $y \neq z$, let

$$B \equiv \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2} \frac{x_2 - z_2}{|x-z|^2} \eta(x)\, dx.$$

Then by the estimation in [**3**, p. 202–204], we have

$$|B| \leq c(1+|y|)^{-1}(1+|z|)^{-1}.$$

Substituting this to (2.3) and noting that $w \geq 0$, we obtain

$$\begin{aligned} |A| &\leq cr^2 \int_{B(-x_0/r, R/r)} \int_{B(-x_0/r, R/r)} \frac{w(x_0+ry)}{1+|y|} \frac{w(x_0+rz)}{1+|z|}\, dy\, dz \\ &= c\left(r \int_{B(-x_0/r, R/r)} \frac{w(x_0+ry)}{1+|y|}\, dy \right)^2. \end{aligned}$$

We restore the original variables by replacing $x_0 + ry$ with y to obtain

$$|A| \leq c \left(\int_{B(0,R)} \frac{w(y)}{r + |y-x_0|}\, dy \right)^2.$$

Therefore,

$$(v_{x_1} v_{x_2})^{**}(x_0) = \sup_{0<r\leq 1} |A| \leq c \left(\int_{B(0,R)} \frac{w(y)}{|y-x_0|}\, dy \right)^2. \tag{2.4}$$

For $x_0 \in \mathbb{R}^2$, we next write

$$g(x_0) = \int_{B(0,R)} \frac{w(y)}{|y-x_0|}\, dy$$

and claim

$$g \in \dot{K}^{1-1/q,2}_{2q,\mathrm{loc}}(\mathbb{R}^2). \tag{2.5}$$

To this end, first choose a cutoff function $\rho \in C_c^\infty(\mathbb{R}^2)$, with $0 \le \rho \le 1$, $\rho = 1$ on $B(0,R)$, $\rho \equiv 0$ on $\mathbb{R}^2 \setminus B(0,2R)$, $|D\rho| \le c/R$. Choose also $\lambda_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ satisfying $0 \le \lambda_\varepsilon \le 1$, $\lambda_\varepsilon = 0$ on $B(x_0,\varepsilon)$, $\lambda_\varepsilon = 1$ on $\mathbb{R}^2 \setminus B(x_0,2\varepsilon)$, $|D\lambda_\varepsilon| \le c/\varepsilon$. Let $M(f)$ denote the Hardy-Littlewood maximal function of f. By [**3**], we have

$$\begin{aligned}
0 \le g(x_0) &\le \int_{\mathbb{R}^2} \rho(y)\frac{w(y)}{|y-x_0|}\,dy \\
&\le -p.\,v.\int_{\mathbb{R}^2} \rho(y)Du(y)\cdot\frac{y-x_0}{|y-x_0|^3}\,dy + cM(|Du|\,|_{B(0,2R)})(x_0) \\
&\quad + c\int_{B(0,2R)} |Du(y)|\frac{1}{|y-x_0|}\,dy \\
&\equiv g_1(x_0)+g_2(x_0)+g_3(x_0).
\end{aligned}$$

Since $y/|y|^3$ is a Calderón-Zygmund kernel and $-1/q < 1-1/q < 2(1-1/(2q))$, by Corollary 2.1 in [**10**], we have

$$\begin{aligned}
\|g_1\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} &\le c\|\rho|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\le c\|\chi_{B(0,2R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned} \tag{2.6}$$

Furthermore, by the same corollary in [**10**], we have

$$\|g_2\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \le c\|\chi_{B(0,2R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}. \tag{2.7}$$

Now, let $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le R/2 < 2^{k_0+1}$. Therefore, $2^{k_0+2} \le 2R < 2^{k_0+3}$. Write

$$\begin{aligned}
g_3(x_0) &= c\int_{\mathbb{R}^2} |Du(y)|\frac{1}{|y-x_0|}\chi_{B(0,2R)}(y)\,dy \\
&\le \sum_{l=-\infty}^{k_0+3}\int_{\mathbb{R}^2} |Du(y)|\chi_l(y)\frac{1}{|y-x_0|}\,dy \\
&= \sum_{l=-\infty}^{k_0+3}\int_{\mathbb{R}^2} |Du(y+x_0)|\chi_l(y+x_0)\frac{1}{|y|}\,dy \\
&\equiv c\sum_{l=-\infty}^{k_0+3}\overline{g}_l(x_0).
\end{aligned}$$

Let $\varepsilon > 0$ to be determined late. We have

$$\begin{aligned}
&\|\chi_{B(0,R/2)}g_3\|_{\dot{K}_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\qquad \le c\left\{\sum_{k=-\infty}^{k_0+1} 2^{2k(1-1/q)}\left(\sum_{l=-\infty}^{k_0+3}\|\chi_k\overline{g}_l\|_{L^{2q}(\mathbb{R}^2)}\right)^2\right\}^{1/2} \\
&\qquad \le c\left\{\sum_{k=-\infty}^{k_0+1} 2^{2k(1-1/q)}\left(\sum_{l=-\infty}^{k_0+3}2^{-2\varepsilon l}\|\chi_k\overline{g}_l\|^2_{L^{2q}(\mathbb{R}^2)}2^{2\varepsilon k_0}\right)\right\}^{1/2} \\
&\qquad \le c(R)\left\{\sum_{l=-\infty}^{k_0+3}\sum_{k=-\infty}^{\min\{l+2,k_0+1\}} 2^{2k(1-1/q)-2\varepsilon l}\|\chi_k\overline{g}_l\|^2_{L^{2q}(\mathbb{R}^2)}\right\}^{1/2} \qquad (2.8) \\
&\qquad\quad + c(R)\left\{\sum_{l=-\infty}^{k_0-2}\sum_{k=l+3}^{k_0+1} 2^{2k(1-1/q)-2\varepsilon l}\|\chi_k\overline{g}_l\|^2_{L^{2q}(\mathbb{R}^2)}\right\}^{1/2} \\
&\qquad \equiv J_1 + J_2.
\end{aligned}$$

For J_1, we first note that when $x_0 \in A_k$ and $k \le l+2$, by the definition of $\overline{g}_l(x_0)$, we have

$$|\overline{g}_l(x_0)| \le \int_{\mathbb{R}^2}|Du(y+x_0)|\chi_l(y+x_0)\frac{1}{|y|}\chi_{\{y\in\mathbb{R}^2:|y|\le 2^{l+3}\}}(y)\,dy.$$

Thus,

$$\begin{aligned}
\|\chi_l\overline{g}_l\|_{L^{2q}(\mathbb{R}^2)} &\le \|\,|Du|\chi_l\|_{L^{2q}(\mathbb{R}^2)}\int_{|y|\le 2^{l+3}}\frac{1}{|y|}\,dy \\
&\le c2^l\|\,|Du|\chi_l\|_{L^{2q}(\mathbb{R}^2)}.
\end{aligned}$$

By choosing $0 < \varepsilon < 1$, we obtain

$$\begin{aligned}
J_1 &\le c\left\{\sum_{l=-\infty}^{k_0+3} 2^{2(1-\varepsilon)l}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\left(\sum_{k=-\infty}^{l+2}2^{2k(1-1/q)}\right)\right\}^{1/2} \\
&\le c\left\{\sum_{l=-\infty}^{k_0+3} 2^{2l(1-1/q)}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\right\}^{1/2} \\
&\le c\|\chi_{B(0,4R)}|Du|\,\|_{\dot{K}_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned}$$

On J_2, we note that when $k \geq l+3$ and $x_0 \in A_k$,

$$\begin{aligned}\overline{g}_l(x_0) &\leq \int_{\mathbb{R}^2} |Du(x_0+y)|\chi_l(x_0+y)\frac{1}{|y|}\chi_{\{y:\,\frac{3}{4}|x_0|\leq|y|\leq\frac{5}{4}|x_0|\}}(y)\,dy\\ &\leq c2^{-k}\int_{\mathbb{R}^2}|Du(x_0+y)|\chi_l(x_0+y)\,dy\\ &\leq c2^{-k+2l(1-1/(2q))}\|\,|Du|\chi_l\|_{L^{2q}(\mathbb{R}^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}J_2 &\leq c\left\{\sum_{l=-\infty}^{k_0-2}2^{2l(1-1/q)}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\left(\sum_{k=l+3}^{k_0+1}2^{2l(1-\varepsilon)}\right)\right\}^{1/2}\\ &\leq c\left\{\sum_{l=-\infty}^{k_0-2}2^{2l(1-1/q)}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\left(\sum_{k=l+3}^{k_0+1}2^{2k(1-\varepsilon)}\right)\right\}^{1/2}\\ &\leq c\left\{\sum_{l=-\infty}^{k_0-2}2^{2l(1-1/q)}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\right\}^{1/2}\\ &\leq c\|\chi_{B(0,4R)}|Du|\,\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)}.\end{aligned}$$

Substituting the estimations on J_1 and J_2 into (2.8), we have

$$\|\chi_{B(0,R/2)}g_3\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)} \leq c\|\chi_{B(0,4R)}|Du|\,\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)}. \tag{2.9}$$

By combining (2.6), (2.7) and (2.9), we obtain

$$\|\chi_{B(0,R/2)}g\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)} \leq c\|\chi_{B(0,4R)}|Du|\,\|_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)} < \infty. \tag{2.10}$$

Thus, the claim (2.5) holds. From (2.10) and (2.4), we deduce

$$\begin{aligned}&\|\chi_{B(0,R/2)}(v_{x_1}v_{x_2})^{**}\|_{\dot{K}^{2(1-1/q),1}_{2q}(\mathbb{R}^2)}\\ &\qquad\leq c(R)\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot{K}^{1-1/q,2}_{2q}(\mathbb{R}^2)}.\end{aligned} \tag{2.11}$$

Now recall from (2.1) that $\theta = u - v$ is harmonic and thus smooth within $B(0,R)$. Consequently, if $x_0 \in B(0,R/2)$, by a similar procedure to [**10**,

p. 205], we obtain that $(\theta_{x_1}\theta_{x_2})^{**}(x_0) \le c\sup_{x\in B(0,3R/4)}|D\theta(x)|^2$; and

$$
\begin{aligned}
\sup_{B(0,3R/4)}|D\theta| &\le c(R)\int_{B(0,R)}|Du(x)|\,dx \\
&\quad + \frac{1}{|B(0,R)|}\int_{B(0,R)}|v(x)|\,dx \\
&\le c(R)\int_{B(0,R)}|Du(x)|\,dx + c(R)\int_{B(0,R)}w(x)\,dx \\
&\le c(R)\int_{B(0,2R)}|Du(x)|\,dx \\
&\le c(R)\sum_{l=-\infty}^{k_0+3}\int_{\mathbb{R}^2}|Du(x)|\chi_l(x)\,dx \qquad (2.12)\\
&\le c(R)\sum_{l=-\infty}^{k_0+3}2^{2l(1-1/(2q))}\|\,|Du|\chi_l\|_{L^{2q}(\mathbb{R}^2)} \\
&\le c(R)2^{k_0}\left\{\sum_{l=-\infty}^{k_0+3}2^{2l(1-1/q)}\|\,|Du|\chi_l\|^2_{L^{2q}(\mathbb{R}^2)}\right\}^{1/2} \\
&\le c(R)\|\chi_{B(0,4R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\chi_{B(0,R/2)}(\theta_{x_1}\theta_{x_2})^{**}\|_{\dot K_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\qquad \le c\left(\sum_{l=-\infty}^{k_0+1}2^{2l}\right)\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \qquad (2.13)\\
&\qquad \le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned}
$$

Similarly, if $x_0 \in B(0,R/2)$, by (2.12), we have

$$
\begin{aligned}
(\theta_{x_1}v_{x_2})^{**}(x_0) &= \sup_{0<r\le 1}\left|\frac{1}{r^2}\int_{\mathbb{R}^2}\theta_{x_1}v_{x_2}\eta\left(\frac{x-x_0}{r}\right)\,dx\right| \\
&\le c\sup_{B(0,3R/4)}|D\theta|^2 + c\Big(\sup_{B(0,3R/4)}|D\theta|\Big)M(\chi_{B(0,R)}|Du|)(x_0) \\
&\le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\quad + c\|\chi_{B(0,4R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}M(\chi_{B(0,R)}|Du|)(x_0).
\end{aligned}
$$

Now, Corollary 2.1 in [**10**] gives us that

$$
\begin{aligned}
&\|\chi_{B(0,R/2)}(\theta_{x_1}v_{x_2})^{**}\|_{\dot K_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\qquad \le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} + c\|\chi_{B(0,4R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\qquad\quad \times \left(\sum_{k=-\infty}^{k_0+1} 2^{2k(1-1/q)}\|\chi_k M(\chi_{B(0,R)}|Du|)\|_{L^q(\mathbb{R}^2)}\right) \\
&\qquad \le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} + c\|\chi_{B(0,4R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\qquad\quad \times \left(\sum_{k=-\infty}^{k_0+1} 2^{k(1-1/q)+k}\|\chi_k M(\chi_{B(0,R)}|Du|)\|_{L^{2q}(\mathbb{R}^2)}\right) \\
&\qquad \le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\qquad\quad + c\|\chi_{B(0,4R)}|Du|\,\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}\|M(\chi_{B(0,R)}|Du|)\|_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} \\
&\qquad \le c\|\chi_{B(0,4R)}|Du|\,\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned}
$$

The same argument shows that

$$
\begin{aligned}
\text{(2.15)}\quad &\|\chi_{B(0,R/2)}(v_{x_1}\theta_{x_2})^{**}\|_{\dot K_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\qquad\qquad \le c\|\chi_{B(0,4R)}|Du|\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)}.
\end{aligned}
$$

By combining the estimations (2.11), (2.13)-(2.15), we obtain

$$
\|\chi_{B(0,R/2)}(u_{x_1}u_{x_2})^{**}\|_{\dot K_q^{2(1-1/q),1}(\mathbb{R}^2)} \le c\|\chi_{B(0,4R)}|Du|\|^2_{\dot K_{2q}^{1-1/q,2}(\mathbb{R}^2)} < \infty,
$$

for each $R \ge 8$. Hence $u_{x_1}u_{x_2} \in \mathcal{H}_{\mathrm{loc}}\dot K_q^{2(1-1/q),1}(\mathbb{R}^2)$.

By rotating variables, similar to [**3**], we can prove that $u_{x_1}^2 - u_{x_2}^2 \in \mathcal{H}_{\mathrm{loc}}\dot K_q^{2(1-1/q),1}$ $(\mathbb{R}^2)$. Finally, estimate (1.2) is a consequence of Proposition 1.

This finishes the proof of Theorem 1. ■

Proof of Theorem 2: We have the ordinary differential equation

$$-\frac{1}{r}(ru')'(r) = w(r). \tag{2.16}$$

We only prove the estimates for $u_{x_1}u_{x_2}$, because, similar to [**3**], those for $u_{x_1}^2 - u_{x_2}^2$ follow then by performing a rotation. Let $\eta \in C_c^\infty(B(0,1))$, $\eta \geq 0$ and $\int_{B(0,1)} \eta(x)\,dx = 1$. Define

$$f(x) = (u_{x_1}u_{x_2})(x) = \frac{x_1x_2}{r^2}(u')^2(r) \quad \text{and} \quad f_i = f\chi_i \quad \text{for} \quad i \in \mathbb{Z}.$$

We also write

$$f^{**}(x) = \sup_{0<r\leq 1}\left|\frac{1}{r^2}\int_{\mathbb{R}^2} \eta\left(\frac{x-y}{r}\right) f(y)\,dy\right|$$

and for $i \in \mathbb{Z}$,

$$f_i^{**}(x) = \sup_{0<r\leq 1}\left|\frac{1}{r^2}\int_{\mathbb{R}^2} \eta\left(\frac{x-y}{r}\right) f_i(y)\,dy\right|.$$

Since $w \in \dot{K}_{q_2,\mathrm{loc}}^{2(1-1/q_2),1}(\mathbb{R}^2) \subset L_{\mathrm{loc}}^1(\mathbb{R}^2)$, the ordinary differential equation (2.16) implies that $v(r) \equiv ru'(r)$ is absolutely continuous on each interval $[0,R]$. In particular, if $2^{i-1} \leq r \leq 2^i$,

$$\left|v(r) - \frac{1}{2^{i-1}}\int_{2^{i-1}}^{2^i} v(r)\,dr\right| \leq \int_{2^{i-1}}^{2^i} |v'(r)|\,dr \leq \int_{2^{i-1}}^{2^i} r|w(r)|\,dr. \tag{2.17}$$

It is easy to verify that if $h \in L^q(\mathbb{R}^2)$, $\operatorname{supp} h \subseteq B(0,r)$ and $\int_{\mathbb{R}^2} h(x)\,dx = 0$, then $|B(0,r)|^{1/q-1}\|h\|_{L^q(\mathbb{R}^2)}^{-1}\, h$ is a central $(2(1-1/q),q)$-atom supported in $B(0,r)$; see [**11**]. Thus, by Theorem 2.1 in [**12**] (see also [**6**]),

$$\begin{aligned}
\|h^{**}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} &\leq \|h^{*}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\leq cr^{2(1-1/q)}\|h\|_{L^q(\mathbb{R}^2)} \\
&\leq cr^2\|h\|_{L^\infty(\mathbb{R}^2)},
\end{aligned}$$

if $h \in L^\infty(\mathbb{R}^2)$. Therefore,

$$\|f_i^{**}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} \leq c2^{2i}\|f_i\|_{L^\infty(\mathbb{R}^2)}.$$

Moreover, $f^{**}_{i+2}(x) = 0$ on $B(0, 2^{i+1} - 1) \supset B(0, 2^i)$ if $i \geq 1$. Now, by (2.17),

$$
\begin{aligned}
2^{2i}\|f_i\|_{L^\infty(\mathbb{R}^2)} &\leq 4 \sup_{2^{i-1}<r\leq 2^i} r^2|u'(r)|^2 \\
&\leq c\left\{2^{-i}\int_{2^{i-1}}^{2^i} |ru'(r)|\,dr + \int_{2^{i-1}}^{2^i} |(ru'(r))'|\,dr\right\}^2 \\
&\leq c\left\{2^{-i/q_1}\int_{2^{i-1}}^{2^i} |r^{1/q_1}u'(r)|\,dr + \int_{2^{i-1}}^{2^i} r|w(r)|\,dr\right\}^2 \\
&\leq c\left\{2^{i(1-2/q_1)}\left(\int_{2^{i-1}}^{2^i} r|u'(r)|^{q_1}\,dr\right)^{1/q_1} + \int_{A_i} |w(x)|\,dx\right\}^2 \\
&\leq c\left\{2^{i(1-2/q_1)}\left(\int_{A_i} |Du(x)|^{q_1}\,dx\right)^{1/q_1} + \int_{A_i} |w(x)|\,dx\right\}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\chi_{B_k} f^{**}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\quad\leq \sum_{j=-\infty}^{\infty} \|\chi_{B_k} f_j^{**}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\quad= \sum_{j=-\infty}^{k+2} \|\chi_{B_k} f_j^{**}\|_{\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)} \\
&\quad\leq c\sum_{j=-\infty}^{k+2} \left\{\left(2^{j(1-2/q_1)}\|\,|Du|\chi_j\|_{L^{q_1}(\mathbb{R}^2)}\right)^2 + \|\,|w|\chi_j\|^2_{L^1(\mathbb{R}^2)}\right\} \\
&\quad\leq c\sum_{j=-\infty}^{k+2} 2^{4j(1/2-1/q_1)}\|\,|Du|\chi_j\|^2_{L^{q_1}(\mathbb{R}^2)} + c\sum_{j=-\infty}^{k+2} \|\,|w|\chi_j\|^2_{L^1(\mathbb{R}^2)} \\
&\quad\leq c\|\chi_{B_{k+2}}|Du|\,\|^2_{\dot{K}_{q_1}^{2(1/2-1/q_1),2}(\mathbb{R}^2)} + c\|\chi_{B_{k+2}}w\|^2_{\dot{K}_{q_2}^{2(1-1/q_2),1}(\mathbb{R}^2)} \\
&\quad< \infty.
\end{aligned}
$$

Hence $u_{x_1}u_{x_2} \in \mathcal{H}_{\mathrm{loc}}\dot{K}_q^{2(1-1/q),1}(\mathbb{R}^2)$. The estimation (1.4) now follows from Proposition 1.

This finishes the proof of Theorem 2. ∎

Proof of Proposition 2: Since $h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$, (1.4) holds by Theorem 3.1 in [**3**]. The remaining thing is to verify that $f \in h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$. Choose $\phi \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\phi \subset B(0,1)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For $t > 0$, set $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$. Then by (1.4), for every $x \in \mathbb{R}^n$ and $0 < t \le 1$, we have

$$\phi_t * f_k(x) \longrightarrow \phi_t * f(x), \quad \text{as} \quad k \longrightarrow \infty;$$

therefore,

$$\begin{aligned}
\varliminf_{k\to\infty} \sup_{1\ge t>0} |\phi_t * f_k(x)| &= \sup_k \inf_{l\ge k} \sup_{1\ge t>0} |\phi_t * f_l(x)| \\
&\ge \sup_{1\ge t>0} \sup_k \inf_{l\ge k} |\phi_t * f_l(x)| \\
&= \sup_{1\ge t>0} \varliminf_{k\to\infty} |\phi_t * f_k(x)| \\
&= \sup_{1\ge t>0} |\phi_t * f(x)|.
\end{aligned}$$

On the other hand, since $\|f_k\|_{h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \le c$, it follows from Theorem 2.1 in [**4**] that

$$\Big\| \sup_{1\ge t>0} |\phi_t * f_k| \Big\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \le c.$$

By Fatou's lemmas of series and integration, we have

$$\begin{aligned}
c &\ge \varliminf_{k\to\infty} \Big\| \sup_{1\ge t>0} |\phi_t * f_k| \Big\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)} \\
&= \varliminf_{k\to\infty} \left\{ \sum_{l=-\infty}^{\infty} 2^{ln(1-1/q)} \Big\| \chi_l \sup_{1\ge t>0} |\phi_t * f_k| \Big\|_{L^q(\mathbb{R}^n)} \right\} \\
&\ge \sum_{l=-\infty}^{\infty} 2^{ln(1-1/q)} \varliminf_{k\to\infty} \Big\| \chi_l \sup_{1\ge t>0} |\phi_t * f_k| \Big\|_{L^q(\mathbb{R}^n)} \\
&\ge \sum_{l=-\infty}^{\infty} 2^{ln(1-1/q)} \Big\| \chi_l \varliminf_{k\to\infty} \sup_{1\ge t>0} |\phi_t * f_k| \Big\|_{L^q(\mathbb{R}^n)} \\
&\ge \sum_{l=-\infty}^{\infty} 2^{ln(1-1/q)} \Big\| \chi_l \sup_{1\ge t>0} |\phi_t * f| \Big\|_{L^q(\mathbb{R}^n)} \\
&= \Big\| \sup_{1\ge t>0} |\phi_t * f| \Big\|_{\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)}.
\end{aligned}$$

Again, from Theorem 2.1 in [**4**], we deduce that $f \in h\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$.

This finishes the proof of Proposition 2. ■

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Primera versió rebuda el 26 de febrer de 1997,
darrera versió rebuda el 17 de març de 1998